Families of G-Constellations Parametrised by Resolutions of Quotient Singularities

submitted by

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for the degree of PhD

of the

University of Bath

2004

...when in eternal lines to time thou growest...

Acknowledgements

Thanks to my family, for being there for me, while I was all over the place for them.

Thanks to Lenochka, for being my summer's day.

Thanks to Lisse, for being *ma chère duchesse*. Thanks to Melli, Vitya, Vadim, Alex, Carl, Hartmut, Paul and Charles for being the friends indeed. Thanks to Vintra and Alena, for making my life rather lively.

Thanks to everyone in Bath and Utrecht for creating a friendly and supportive environment to work in.

Thanks to Miles Reid, Alastair Craw and Akira Ishii for enlightening discussions and for their work on G-constellations which inspired this thesis.

Thanks to Gregory Sankaran and Eduard Looijenga for the discussions which taught me many tricky aspects of scheme theory and algebraic topology.

And a very special thanks to my supervisor Alastair King for all the inspiration, all the insights, all the discussions, all the shared knowledge, all the support and for the endless patience.

Dedicated to Marcel Proust and e.e. cummings.

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Chapter 1

Introduction

1.1 Background

Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite subgroup and X the quotient space $\mathbb{C}^3/G$. Nakamura made a study of G-clusters, the G-invariant subschemes of dimension 0 whose coordinate ring with the induced G-action is the regular representation V_{reg} of G. He introduced the scheme G-Hilb, which parametrises all G-clusters and showed [Nak00] that, in the case of G abelian, it is a crepant resolution of $\mathbb{C}^3/G$. He conjectured that the same holds for the non-abelian case.

Craw and Reid [CR02] introduced an alternative explicit calculation of G-Hilb $\mathbb{C}^3$ and, in his thesis [Cra01], Craw introduced the concept of G-constellation as a generalisation of G-cluster. A G-constellation is a G-equivariant Artinian coherent sheaf whose global sections form the regular representation of G. In particular, the structure sheaf of any G-cluster is a G-constellation.

G-constellations can be interpreted in terms of representations of the McKay quiver of G. This allows for the use of an earlier result of King [Kin94] on GIT construction of moduli spaces of quiver representations to introduce the stability conditions known as θ-stability on G-constellations and to construct their moduli spaces M_θ. In a quiver-theoretic context, Kronheimer [Kro89] and Sardo-Infirri [SI96a], [SI96b] have already considered these moduli spaces and have studied the chamber structure in the space Π of stability parameters θ, where all values of θ in the same chamber yield the same M_θ. Bridgeland, King and Reid [BKR01] use derived category methods to show that G-Hilb is a crepant resolution of X for any finite $G \subset \mathrm{SL}_3(\mathbb{C})$. Their method can be used to show that, for any chamber in Π, M_θ is a crepant resolution. However it yields little information about either the structure of the chamber space or the geometry of the M_θ.

Craw in his thesis conjectured that every projective crepant resolution of X can be realised as a moduli space M_θ of θ-stable G-constellations for some chamber in Π. A recent paper by Craw and Ishii [CI02] proves this for all abelian $G \subset \mathrm{SL}_3(\mathbb{C})$.

This thesis divides thematically into two major parts. Chapters 2 to 5, treat the classification of natural families of G-constellations that a given resolution can parametrise. Chapter 6 works towards establishing the existence of a simple family of G-constellations on a given resolution.

1.2 Generically natural families

Rather than constructing a resolution as a moduli space of G-constellations, we take an arbitrary (not necessarily projective or crepant) resolution of X and study what families of G-constellations it can parametrise.

Let G be any finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X be a quotient scheme $\mathbb{C}^n/G$ and Y be a resolution of X.

$$\begin{tikzcd} Y \arrow[dr, "\pi"] & & \mathbb{C}^n \arrow[dl, "q"'] \\ & X & \end{tikzcd}$$

Let R denote the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$ of $\mathbb{C}^n$. A (G, R)-*module* is a G-representation V together with a G-equivariant action of R. The categories of finite length G-equivariant coherent sheaves on $\mathbb{C}^n$ and of (G, R)-modules are equivalent and we choose to work in the latter category. So by G-constellation we shall usually mean a (G, R)-module, whose underlying G-representation is the regular representation V_{reg}. In the most naive sense, a family of G-constellations parametrised by Y is a locally free sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules, equipped with equivariant G and R actions, whose fibre $\mathcal{F}_{|p}$ (the pullback of $\mathcal{F}$ to a point scheme $p \hookrightarrow Y$) at any point p of Y is a G-constellation.

In Chapter 2 we develop a criterion, which, out of all the possible families of G-constellations, picks out the geometrically natural families, in which the G-constellation $\mathcal{F}_{|p}$, parametrised by a point p of Y in $\mathcal{F}$, is geometrically related to p itself. The families satisfying this criterion (Definition 2.4) we call `generically natural families` or `gnat-families`, for short.

For an example of the kind of geometric relation between p and $\mathcal{F}_{|p}$ which we might want, consider the support of a G-constellation $\mathcal{F}_{|p}$, viewed as a G-equivariant finite length sheaf on $\mathbb{C}^n$. The support is a finite union of G-orbits. We show (Lemma 2.6) that, in any gnat-family $\mathcal{F}$, the set-theoretic support of a G-constellation $\mathcal{F}_{|p}$, parametrised by a point $p \in Y$, is precisely the orbit $q^{-1}(\pi(p))$ in $\mathbb{C}^3$. In particular, any free orbit Z of G is a G-cluster and by dimension considerations its structure sheaf $\mathcal{O}_Z$ is the only G-constellation whose support is the support of Z. On the other hand, if $U \subset X$ is any open set such that G acts freely on $q^{-1}(U)$, then $q_*(\mathcal{O}_{\mathbb{C}^n})|_U$ is a natural family of G-clusters parametrised by U. We show (Lemma 2.6) that restricted to $\pi^{-1}(U)$ any gnat-family is isomorphic (up to tensoring up by a G-invariant line-bundle) to the natural family $\pi^*(q_*(\mathcal{O}_{\mathbb{C}^n}))$.

It turns out that the naturality criterion we develop can be reduced to merely asking for $\mathcal{F}$ to agree with the natural family $\pi^*(q_*(\mathcal{O}_{\mathbb{C}^n}))$ generically, i.e. for their stalks at the generic point of Y to be isomorphic as G, R and $K(Y)$ modules. Hence the name `gnat-family`. We then show (Proposition 2.5) that indeed any family $\mathcal{F}$ which satisfies this requirement on its stalk at the generic point of Y has all the desired properties of geometrical naturality: the support of the G-constellation $\mathcal{F}_{|p}$ parametrised by $p \in Y$ is indeed $q^{-1}(\pi(p))$ and $\mathcal{F}$ agrees with $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ wherever G acts freely. Moreover, $\mathcal{F}$ can be G, R and $\mathcal{O}_Y$ equivariantly embedded into the generic stalk of $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$: the constant sheaf $K(\mathbb{C}^n)$ on Y.

Now for abelian G, any family of G-constellations is a direct sum of invertible G-eigensheaves. Therefore any gnat-family $\mathcal{F}$ splits into a direct sum of invertible $\mathcal{O}_Y$-submodules of $K(\mathbb{C}^n)$.

In Chapter 3 we extend the construction of Cartier divisors on Y, as global sections of $K^*(Y)/\mathcal{O}_Y^*$, by defining a *G-Cartier divisor* to be a global section of $K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$, where $K_G^*(\mathbb{C}^n)$ is the group of all nonzero G-homogeneous rational functions on $\mathbb{C}^n$. And just in a same way that a Cartier divisor corresponds to an invertible sub-$\mathcal{O}_Y$-module of $K(Y)$, a G-Cartier divisor corresponds to an invertible sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$.

To make a link with Weil divisors, we extend a valuation at a prime divisor from $K(Y)$ to $K^*_G(\mathbb{C}^n)$ and show this is a natural notion. We then define G-Weil divisors (Definition 3.5) as a subset of $\mathbb{Q}$-Weil divisors on Y, in such a way as to have the correspondence between G-Weil and G-Cartier divisors in place when Y is smooth.

In Chapter 4 (Propositions 4.2 and 4.3) we take a detour to translate some of the concepts introduced in Chapter 3 into the language of toric geometry used to describe toric resolutions of X. Toric resolutions make for good explicit calculations on Y and in Example 4.1 we introduce Y on which all of the examples in Chapters 4 and 5 are calculated: a single toric flop of G-Hilb, where G is the cyclic subgroup of $\mathrm{GL}_3(\mathbb{C})$ of order 8 traditionally denoted $\frac{1}{8}(1, 2, 5)$.

In Chapter 5 we return to gnat-families. Let $\mathcal{F}$ be any gnat-family, χ any character of G and $\mathcal{F}_\chi$ a corresponding eigensheaf. Inclusion into $K(\mathbb{C}^n)$ as a (G, R)-submodule of its generic stalk induces an inclusion of $\mathcal{F}_\chi$ into $K(\mathbb{C}^n)$ and hence defines a G-Cartier divisor and consequently a G-Weil divisor D_χ.

Conversely, given a set $\{D_\chi\}_{\chi \in G^\vee}$, where $G^\vee$ is a character group $\mathrm{Hom}(G, \mathbb{C}^*)$ and each D_χ is a χ-Weil divisor, we could ask when is $\mathcal{O}_Y$-submodule $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$ of $K(\mathbb{C}^n)$ a gnat-family. We show in Proposition 5.5 that this is equivalent to the condition that for any G-homogeneous $f \in R$, the Weil divisor

$$D_\chi + (f) - D_{\chi\rho(f)}$$

is effective, where $\rho(f) \in G^\vee$ is the homogeneous weight of f and (f) the principal G-Weil divisor of f. This condition can be thought of as demanding that the action of R on $\bigoplus \mathcal{L}(-D_\chi)$ by multiplication in $K(\mathbb{C}^n)$ is everywhere regular on Y. It suffices to check the effectiveness of the divisor above with $f = x_i$ for each of the basic monomials x_i, so we establish a 1-to-1 correspondence between isomorphism classes of gnat-families and sets $\{D_\chi\}_{\chi \in G^\vee}$ of G-Weil divisors satisfying a finite number of inequalities.

It is usual in moduli problems to consider families up to equivalence, namely twisting by a line bundle. We show that any equivalence class of gnat-families contains a unique family with $D_{\chi_0} = 0$ in the corresponding divisor set, where χ_0 is the trivial character. We say that a gnat-family is *normalised*if it satisfies $D_{\chi_0} = 0$. We then define `maximal shift` divisors $\{M_\chi\}_{\chi \in G^\vee}$ (Definition 5.18), show that $\mathcal{F}_{\max} = \oplus(-\mathcal{M}_\chi)$ is a normalised gnat-family (Lemma 5.19) and then show (Proposition 5.20) that $\mathcal{F}_{\max}$ provides a bound on all normalised gnat-families. Specifically, for any normalised gnat-family, we prove that the corresponding G-Weil divisor set $\{D_\chi\}$ satisfies

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}}$$

In particular, this implies that the number of equivalence classes is finite, since the result proved in Corollary 3.14 implies that the only nonzero summands of M_χ are the exceptional divisors and the birational transforms in Y of images in X of coordinate strata $x_i = 0$ of X.

Thus we obtain the following classification theorem (Theorem 5.30):

Theorem (Classification)**.** *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X the quotient of $\mathbb{C}^n$ by the action of G and Y a resolution of X. Then every gnat-family on Y, up to isomorphism, is of the form $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$, where each D_χ is a χ-Weil divisor and the set $\{D_\chi\}$ satisfies the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad \forall\, \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Here $\rho(f) \in G^\vee$ is the homogeneous weight of f. Conversely for any such set $\{D_\chi\}$, $\bigoplus \mathcal{L}(-D_\chi)$ is a gnat-family.

Moreover, each equivalence class of gnat-families has precisely one family with $D_{\chi_0} = 0$. The divisor set $\{D_\chi\}$ corresponding to such a family satisfies the inequalities

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}}$$

where $\{M_\chi\}$ is a fixed divisor set depending only on G and Y. In particular, the number of equivalence classes of families is finite.

1.3 Orthonormal Families of G-Constellations

In Chapter 6, we turn our attention to the case $G \subset \mathrm{SL}_3(\mathbb{C})$ and Y a crepant resolution of X. In case when gnat-family $\mathcal{F}$ is `orthonormal`(i.e. `orthogonal` and `simple`, see Definition 6.1), we explain in 6.1 that a minor modification of the machinery of [BKR01] allows to establish a derived category equivalence

$$D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^3)$$

of bounded derived categories of coherent sheaves on Y and G-equivariant coherent sheaves on $\mathbb{C}^3$.

Craw and Ishii proved in [CI02] that every projective crepant resolution of X is realised as a moduli space M_θ of θ-stable G-constellations. It is easy to show that, for any θ, the tautological family of θ-stable G-constellations on M_θ is orthonormal. Thus for any projective crepant resolution Y of X, one of the gnat-families on Y is orthonormal. We ask (Question 6.2) that the same is also true for nonprojective crepant resolutions of X.

As a first step towards answering Question 6.2, we study simplicity of gnat-families on a given crepant resolution Y of X.

First, for any G-constellation V, we define (Definition 6.3) a subgraph Γ_V of the McKay quiver of G with the following property: V is simple if and only if Γ_V is connected. This was first introduced by Craw and Ishii in [CI02], Section 10.2. Then, in Sections 6.2 and 6.3, we prove that for any gnat-family $\mathcal{F}$ $\Gamma_{\mathcal{F}_{|p}}$ stays constant as $p \in Y$ varies along any given orbit of the torus T, the quotient by G of the maximal torus of $\mathrm{SL}_3(\mathbb{C})$ containing G. Thus, we define $\Gamma_{\mathcal{F},\sigma}$ to be $\Gamma_{\mathcal{F}_p}$ for any p on S_σ, the orbit of T which corresponds to a cone σ in the toric fan of Y.

Therefore to show that a given $\mathcal{F}$ is simple it suffices to show that Γ_σ is connected for every orbit S_σ of T in Y. Note that for the open orbit S_0 this is trivial, as the restriction of $\mathcal{F}$ to S_0 is a family of G-clusters, and the structure sheaf of any G-cluster is generated as an R-module by its χ_0-eigensheaf and is hence simple.

In Section 6.4, we summarise the embedding of the McKay quiver of G into a real 2-torus T_G first introduced by Reid in [Rei97]. We shall rely heavily on the topological properties of T_G, so in Section 6.5, we describe how the McKay quiver of G provides a CW-complex structure $\mathcal{T}$ on T_G. Then in Sections 6.6 and 6.8 we explain how various numerical data defining the family $\mathcal{F}$ can be described in terms of chains and cochains in chain complex $\mathcal{C}^{\mathcal{T}}_\bullet$ of the CW complex $\mathcal{T}$, and set up the necessary formalities to do homology and cohomology calculations on T_G.

In Section 6.7, we prove (Corollary 6.35) that for any $\mathcal{F}$ and any codimension 1 orbit S_σ in the fan of Y, the graph $\Gamma_{\mathcal{F},\sigma}$ is connected. Thus any gnat-family $\mathcal{F}$ is simple along all codimension 1 orbits. Moreover, in Proposition 6.40 we prove that graph $\Gamma_{\mathcal{F},\sigma}$ uniquely determines the codimension 1 orbit S_σ.

In Section 6.9, we show (Corollary 6.62) that if $\Gamma_{\mathcal{F},\sigma}$ is disconnected for some codimension 2 orbit S_σ, then $\mathcal{F}$ can be modified to produce another gnat-family $\mathcal{F}'$, such that $\Gamma_{\mathcal{F}',\sigma}$ is connected. Finally, we demonstrate in Corollary 6.64 that for the gnat-family $\mathcal{F}_{\max} = \bigoplus \mathcal{L}(-M_\chi)$ produced by maximal shift divisors M_χ and for any codimension 2 orbit S_σ graph $\Gamma_{\mathcal{F}_{\max},\sigma}$ is connected. Thus, for any Y there exists at least one gnat-family which is simple along all codimension 2 orbits.

In Section 6.10, we prove the main theorem of Chapter 6:

Theorem (Theorem 6.83)**.** *Let G be a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$. Let Y be any crepant toric resolution of $X = \mathbb{C}^3/G$. Let $\mathcal{F} = \bigoplus \mathcal{L}(-D_\chi)$ be a gnat-family on Y. Let $\sigma = \langle e_i, e_j, e_k\rangle$ be any three-dimensional cone in the fan of Y.*

Then there exists an algorithm which modifies $\mathcal{F}$ until it produces a new gnat-family $\mathcal{F}'$ which is simple restricted to A_σ.

Finally, in Section 6.12 we give a concrete example, for the group $G = \frac{1}{6}(1,1,4) \oplus \frac{1}{2}(1,0,1)$, of a non-projective crepant resolution Y and a gnat-family which is (globally) simple on Y.

Chapter 2

Generically natural families

2.1 G-Constellations and Families

Let G be a finite abelian group (possibly containing quasi-reflections) and let V_{giv} be an n-dimensional faithful representation of G. We identify the symmetric algebra $S({V_{\mathrm{giv}}}^{\vee})$ with the coordinate ring R of $\mathbb{C}^n$ via a choice of such an isomorphism that the induced action of G on $\mathbb{C}^n$ is diagonal. By the dual action of G on R we shall mean the left action given by

$$g \cdot f(\mathbf{v}) = f(g^{-1} \cdot \mathbf{v}) \qquad \text{for all } \mathbf{v} \in \mathbb{C}^n. \tag{2.1}$$

Corresponding to the inclusion $R^G \subset R$ of the subring of G-invariant functions we have the quotient map $q\colon \mathbb{C}^n \to X$, where $X = \operatorname{Spec} R^G$ is the quotient space. This space is generally singular. So we are typically interested in taking resolutions $\pi\colon Y \to X$ of it.

$$\begin{tikzcd}
Y \arrow[dr, "\pi"'] & & \mathbb{C}^n \arrow[dl, "q"] \\
& X &
\end{tikzcd}$$

We aim to study ways in which Y can parametrise families of G-*constellations.*

Definition 2.1 ([CI02])**.** A G-$\texttt{constellation}$ is a G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ such that $H^0(\mathcal{F})$ is isomorphic as a $\mathbb{C}[G]$-module to the regular representation V_{reg}.

Of course as $\mathcal{F}$ is coherent, it is uniquely determined by the module $H^0(\mathcal{F})$ via the associated sheaf $\tilde{\bullet}$ construction ([Har77], p.110). The actions of G and R on $\mathcal{F}$ are entirely determined by their induced actions on $H^0(\mathcal{F})$. We shall adopt this more algebraic point of view, and consider the following class of objects:

Definition 2.2. A (G,R)-`module` is a $\mathbb{C}[G]$-module V together with an equivariant R-action, that is

$$g \cdot (f \cdot \mathbf{v}) = (g \cdot f) \cdot (g \cdot \mathbf{v}) \tag{2.2}$$

must hold for all $\mathbf{v} \in \mathrm{V}$, $g \in G$ and all $f \in R$.

A morphism of (G,R)-modules is a G and R equivariant linear map of the underlying vector spaces.

The functors $\tilde{\bullet}$ and $H^0(\bullet)$ provide an equivalence between the categories of finite length coherent G-equivariant sheaves on $\mathbb{C}^n$ and of (G,R)-modules, thus we can can use both concepts interchangeably.

Any R-action on V is defined by an element of $\mathrm{Hom}_{\mathbb{C}}(R \otimes_{\mathbb{C}} V, V)$. As $R = S({V_{\mathrm{giv}}}^{\vee})$ it is sufficient to consider restrictions to $\mathrm{Hom}_{\mathbb{C}}({V_{\mathrm{giv}}}^{\vee} \otimes V, V)$. The condition (2.2) is precisely equivalent to asking for this homomorphism to be G-equivariant.

Conversely, $\alpha \in \mathrm{Hom}_G({V_{\mathrm{giv}}}^{\vee} \otimes V, V)$ defines an R-action on V if and only if it satisfies

$$\alpha(v_1 \otimes \alpha(v_2 \otimes v)) = \alpha(v_2 \otimes \alpha(v_1 \otimes v)) \tag{2.3}$$

Thus we see that there exists a one-to-one correspondence between all the (G,R)-modules with an underlying $\mathbb{C}[G]$-module V and the elements of $Z_{R,G} \subseteq \mathrm{Hom}_G({V_{\mathrm{giv}}}^{\vee} \otimes V, V)$ satisfying the commutator conditions (2.3).

Further, it can be seen that the R-structures of two isomorphic (G,R)-modules on V differ by conjugation by an element of $\mathrm{Aut}_G(V)$. Therefore we have a one-to-one correspondence between isomorphism classes of (G,R)-modules with underlying $\mathbb{C}[G]$-module V and the orbits of $\mathrm{Aut}_G(V)$ in $Z_{R,G}$.

Definition 2.3. A `family of` (G,R)`-modules parametrised by a scheme` S is a locally free sheaf $\mathcal{F}$ of $\mathcal{O}_S$-modules with G and R acting by $\mathcal{O}_S$-linear endomorphisms, so that

$$g \cdot (f \cdot s) = (g \cdot f) \cdot (g \cdot s) \tag{2.4}$$

for all $g \in G$, $f \in R$ and any local section s of $\mathcal{F}$.

We shall say that two families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if there exists an invertible sheaf $\mathcal{L}$ on S such that $\mathcal{F}$ is (G,R)-equivariantly isomorphic to $\mathcal{F}' \otimes \mathcal{L}$.

We shall call $\mathcal{F}$ a `family of G-constellations` if its fibre $\mathcal{F}_{|p}$ at any point $p \in Y$ is a G-constellation, in a sense that, as a (G,R)-module the underlying $\mathbb{C}[G]$-module is the regular representation. To obtain, from such an $\mathcal{F}$, a family of G-constellations in the sense of sheaves on $\mathbb{C}^n$, we may apply the $\tilde{\bullet}$ construction relative to S, yielding a coherent $\mathcal{O}_S \times \mathbb{C}^n$-module $\mathcal{U}_{\mathcal{F}}$, which is flat over S because $\mathcal{F}$ is a locally free $\mathcal{O}_S$ module. For any point $p \in Y$, the restriction of $\mathcal{U}_{\mathcal{F}}$ to $\{p\} \times \mathbb{C}^n$ is a G-equivariant coherent sheaf on $\mathbb{C}^n$ which is precisely the associated sheaf of the (G,R)-module $\mathcal{F}_{|p}$.

2.2 Naturality criterion

Any sheaf $\mathcal{F}$ with a G-action must split into G-eigensheaves, which are locally free if $\mathcal{F}$ is. In particular, we see that for an abelian G any family of G-constellations must split as

$$\bigoplus_{\chi \in G^\vee} \mathcal{L}_\chi$$

where G acts on each invertible sheaf $\mathcal{L}_\chi$ by the character $\chi \in G^\vee$. Recall that $G^\vee$ is the character group $\mathrm{Hom}(G, \mathbb{C}^*)$.

Any free G-orbit $Z \subset \mathbb{C}^n$ is a G-cluster, its coordinate ring $H^0(\mathcal{O}_Z)$ a G-constellation. Considering $H^0(\mathcal{O}_Z)$ as the fibre of $q_*\mathcal{O}_{\mathbb{C}^n}$ at $x = q(Z) \in X$, we see that over any $U \subset X$ such that G acts freely on $q^{-1}(U)$, we have a natural family of G-constellations $(q_*\mathcal{O}_{\mathbb{C}^n})|_U$. Consequently, we have a natural family $(\pi^* q_* \mathcal{O}_{\mathbb{C}^n})|_{q^{-1}(U)}$ of G-constellations on $\pi^{-1}(U) \subset Y$.

The generic stalk of $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ is $K(Y) \otimes_{K(X)} K(\mathbb{C}^n)$ where the $K(X)$ acts on $K(Y)$ via the homomorphism $\pi_{\mathrm{gen}} \colon K(X) \xrightarrow{\sim} K(Y)$ induced by the morphism π. We now proceed to single out a class of families of G-constellations on Y, which agree with the natural family $\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ generically. Note that although

$\pi^* q_* \mathcal{O}_{\mathbb{C}^n}$ is not a family of G-constellations on the whole of Y, it is a family of G-constellations on an open set of Y and hence this notion is well-defined.

Definition 2.4. Let $\pi \colon Y \to X = \mathbb{C}^n/G$ be a birational morphism. Let p_Y denote the generic point of Y. A `generically natural family (or gnat-family for short) across` Y is a family of G-constellations parametrised by Y for which there exists a G, R and $K(Y)$-equivariant isomorphism

$$\mathcal{F}_{|p_Y} \xrightarrow{\sim} (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{p_Y} \tag{2.5}$$

We now show that any family which agrees with the natural one generically possesses several other important naturality properties.

Proposition 2.5. *Let $\pi : Y \to X$ be a birational morphism and let $\mathcal{F}$ be a family of G-constellations on Y. Then the following are equivalent:*

1. *$\mathcal{F}$ is a gnat-family.*
2. *$\mathcal{F}$ can be G, R and $\mathcal{O}_Y$-equivariantly embedded into $K(\mathbb{C}^n)$, viewed as a constant sheaf of $\mathcal{O}_Y$ and (G, R)-modules on Y. The action of $\mathcal{O}_Y$ on $K(\mathbb{C}^n)$ is determined by the isomorphism $K(X) \simeq K(Y)$ induced by the birational morphism π.*
3. *For any open $U \subseteq Y$, $s \in \mathcal{F}(U)$ and $f \in R^G$ we have*

$$f \cdot s = fs \tag{2.6}$$

   where on the left-hand side f acts as an element of R and on the right-hand side as a section of $\mathcal{O}_Y$, via the inclusion $\pi^{-1}\mathcal{O}_X \hookrightarrow \mathcal{O}_Y$.

4. *For any open $U \subset X$ such that G acts freely on $q^{-1}U$,*

$$\mathcal{F}|_{\pi^{-1}U} \simeq \pi^* q_* \mathcal{O}_{\mathbb{C}^n}|_{\pi^{-1}U} \otimes \mathcal{L} \tag{2.7}$$

   for some invertible sheaf $\mathcal{L}$ on $\pi^{-1}U$.

Before tackling this proposition, we prove a useful lemma, which provides a nice geometrical interpretation of the condition (2.6).

Lemma 2.6. *Let $\mathcal{F}$ be a family of G-constellations on Y satisfying (2.6). Then for any $p \in Y$ we have a scheme-theoretic inclusion*

$$\operatorname{Supp} \mathcal{F}_{|p} \subseteq q^{-1}\pi(p) \tag{2.8}$$

where $\operatorname{Supp} \mathcal{F}_{|p}$ *is the support of the corresponding G-equivariant coherent sheaf on $\mathbb{C}^n$.*

Moreover, set-theoretically we have equality. Further, if G acts freely on $q^{-1}(p)$, we have

$$\mathcal{F}_{|p} \simeq (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{|p}$$

as G-constellations.

Proof. Given an arbitrary G-constellation V, the support of V as a G-equivariant coherent sheaf on $\mathbb{C}^n$ is the vanishing set of the ideal $\operatorname{Ann}_R V \subset R$. On the other hand, $q^{-1}\pi(p)$ is the vanishing of the ideal in R generated by $\mathfrak{m}_{\pi p} \in R^G$. So scheme-theoretically (2.8) is equivalent to

$$\operatorname{Ann}_{R^G} k(\pi p) \subset \operatorname{Ann}_R \mathcal{F} \otimes_{\mathcal{O}_Y} k(p)$$

which follows immediately from (2.6).

To show the set-theoretic equality, we observe from (2.2) that the ideal $\operatorname{Ann}_R \mathcal{F}_p$ is G-invariant, and so, set-theoretically $\operatorname{Supp} \mathcal{F}_{|p}$ is a union of G-orbits in $\mathbb{C}^n$. But (2.8) now implies that it is contained in a single orbit: the closed points of $q^{-1}\pi(p)$. Therefore we have equality.

For the last bit, we observe that $\mathcal{F}_{|p}$ is a finite length sheaf on $\mathbb{C}^n$ and so splits as a direct sum

$$\bigoplus_{x \in \operatorname{Supp} \mathcal{F}_{|p}} (\mathcal{F}_{|p})_{|x}$$

of its fibres at each closed point in its support. But as G acts freely on $q^{-1}\pi(p)$, the size of the orbit is $|G|$. Since this is also the dimension of $\mathcal{F}_{|p}$, each $(\mathcal{F}_{|p})_{|x}$ must be 1-dimensional and hence

$$\mathcal{F}_{|p} = \bigoplus_{x \in q^{-1}\pi(p)} (\mathcal{O}_{\mathbb{C}^n})_{|x} \simeq (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{|p}$$

$\square$

Proof of Proposition 2.5. 4 $\Rightarrow$ 1 is obtained by considering the restriction of the isomorphism (2.7) to stalks at p_Y.

1 $\Leftrightarrow$ 2: consider the sheaf $\mathcal{F} \otimes_{\mathcal{O}_Y} K(Y)$. On any open U where $\mathcal{F}$ is a free $\mathcal{O}_Y$-module, $\mathcal{F} \otimes_{\mathcal{O}_Y} K(Y)$ is the constant sheaf $\mathcal{F}_{p_Y}$ for which we have the G, R and $K(Y)$-equivariant isomorphism (2.5) to the constant sheaf $K(\mathbb{C}^n)$. A sheaf constant on an open cover must be constant globally as Y is irreducible. Now the natural map $\mathcal{F} \hookrightarrow \mathcal{F} \otimes K(Y)$ becomes the required embedding.

For 2 $\Rightarrow$ 3 it is sufficient to prove that (2.6) holds for constant sheaf $K(\mathbb{C}^n)$. On the LHS of (2.6) the action of R^G is induced by inclusion $R^G \hookrightarrow R \to K(\mathbb{C}^n)$. On the RHS, we first embed R^G into $K(Y)$ by

$$R^G \hookrightarrow K(X) \xrightarrow{\pi_{\mathrm{gen}}} K(Y)$$

and then consider $K(\mathbb{C}^n)$ to be $K(Y)$-module via

$$K(Y) \xrightarrow{\pi_{\mathrm{gen}}^{-1}} K(X) \hookrightarrow K(\mathbb{C}^n)$$

Thus the actions of R^G on both sides of (2.6) are both simply the multiplication in $K(\mathbb{C}^n)$.

So we are left with proving 3 $\Rightarrow$ 4.

We begin with a local version: if $p \in \pi^{-1}(U) \subset Y$, then $\mathcal{F}_p \simeq (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_p$. That is the stalks at p are (G, R)-equivariantly isomorphic.

Now $(\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_p$ (which we can write as $R \otimes_{R^G} \mathcal{O}_{Y,p}$) is a free $\mathcal{O}_{Y,p}$-module of rank $|G|$. This is because G acting freely on $q^{-1}\pi(p)$ implies that the quotient map q is flat and $|G|$-to-one at $\pi(p)$. $\mathcal{F}_p$ is also a free $\mathcal{O}_{Y,p}$-module of rank $|G|$, because $\mathcal{F}$ is a family of G-constellations. Therefore we can consider the determinant of any (G, R)-equivariant $\mathcal{O}_{Y,p}$-morphism between the two, and it would suffice to find a morphism whose determinant is invertible.

Consider the map $\theta : (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_p \to \mathcal{F}_p$ defined by

$$m \otimes f \to m.(f s_0) \quad m \in R,\ f \in \mathcal{O}_{Y,p} \tag{2.9}$$

where s_0 is a fixed choice of any $\mathcal{O}_{Y,p}$-generator of the χ_0-eigenspace of $\mathcal{F}_p$.

This map is a well-defined $\mathcal{O}_{Y,p}$-module map, that is, it descends from the set-theoretic product $R \times \mathcal{O}_{Y,p}$ to the tensor product, precisely because both $\mathcal{F}_p$

and $R \otimes \mathcal{O}_{Y,p}$ satisfy (2.6). It is G-equivariant because $1 \mapsto s_0$ ensures that χ_0-eigenspace maps to χ_0-eigenspace and (2.2) forces the rest. Finally not only θ is defined to be R-action equivariant, but the reader can verify that it is the unique element of $\mathrm{Hom}_{(G,R)}(R \otimes \mathcal{O}_{Y,p}, \mathcal{F}_p)$ which maps 1 to s_0. Note that in particular, this shows that

$$\mathrm{Hom}_{(G,R)}(R \otimes \mathcal{O}_{Y,p}, \mathcal{F}_p) \simeq (\mathcal{F}_p)_{\chi_0} \simeq \mathcal{O}_{Y,p} \qquad (\dagger)$$

θ is a (G,R)-equivariant morphism. It descends to the (G,R)-equivariant morphism

$$\overline{\theta} : (\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{|p} \to \mathcal{F}_{|p}$$

on fibres. Similarly to ($\dagger$),

$$\mathrm{Hom}_{(G,R)}((\pi^* q_* \mathcal{O}_{\mathbb{C}^n})_{|p}, \mathcal{F}_p) \simeq \mathbb{C};$$

i.e. all (G,R)-equivariant morphisms between the two are scalar multiples of each other. Since by Lemma 2.6, the two fibres are (G,R)-equivariantly isomorphic, we have that unless $\overline{\theta}$ is a zero map, it is an isomorphism. But it maps [1] to $[s_0]$, and the latter can not be 0 by the choice of s_0. So $\det \overline{\theta} \neq 0$ implying that $\det \theta \in \mathcal{O}^*_{Y,p}$, as required.

The isomorphisms on stalks give isomorphisms $\theta_i : R \otimes_{R^G} \mathcal{O}_{U_i} \to \mathcal{F}|_{U_i}$ on an open cover $\{U_i\}$ of U, as both sheaves are locally free and of finite rank. Then on each intersection $U_i \cap U_j$, $\theta_i \circ \theta_j^{-1}$ is a (G,R)-automorphism of $R \otimes_{R^G} \mathcal{O}_{U_i \cap U_j}$. Any such, by an argument identical to ($\dagger$), is a multiplication by an element of $\mathcal{O}^*_{U_i \cap U_j}$,which concludes the proof. □

From now on, we shall concern ourselves only with those families of G-constellations which are generically natural.

Chapter 3

Line bundles and G-Cartier divisors

3.1 Valuations of G-homogeneous functions

As we deal with families of G-constellations which are subsheaves of $K(\mathbb{C}^n)$, it would be useful to have a language similar to that of the Cartier divisors to describe the invertible sub-$\mathcal{O}_Y$-modules of $K(\mathbb{C}^n)$ with nontrivial G-action. In this section we extend the familiar construction of Cartier divisors using the larger group of nonzero G-homogeneous rational functions, which we shall denote by $K^*_G(\mathbb{C}^n)$, instead of the group of nonzero invariant rational functions $K^*(Y)$.

Definition 3.1. We shall say that a rational function $f \in K(\mathbb{C}^n)$ is G-`homogeneous of weight` $\chi \in G^\vee$ if such that

$$g \cdot f = \chi(g^{-1})f \quad \text{for all } g \in G \tag{3.1}$$

We denote by $K_\chi(\mathbb{C}^n)$ the subset of $K(\mathbb{C}^n)$ of G-homogeneous elements of a specific weight χ and by the $K_G(\mathbb{C}^n)$ the subset of $K(\mathbb{C}^n))$ of all the G-homogeneous elements. We shall use R_χ and R_G to mean $R \cap K_\chi(\mathbb{C}^n)$ and $R \cap K_G(\mathbb{C}^n)$ respectively.

The choice of a sign in this definition is motivated as follows: we want a function $p \in R$ to be G-homogeneous of weight $\chi \in G^\vee$ if $p(g \cdot v) = \chi(g)p(v)$ for any $g \in G$ and $v \in \mathbb{C}^n$. E.g. the usual concept of a homogeneous polynomial, whose degree, an integer number, is precisely its weight as a character of $\mathbb{C}^*$ acting diagonally on $\mathbb{C}^n$. In view of (2.1), this means we must have $\chi(g^{-1})$ instead of $\chi(g)$ in (3.1).

Now consider $K_G^*(\mathbb{C}^n)$, the invertible elements of $K_G(\mathbb{C}^n)$. Using the fact that $K(Y) = K(X) = K(\mathbb{C}^n)^G$, we have a short exact sequence of multiplicative groups:

$$1 \to K^*(Y) \to K_G^*(\mathbb{C}^n) \to G^\vee \to 1 \tag{3.2}$$

What makes this enlargement of $K^*(Y)$ useful is that we can still define a valuation of a G-homogeneous rational function at a prime Weil divisor.

Definition 3.2. Let $D \subset Y$ be a prime Weil divisor on Y. Given any $f \in K_G^*(\mathbb{C}^n)$, we choose any $n \in \mathbb{Z}$ such that f^n is invariant, i.e. $f^n \in K(Y)$. For instance, $n = |G|$. Then we define

$$v_D(f) = \frac{1}{n} v_D(f^n) \in \mathbb{Q} \tag{3.3}$$

where $v_D(f^n)$ is the ordinary valuation of f^n in the local ring $\mathcal{O}_{D,Y}$ of the generic point of D. This is well defined since for any $g \in K(Y)$, we have $v_D(g^k) = k v_D(g)$.

In what follows, we write

$$\{q\} = q - [q]$$

for the fractional part of $q \in \mathbb{Q}$. Generally, the valuations defined above are $\mathbb{Q}$-valued. However, if f and g in $K_G^*(\mathbb{C}^n)$ are both χ-homogeneous, then f/g is G-invariant and hence for any Weil divisor D on Y, $v_D(f) - v_D(g) \in \mathbb{Z}$. Therefore the fractional part of $v_D(f)$ is independent of the choice of f in $K_\chi^*(\mathbb{C}^n)$.

Definition 3.3. For any prime divisor P on Y, we define $v(P, \chi)$ to be the number $\{v_P(f)\} \in \mathbb{Q} \cap [0, 1)$, where f is any element of $K_\chi^*(\mathbb{C}^n)$.

3.2 G-Cartier and G-Weil divisors

We can now replicate, almost word-for-word, the definitions in [Har77], pp. 140–141.

Definition 3.4. A G-`Cartier divisor` on Y is a global section of the sheaf of multiplicative groups $K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$, i.e. the quotient of the constant sheaf $K_G^*(\mathbb{C}^n)$ on Y by the sheaf $\mathcal{O}_Y^*$ of invertible regular functions.

As usual, such a section can be described by a choice of an open cover $\{U_i\}$ of Y and functions $\{f_i\} \subseteq K_G^*(\mathbb{C}^n)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_Y^*)$. Observe that, as their ratios are invariant, the f_i must all be homogeneous of the same weight $\chi \in G^\vee$. In this case, we say that the divisor is χ-`Cartier`.

As with ordinary Cartier divisors, a G-Cartier divisor is principal if it lies in the image of the natural map $K_G^*(\mathbb{C}^n) \to K_G^*(\mathbb{C}^n)/\mathcal{O}_Y^*$ and two divisors are linearly equivalent if their difference is principal.

However when defining the corresponding enlargement of the group of Weil divisors, we have to be a little bit careful.

Definition 3.5. A χ-`Weil divisor` on Y is a finite sum $\sum q_i D_i$ (where $q_i \in \mathbb{Q}$) of prime Weil divisors on Y such that

$$q_i - v(D_i, \chi) \in \mathbb{Z} \tag{3.4}$$

for all i.

We shall further use the term G-`Weil divisor` to refer to all χ-divisors for any $\chi \in G^\vee$.

Definition 3.6. For any $f \in K_G^*(\mathbb{C}^n)$, we define `the principal` G-`Weil divisor of` f to be

$$\operatorname{div}(f) = \sum v_P(f) P$$

with the sum taken over all prime Weil divisors P on Y. This sum is finite as $f^{|G|}$ is a regular function on Y and hence only has non-zero valuations at finitely many prime divisors.

Given any $\chi, \chi' \in G^\vee$, we can see that, for any prime divisor D,

$$v(D, \chi) + v(D, \chi') - v(D, \chi\chi') \in \mathbb{Z}$$

as it is equal to the valuation at D of an invariant function. Hence G-Weil divisors form an additive group. We define two G-Weil divisors to be linearly equivalent if their difference is principal and a divisor $\sum q_i D_i$ to be effective if all $q_i \geq 0$.

Recall ([Har77], Proposition 6.11) that there is an injective homomorphism from the group of Cartier divisors to the group of Weil divisors which is an isomorphism when Y is smooth. The definition extends naturally to an injective homomorphism from the group of G-Cartier divisors to the group of G-Weil divisors, but some care needs to be taken to show that it is surjective when Y is smooth.

Definition 3.7. Define a map ϕ from the group of G-Cartier divisors to the group of G-Weil divisors on Y by

$$\{(f_i, U_i)\} \mapsto \sum k_D D$$

where the sum is taken over all prime Weil divisors D on Y and $k_D = v_D(f_i)$ for any f_i such that $U_i \cap D$ is not empty. Once again the sum is finite, as each f_i has nonzero valuation only on finitely many prime Weil divisors.

Proposition 3.8. *Let ϕ be the injective homomorphism defined above. If Y is smooth, then ϕ is an isomorphism.*

Proof. We need surjectivity. So suppose we have a χ-Weil divisor D on Y. Take any $f \in K^*_\chi(\mathbb{C}^n)$. Then $D - (f)$ is an ordinary Weil divisor and as Y is smooth, it has a Cartier divisor $\{(U_i, g_i)\}$ corresponding to it as before. Then $\{(U_i, g_i f)\}$ is the χ-Cartier divisor which ϕ maps to D. □

The point of introducing G-Cartier divisors is that they correspond to invertible sheaves which carry a G-action in the same way that ordinary Cartier divisors correspond to the ordinary invertible sheaves.

Indeed consider D, the χ-Cartier divisor on Y specified by a collection $\{(U_i, f_i)\}$ where U_i form an open cover of Y and $f_i \in K^*_\chi(\mathbb{C}^n)$. We define an invertible sheaf $\mathcal{L}(D)$ on Y as the sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$ generated by f_i^{-1} on U_i. Observe that we have an action of G on $\mathcal{L}(D)$, restricted from the one on $K(\mathbb{C}^n)$, and it acts on every section by the character χ.

Proposition 3.9. *The map $D \to \mathcal{L}(D)$ gives an isomorphism between the group G-Cl of G-Cartier divisors up to linear equivalence and the group G-Pic of invertible G-sheaves on Y.*

Proof. A standard argument from [Har77], Corollary 6.15, shows that it is an injective homomorphism. To show that it is an isomorphism, we need to be able to embed any invertible G-sheaf $\mathcal{L}$ where G acts by some $\chi \in G^\vee$ as a sub-$\mathcal{O}_Y$-module into $K(\mathbb{C}^n)$.

Given such $\mathcal{L}$, we consider the sheaf $\mathcal{L} \otimes_{\mathcal{O}_Y} K(Y)$. On every open set U_i where $\mathcal{L}$ is trivial, it is G-equivariantly isomorphic to the constant sheaf $K_\chi(\mathbb{C}^n)$. On an irreducible scheme a sheaf constant on an open cover is constant itself, so as Y is irreducible we have $\mathcal{L} \otimes_{\mathcal{O}_Y} K(Y) \simeq K_\chi(\mathbb{C}^n)$ and a particular choice of this isomorphism gives the necessary embedding as

$$\mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_Y} K(Y) \simeq K_\chi(\mathbb{C}^n) \subset K(\mathbb{C}^n)$$

□

3.3 Ramification of the quotient map

A curious thing about G-divisors and valuations of G-homogeneous functions is the fact that on the quotient space X every prime Weil divisor is a principal divisor of some G-homogeneous function. In particular, every G-Weil divisor is G-Cartier.

Proposition 3.10. *Let P be a prime Weil divisor on X. Then there exists an $f \in R_G^*$ such that $P = \operatorname{div} f$, that is*

$$v_D(f) = \begin{cases} 1, & \textit{when } D = P \\ 0, & \textit{when } D \neq P \end{cases}$$

for any prime divisor D on Y.

Proof. Let $I_P \subset R^G$ be the prime ideal of height 1 corresponding to P. Consider the ring extension $R^G \subseteq R$. By a classical result of Emmy Noether ([Ben94], Theorem 1.3.1), this extension is integral. This then implies ([Mat86], Theorem 9.3) that there exists a prime ideal I' of height 1 in R lying over I_P, that is $I_P = I' \cap R^G$ and that every other prime ideal lying over I_P is conjugate to I' by an element of G. As R is an UFD, every prime ideal of height one is principal and so there exists some $y' \in R$ such that $I_P = (y') \cap R^G$.

So take $g_0 = 1, g_1, \ldots, g_k \in G$ to be such that the principal ideals (y'), $(g_1 \cdot y')$, $\ldots$, $(g_k \cdot y')$ are all the distinct prime ideals lying over I_P. Then we claim that $y = \prod g_i \cdot y'$ is a G-homogeneous function and that $I_P = (y) \cap R^G$. Indeed, $(h \cdot y) = \bigcap((hg_i) \cdot y')$. The ideals $((hg_i) \cdot y')$ are all distinct prime ideals lying over I_P and therefore

$$(h \cdot y) = \bigcap((hg_i) \cdot y') = \bigcap(g_i \cdot y') = (y)$$

which implies $h \cdot y \in \mathbb{C}^* y$. For the second claim, observe that $I_P = g_i \cdot I_P = (g_i \cdot y') \cap R^G$ for all i. Consequently $I_P = (\bigcap(g_i \cdot y')) \cap R^G = (y) \cap R^G$.

Thus we have $I_P = (y) \cap R^G$. Note that (y) is the vanishing ideal of the preimage of P in $\mathbb{C}^n$. Now let k be the ramification index of the valuation ring extension $R^G_{I_P} \subset R_{(y)}$. Then for any $w \in K(\mathbb{C}^n)^G$ we have $v_P(w) = \frac{1}{k} v_{(y)}(w)$, which immediately extends to the $\mathbb{Q}$-valued valuation $v_P(w)$ of any G-homogeneous $w \in K^*_G(\mathbb{C}^n)$. In particular, we see that $v_P(y) = \frac{1}{k}$. Now take any other prime divisor D on Y. We have $I_D = (u) \cap R^G$ for some prime $u \in R$. If now $v_D(y) \neq 0$, then as y is regular we have $y \in (u)$ and so $g_i \cdot y' \in (u)$ for some i. Then $(u) = (g_i \cdot y)$ and $D = P$.

Now taking $f = y^k$ finishes the proof.

$\square$

In the course of the proof of Proposition 3.10, we see that the valuations of G-homogeneous functions are actually noninteger only at ramification divisors of q. We now contemplate along which actual divisors the ramification can occur.

Proposition 3.11. *There are only finitely many prime divisors P on X with ramification index greater than 1. More precisely, if we write the ideal of each such P as $(y) \cap R^G$ for $y \in R^*_G$ as in Proposition 3.10, then we will have at most one y of weight χ for each character $\chi \in G^\vee$.*

Explicitly, the ramification can only occur along the images of coordinate hyper-planes (x_1), $\ldots$, (x_n) of $\mathbb{C}^n$ and in the case of $G \subset \mathrm{SL}_n(\mathbb{C})$ ramification never occurs at all.

Proof. For each character $\chi \in G^\vee$ fix a G-homogeneous function $f_\chi \in R$ of weight χ. We further demand that it is minimal such, in a sense that no element of R^G other than 1 divides it. We shall now show that ramification could only occur along one of the $(f_\chi) \cap R^G$ and only when f_χ is the unique function satisfying these conditions.

To see it, take any prime divisor P on X. Write $I_P = (y) \cap R^G$ for $y \in R_G^*$ as in Proposition 3.10. Unless $f_\chi \in (y)$, $v_{(y)}(f_\chi) = 0$ and hence $v_{(y)}(\frac{y}{f_\chi}) = 1$ and so there is no ramification along P. But if $f_\chi \in (y)$ then minimality condition forces $f_\chi = y$.

Explicitly, when G is abelian we know that the character map $\rho : \mathbb{Z}^n \to G^\vee$ is surjective (see Section 4.1, (4.2)). Given a character $\chi \in G$, there exists $m \in \mathbb{Z}^n$ such that $x^m = \prod x_i^{m_i}$ is G-homogeneous of weight χ. Then above implies that ramification can only occur along $(y) \cap R^G$ if y is monomial. But recalling proof of Proposition 3.10, $y = \prod g_i.y'$ where y' is prime. This implies y' must be one of the basic monomials x_i.

In the case $G \subset \mathrm{SL}_n(\mathbb{C})$, we know that $x_1 \dots x_n$ is invariant. As we also have $v_{(x_i)}(x_1 \dots x_n) = 1$, there is also no ramification along any of $(x_i) \cap R^G$.

□

Propositions 3.10 and 3.11 have an immediate corollary in terms of the numbers $v(P, \chi)$, introduced in the Definition 3.3, on X.

Corollary 3.12. *For any P, a prime Weil divisor on X which is not a ramification divisor of q, and $\chi \in G^\vee$, there exists a monomial $m \in R_\chi$ such that $v_P(m) = 0$. Consequently*

$$v(P, \chi) = 0$$

Proof. Unless $P = (x_i) \cap R^G$, one can take m to be any monomial in R of weight χ. If $P = (x_i) \cap R^G$, then, unless there is ramification at P, there exists a $p \in R^G$ whose valuation at (x_i) in $\mathbb{C}^n$ is 1. Note that we can take p to be monomial by considering its monomial summands. Then $\frac{p}{x_i} \in R_{\chi^{-1}}$ and $v_P(\frac{p}{x_i}) = 0$, so we can take $m = \frac{p}{x_i}^{|G|-1}$. □

Let us look at some concrete examples of the ramification occurring and not occurring.

Example 3.13. First consider $G = \frac{1}{3}(1, 2)$, the group of 3rd roots of unity embedded into $\mathrm{SL}_2(\mathbb{C})$ by

$$\xi \mapsto \begin{pmatrix} \xi^1 & \\ & \xi^2 \end{pmatrix}$$

If we write χ_k for the character of G given by $\xi \mapsto \xi^k$, then x is of weight χ_1 and y of weight χ_2.

Let P be the image in X of the hyper-plane $x = 0$. It is a prime Weil divisor (but not a Cartier one) given by $(x^3, xy) = (x) \cap R^G$. $v_{(x)}(xy) = 1$, so there is no ramification. And consequently, $v_P(x) = v_{(x)}(x) = 1$ as $x^3 = (xy)^3 y^{-3}$.

Now take $G = \frac{1}{4}(1, 2)$, the group of 4th roots of unity embedded into $\mathrm{SL}_2(\mathbb{C})$ by

$$\xi \mapsto \begin{pmatrix} \xi^1 & \\ & \xi^2 \end{pmatrix}$$

Then the divisor P is given by (x^4, x^2y). So we see that index of ramification is $v_{(x)}(x^2y) = 2$ and correspondingly $v_P(x) = \frac{1}{2}v_{(x)}(x) = \frac{1}{2}$.

Corollary 3.14. *Let $\pi\colon Y \to X$ be a resolution and P a prime Weil divisor on Y, which is neither exceptional nor a proper transform of a ramification divisor of q in X. Then for any $\chi \in G^\vee$ there exists $m \in R_\chi$ such that $v_P(m) = 0$, implying*

$$v(P, \chi) = 0$$

Proof. This is a straightforward consequence of Corollary 3.12. Consider $P' = \pi(P)$, the image of P in X. Unless P is exceptional, P' is a prime Weil divisor on X. Its generic point lies in the open set on which the resolution map is an isomorphism, which implies that for any $f \in K(\mathbb{C}^n)$, $v_P(f) = v_{P'}(f)$. Now Corollary 3.12 gives the result.

$\square$

Chapter 4

Toric Picture

4.1 Basics

In this section we give a brief exposition of the necessary toric background and then translate some of the results of Chapter 3 into the toric language. A more thorough exposition of toric geometry in general can be found in [Dan78] and of toric geometry as related to quotient singularities in [IR96].

The group G is abelian, so consider the maximal torus $(\mathbb{C}^*)^n \subset \mathrm{GL}_n(\mathbb{C})$ containing G. We have an exact sequence of abelian groups:

$$0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^n \longrightarrow T \longrightarrow 0 \tag{4.1}$$

where T is the quotient torus which acts on the quotient space X.

By applying $\mathrm{Hom}(\bullet, \mathbb{C}^*)$ to (4.1) we obtain an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^n \xrightarrow{\rho} G^\vee \longrightarrow 0 \tag{4.2}$$

where $\mathbb{Z}^n$ is thought of as the lattice of exponents of Laurent monomials. Thus given $m = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we write x^m for $x_1^{k_1} \dots x_n^{k_n}$. M is the sublattice in $\mathbb{Z}^n$ of (exponents of) G-invariant Laurent monomials.

Note that each Laurent monomial is a G-homogeneous function and ρ is precisely the weight map, that is $x^m(g.\mathbf{v}) = \rho(m)(g)\, x^m(\mathbf{v})$ for any $\mathbf{v} \in \mathbb{C}^n$.

Applying $\mathrm{Hom}(\bullet, \mathbb{Z})$ to (4.2) we obtain

$$0 \longrightarrow (\mathbb{Z}^n)^\vee \longrightarrow L \longrightarrow \mathrm{Ext}^1(G^\vee, \mathbb{Z}) \longrightarrow 0$$

where we write $(\mathbb{Z}^n)^\vee$ for the dual lattice of $\mathbb{Z}^n$, L for the dual of M and note that $\mathrm{Hom}(G^\vee, \mathbb{Z}) = 0$ as $G^\vee$ is finite and $\mathrm{Ext}^1(\mathbb{Z}^n, \mathbb{Z}) = 0$ as $\mathbb{Z}^n$ is free.

Thus we see that $L/(\mathbb{Z}^n)^\vee \simeq \mathrm{Ext}^1(G^\vee, \mathbb{Z})$. Taking an injective resolution of $\mathbb{Z}$

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

we see that $\mathrm{Ext}^1(G^\vee, \mathbb{Z}) \simeq \mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$ as $\mathrm{Hom}(G^\vee, \mathbb{Q}) = 0$. Now a choice of a map $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}^*$ which is equivalent to a simultaneous choice of a primitive m-th root of unity for all $m \in \mathbb{N}$, would give us

$$L/(\mathbb{Z}^n)^\vee \simeq \mathrm{Hom}(G^\vee, \mathbb{C}^*) = G$$

allowing us to identify points in $L/(\mathbb{Z}^n)^\vee$ with elements of G.

Tautologically, we have a $\mathbb{Z}$-valued pairing between M and L. This pairing extends naturally to a $\mathbb{Q}$-valued pairing between $\mathbb{Z}^n$ and L. For the purposes of the exposition to follow, it will be convenient to think of elements of L as functions on the monomial lattices $M \hookrightarrow \mathbb{Z}^n$. Henceforth, given $l \in L$ and $m \in \mathbb{Z}^n$, we write $l(m)$ to denote the pairing above.

For any cone $\tau \subset \mathbb{Z}^n \otimes \mathbb{R}$, $\tau \cap M$ and $\tau \cap \mathbb{Z}^n$ are abelian semigroups. We write $\mathbb{C}[\tau \cap M]$ and $\mathbb{C}[\tau \cap \mathbb{Z}^n]$ for the $\mathbb{C}$-algebras generated by the corresponding Laurent monomials. Whenever we omit the lattice, writing $\mathbb{C}[\tau]$, it should be assumed that the lattice is M.

Let L_+ be the dual of the cone M_+ of regular Laurent monomials in M (similarly, we use $\mathbb{Z}^n_+$ and ${(\mathbb{Z}^n)^\vee}_+$). The fan of X in L consists of a single three-dimensional cone L_+ and all its subfaces. The fan of any toric resolution of X is given by a subdivision of L_+ into basic cones.

Fix such a toric resolution Y. Write $\mathfrak{F}$ for the set of basic cones which make up the fan of Y. We denote by A_σ the toric variety Spec $\mathbb{C}[\sigma^\vee]$ corresponding to the cone σ in $L \otimes \mathbb{R}$. Then Y is constructed in toric geometry by gluing together $\{A_\sigma\}_{\sigma \in \mathfrak{F}}$: A_{σ_1} and A_{σ_2} are glued along $A_{\sigma_1 \cap \sigma_2} =$ Spec $\mathbb{C}[(\sigma_1 \cap \sigma_2)^\vee]$. Thus $\{A_\sigma\}_{\sigma \in \mathfrak{F}}$ is an open affine cover of Y.

Now write $\mathfrak{E} \subset L$ for the 1-skeleton of the fan of Y. In toric geometry, each element of $\mathfrak{E}$ corresponds either to an exceptional divisor on Y or the proper transform of one of the coordinate hyper-planes in X. For $e_i \in \mathfrak{E}$, write E_i for the divisor on Y corresponding to it.

It is often important to know whether the resolution is crepant or not. The discrepancy of each E_i depends only on e_i and not on the choice of Y. If the coordinates of the element e_i of $\mathfrak{E}$ are $(k_1, \ldots, k_n) \in L \subset \mathbb{Q}^n$, then ([IR96], 1.4 and [Rei87], Prop. 4.8 for technicalities) the discrepancy of the divisor E_i is $(\sum k_i) - 1$, so the crepant divisors correspond to the elements of L which lie in the junior simplex:

$$\Delta = \{(k_1, \ldots, k_n) \in L \otimes \mathbb{R} \mid k_i > 0 \text{ and } \sum k_i = 1\}$$

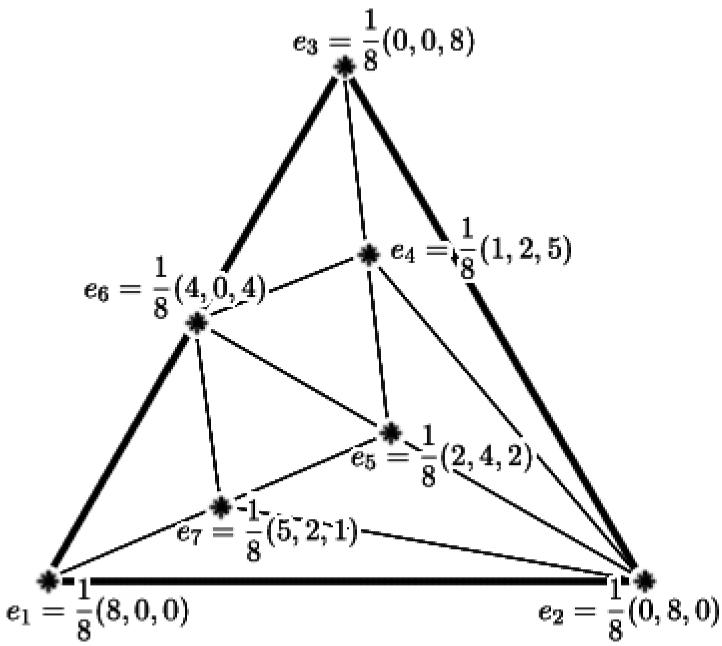
Note that if a basic cone contains $e \in \Delta \cap L$, then e must be one of its generators. So, for any resolution, $\Delta \cap L$ is a subset of $\mathfrak{E}$ and the crepant ones are precisely those for which this inclusion is an equality.

Example 4.1. Let the group G be $\frac{1}{8}(1, 2, 5)$, the group of 8th roots of unity embedded into $\mathrm{SL}_3(\mathbb{C})$ by

$$\xi \mapsto \begin{pmatrix} \xi^1 & & \\ & \xi^2 & \\ & & \xi^5 \end{pmatrix}$$

We shall write χ_k for the character of G given by $\xi \mapsto \xi^k$. So x has weight χ_1, y weight χ_2 and z weight χ_5.

The lattice L is generated in $(\mathbb{Z}^3)^\vee \otimes \mathbb{Q}$ by elements of $(\mathbb{Z}^3)^\vee$ and $\frac{1}{8}(1, 2, 5)$. The cone L_+, the positive octant, is the fan of X. A crepant resolution of Y is given by a triangulation of the junior simplex Δ into basic triangles. For the subsequent examples, we choose the following triangulation:



So $\mathfrak{E} = \Delta \cap L = \{e_1, \ldots, e_7\}$.
And the basic cones of the fan $\mathfrak{F}$ of Y are

$$\mathfrak{F} = \Big\{ \langle e_1, e_2, e_7 \rangle, \langle e_7, e_2, e_5 \rangle, \langle e_4, e_2, e_5 \rangle, \langle e_4, e_3, e_2 \rangle, \langle e_3, e_4, e_6 \rangle, \langle e_4, e_6, e_5 \rangle, \langle e_6, e_5, e_7 \rangle, \langle e_1, e_6, e_7 \rangle \Big\}$$

This is the setup for one example discussed on several subsequent occasions throughout Chapters 4 and 5.

4.2 Valuations

We now establish two simple results which translate the notions defined in the Chapter 3 into toric language.

Proposition 4.2. *Let Y be a toric resolution of X, $\mathfrak{F}$ its fan and $\mathfrak{E}$ the 1-skeleton of $\mathfrak{F}$. For any $e_i \in \mathfrak{E}$ and $m \in \mathbb{Z}^n$,*

$$v_{E_i}(x^m) = e_i(m) \in \mathbb{Q} \tag{4.3}$$

Proof. Take any basic cone $\sigma \in \mathfrak{F}$ such that $e_i \in \sigma$. Without loss of generality $i = 1$ and $\sigma = \langle e_1, \ldots, e_n \rangle$. Let $\check{e}_1, \ldots, \check{e}_n$ be the dual basis in M.

For any $m \in \mathbb{Z}^n$, $|G|m \in M$. Using the dual basis,

$$|G|m = \sum_{j=1}^{n} |G| e_j(m)\, \check{e}_j$$

therefore

$$x^{|G|m} = (x^{\check{e}_1})^{|G|e_1(m)} \ldots (x^{\check{e}_n})^{|G|e_n(m)}$$

The restriction of the exceptional divisor E_1 to A_σ is given by the principal Weil divisor div $x^{\check{e_1}}$. Thus the local ring of E_1 is the coordinate ring of A_σ localised at the ideal $(x^{\check{e}_1})$, and so the valuation of $x^{|G|m} \in \mathcal{O}_Y$ is $|G|e_1(m)$. By definition, $v_{E_1}(x^m) = \frac{1}{|G|} v_{E_1}(x^{|G|m}) = e_1(m)$. □

The second result establishes which compatibility conditions a set of monomials $\{x^{m_\sigma}\}_{\sigma \in \mathfrak{F}}$ must satisfy for it to define a G-Cartier divisor. When the conditions are satisfied, we further establish the form which the corresponding G-Weil divisor must take.

Proposition 4.3. *A set $\{x^{m_\sigma}\}_{\sigma \in \mathfrak{F}} \subset \mathbb{C}[\mathbb{Z}^n]$ of Laurent monomials defines a G-Cartier divisor $\{(A_\sigma, x^{m_\sigma})\}_{\sigma \in \mathfrak{F}}$ on Y if and only if for any $e_i \in \mathfrak{E}$*

$$e_i(m_\sigma) = e_i(m_\tau) \text{ for all } \sigma, \tau \ni e_i \tag{4.4}$$

When (4.4) holds, denote by q_i the value of $e_i(m_\sigma)$ for any $\sigma \ni e_i$. Then, under the isomorphism ϕ from Proposition 3.8, $\{(A_\sigma, x^{m_\sigma})\}_{\sigma \in \mathfrak{F}}$ corresponds to the G-Weil divisor

$$\sum_{e_i \in \mathfrak{E}} q_i E_i$$

Proof. Observe that if $\sigma, \tau \in \mathfrak{E}$ are such that e_i belongs to both, then the generic point p_{E_i} of E_i lies in $A_\sigma \cap A_\tau$. If $\{(A_\sigma, x^{m_\sigma})\}$ is a G-Cartier divisor, then $x^{m_\sigma}/x^{m_\tau} \in \mathcal{O}^*(A_\sigma \cap A_\tau)$, so we have $v_{E_i}(x^{m_\sigma}/x^{m_\tau}) = 0$ and hence

$$e_i(m_\sigma) = v_{E_i}(x^{m_\sigma}) = v_{E_i}(x^{m_\tau}) = e_i(m_\tau)$$

Conversely suppose we have $e_i(m_\sigma) = e_i(m_\tau)$ for all $e_i \in \sigma \cap \tau$. Then $m_\sigma -$

$m_\tau \in (\sigma \cap \tau)^\perp$, and hence x^{m_σ}/x^{m_τ} is invertible in $\mathbb{C}[(\sigma \cap \tau)^\vee] = \mathcal{O}_Y(A_\sigma \cap A_\tau)$ as required.

For the last part, recall that $\phi(\{(A_\sigma, x^{m_\sigma})\})$ is defined as the sum $\sum n_D D$ over all prime divisors on Y where $n_D = v_D(x^{m_\sigma})$ for any σ such that $D \cap A_\sigma \neq \emptyset$. So it suffices to prove that, for all $\sigma \in \mathfrak{F}$, the restrictions of the principal divisor (x^{m_σ}) and $\sum_{i \in \mathfrak{E}} q_i E_i$ to A_σ are identical.

Without loss of generality, we can take $\sigma = \langle e_1, \ldots, e_n \rangle$. Then $\mathcal{O}_{A_\sigma} = \mathbb{C}[t_1, \ldots, t_n]$ where $t_i = x^{\check{e}_i}$. We have $x^{m_\sigma} = \prod_{e_i \in \sigma} t_i^{q_i}$ and recall (proof of Proposition 4.2) that $E_i|_{A_\sigma} = (t_i)$. Therefore

$$(x^{m_\sigma})|_{A_\sigma} = \sum_{e_i \in \sigma} q_i \ (t_i) = (\sum_{e_i \in \sigma} q_i \ E_i)|_{A_\sigma}$$

and the result follows. □

Remarks. 1. Observe that the 'only if' part of the proof is completely general and doesn't rely on the toric technology. It is the standard argument used to show that the morphism ϕ taking Cartier divisors to Weil divisors is well-defined.

On the other hand the 'if' argument is toric-specific and relies heavily on the fact that the invertible functions on $A_\sigma \cap A_\tau$ are precisely the monomials in $(\sigma \cap \tau)^\vee$.

2. Note that, in particular, we have proved that for any $m \in \mathbb{Z}^n$, the sum

$$\sum_{i \in \mathfrak{E}} v(E_i, x^m) E_i$$

is a valid G-Weil divisor on Y. Recalling the definition of G-Weil divisors, this provides an independent proof that for any prime divisor D which is not E_i for some $i \in \mathfrak{E}$, we have

$$v(D, \chi) = 0$$

for all $\chi \in G^\vee$, since $v(D, \chi)$ is defined as the fractional part of the valuation of any homogeneous rational function of weight χ on D.

Example 4.4. To illustrate the above, in the context of the Example 4.1, we calculate explicitly the χ_6-Cartier divisor corresponding to the χ_6-Weil divisor

$$D = \frac{7}{4}E_4 + \frac{1}{2}E_5 - \frac{1}{4}E_7$$

Consider the cone $\sigma = \langle e_4, e_5, e_6 \rangle$. Calculating the dual basis which generates the abelian semigroup $\check{\sigma} \cap M$, we get

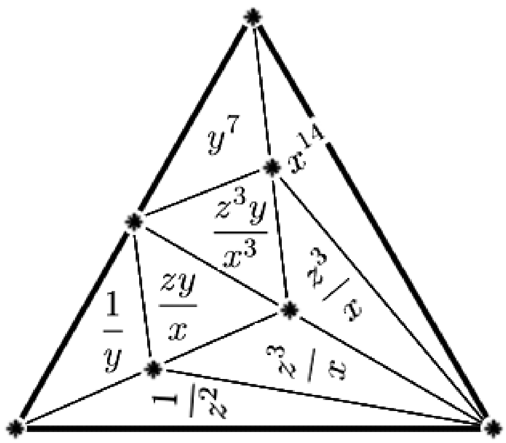
$$\check{e}_4 = (-2, 0, 2), \quad \check{e}_5 = (1, 2, -1), \quad \check{e}_6 = (2, -1, 0)$$

So $A_\sigma = \operatorname{Spec} \mathbb{C}[\frac{z^2}{x^2}, \frac{xy^2}{z}, \frac{x^2}{y}]$ and the restrictions of E_4, E_5 and E_6 to A_σ are given by $(\frac{z^2}{x^2})$, $(\frac{xy^2}{x^2})$ and $(\frac{x^2}{y})$ respectively. To specify D on A_σ we need $f \in K\chi_6(\mathbb{C}^3)$ such that $v_{E_4}(f) = \frac{7}{4}$, $v_{E_5}(f) = \frac{1}{2}$ and $v_{E_6}(f) = 0$, so we take

$$\left(\frac{z^2}{x^2}\right)^{7/4} \left(\frac{xy^2}{z}\right)^{1/2} \left(\frac{x^2}{y}\right)^{0} = \frac{z^3 y}{x^3}$$

to be f.

Repeating the same calculations for the remaining cones in the fan $\mathfrak{F}$ we get the χ_6-Cartier divisor given by



and we can indeed see that, as all the monomials representing the divisor have weight χ_6, their ratios are all invariant and the sub-$\mathcal{O}_Y$ module of $K(\mathbb{C}^n)$ they generate is an invertible sheaf on Y with the natural action of G by χ_2.

4.3 $v(E_i, \chi)$ and $\mathrm{Ext}^1(G^\vee, \mathbb{Z})$

Consider again $\mathrm{Hom}(\bullet, \mathbb{Z})$ of (4.2):

$$0 \longrightarrow (\mathbb{Z}^n)^\vee \longrightarrow L \longrightarrow \mathrm{Ext}^1(G^\vee, \mathbb{Z}) \longrightarrow 0$$

Since (4.2) is a projective resolution of $G^\vee$, then, by definition, $\mathrm{Ext}^1(G^\vee, \mathbb{Z})$ is H^1 of $\mathrm{Hom}(\bullet, \mathbb{Z})$ applied to it. Thus we have a canonical identification of elements of $\mathrm{Ext}^1(G^\vee, \mathbb{Z})$ with $L/\mathbb{Z}^n$, i.e. with the elements of L in the unit cube.

On the other hand, we saw in section 4.1 that H^1 of $\mathrm{Hom}(G^\vee, \bullet)$ applied to an injective resolution of $\mathbb{Z}$ is $\mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$, and therefore the standard isomorphism identifies $\mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$ with $\mathrm{Ext}^1(G^\vee, \mathbb{Z})$, which we shall denote by

$$\theta_{\mathrm{Ext}} : \quad \mathrm{Ext}^1(G^\vee, \mathbb{Z}) \to \mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$$

For a given point $e_i \in L$, we could ask which map $G^\vee \to \mathbb{Q}/\mathbb{Z}$ does θ_{Ext} identify the corresponding element of $\mathrm{Ext}^1(G^\vee, \mathbb{Z})$ with. To establish this, we first recall the construction of θ_{Ext}: we take an injective resolution

$$I : 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

of $\mathbb{Z}$ and a projective resolution

$$P : 0 \to M \to \mathbb{Z}^n \to G^\vee \to 0$$

of $G^\vee$ and form $\mathrm{Tot}(\mathrm{Hom}(P, I))$, the total chain complex of the double chain complex $\mathrm{Hom}(P, I)$. We then have maps

$$\mathrm{Hom}(G^\vee, I) \to \mathrm{Tot}(\mathrm{Hom}(P, I)) \leftarrow \mathrm{Hom}(P, \mathbb{Z})$$

as follows

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Q}) \arrow[r] & \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}(M, \mathbb{Q}) \arrow[r] & \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & (\mathbb{Z}^n)^\vee \arrow[r] \arrow[u] & L \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and for general homological algebra reasons we know these maps to be quasi-isomorphisms, inducing isomorphisms on the cohomology groups. In particular, $\theta_{\text{Ext}} : L/(\mathbb{Z}^n)^\vee \to \mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$.

Explicitly, for any $\alpha \in L$, class $[\alpha] \in L/(\mathbb{Z}^n)^\vee$ maps to class $[0 \oplus \alpha_1]$ in $H^1(\mathrm{Tot}(\mathrm{Hom}(P, I))$, where α_1 is the image of α in $\mathrm{Hom}(M, \mathbb{Q})$. Since $\mathrm{Hom}(\bullet, \mathbb{Q})$ is exact, α_1 pulls back to some $\alpha_2 \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Q})$. Let now α_3 be the image of α_2 in $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z})$ and observe that $[-\alpha_3 \oplus 0] = [0 \oplus \alpha_1]$, since $\alpha_3 \oplus \alpha_1$ is an image of α_2 in $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}(M, \mathbb{Q})$. And now, since $\mathrm{Hom}(\bullet, \mathbb{Q}/\mathbb{Z})$ is exact and image of $-\alpha_3$ in $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ vanishes, $-\alpha_3$, and hence $[-\alpha_3 \oplus 0]$, pulls back to some $\alpha_4 \in \mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$.

Let now $\alpha = e_i$ for some $e_i \in \mathfrak{E}$. Recall Definition 3.3 and consider the induced map $v(E_i, \bullet) \in \mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$ which for any $\chi \in G^\vee$ gives a fractional part of a valuation of any G-homogeneous monomial of weight χ at E_i. Now observe that the above calculation of θ_{Ext} gives $\theta_{\text{Ext}}([e_i])$ to be precisely $-v(E_i, \bullet)$.

Indeed, if we set $\alpha = e_i$, then the pullback α_2 to $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Q})$ is precisely the map $m \mapsto e_i(m)$, which, by Proposition 4.2, is the valuation map v_{E_i}, which gives the rational valuation of a given G-homogeneous monomial at exceptional divisor E_i. Taking just the fractional part of v_{E_i} gives us α_3 in $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Q}/\mathbb{Z})$. The image of α_3 in $\mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ vanishing corresponds to the fact that the valuation map is integer-valued at invariant monomials, and in Definition 3.3 we use that to define $v(E_i, \bullet)$ to be precisely the pullback of α_3 to $\mathrm{Hom}(G^\vee, \mathbb{Q}/\mathbb{Z})$, i.e. $-\theta_{\text{Ext}}(e_i)$.

4.4 Representations of the McKay Quiver

We now introduce a useful way to visualise the mechanics of a family of G-constellations over a particular toric affine piece of Y. Suppose we have a family $\mathcal{F}$ of G-constellations on Y and a cone σ in the fan $\mathfrak{F}$. In this section, we take a close look at the structure of $\mathcal{F}$ restricted to the corresponding affine piece A_σ.

Over A_σ the sheaf $\mathcal{F}$ is trivialised and we have

$$\mathcal{F}(A_\sigma) \simeq \mathbb{C}[\sigma^\vee] \otimes_{\mathbb{C}} V_{\mathrm{reg}} \simeq \bigoplus_{\chi} F_\chi$$

where each F_χ is isomorphic to $\mathbb{C}[\sigma^\vee]$ and G acts on it by χ. The whole structure of $\mathcal{F}$ as a family of G-constellations on A_σ is contained in the way that R acts on the F_χ. An effective method to visualise the mechanics of this is to consider the representations of the McKay quiver of G. We briefly summarise the necessary background. For a more detailed exposition of the following material see [Kin94].

Definition 4.5. A `quiver` consists of a vertex set Q_0, an arrow set Q_1 and two maps $h\colon Q_1 \to Q_0$ and $t\colon Q_1 \to Q_0$ giving the head $hq \in Q_0$ and the tail $tq \in Q_0$ of each arrow $q \in Q_1$.

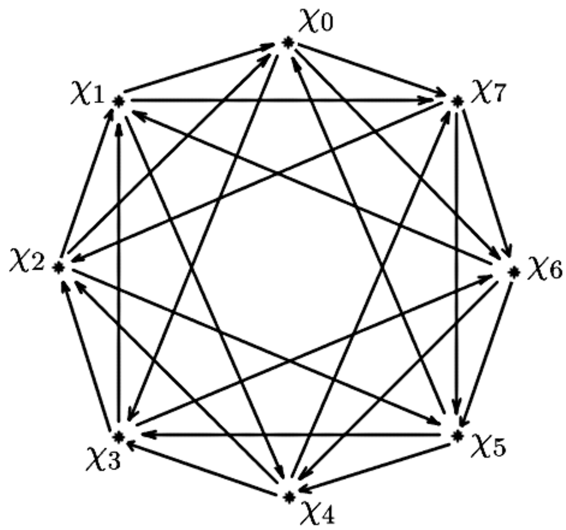
Definition 4.6. Let G be a finite subgroup of $\mathrm{GL}(V_{\mathrm{giv}})$. Then the `McKay quiver of` G is the quiver with the vertex set Q_0 labelled by the irreducible representations ρ of G and the arrow set Q_1 which has precisely $\dim \mathrm{Hom}_G(\rho_i, \rho_j \otimes V_{\mathrm{giv}})$ arrows going from the vertex ρ_i to the vertex ρ_j.

Example 4.7. 1. In our case, G is abelian and $V_{\mathrm{giv}} = \mathbb{C}^n$. So ${V_{\mathrm{giv}}}^\vee$ decomposes into irreducible representations as $\bigoplus \mathbb{C}x_i$, where the x_i are the basic monomials. If we write U_χ for the representation corresponding to $\chi \in G^\vee$, we have

$$\mathrm{Hom}_G(U_{\chi_i}, U_{\chi_j} \otimes \mathbb{C}^n) = \bigoplus_{x_k \mid \chi_i \rho^{-1}(x_k) = \chi_j} \mathrm{Hom}_G(x_k \otimes U_{\chi_i}, U_{\chi_j}) \qquad (4.5)$$

where by $x_k \otimes U_{\chi_i}$, we denote the space $\mathbb{C}x_k \otimes_{\mathbb{C}} U_{\chi_i}$. Each of the spaces $\mathrm{Hom}_G(x_k \otimes U_{\chi_i}, U_{\chi_j})$ is one-dimensional and so has one arrow from χ_i to χ_j corresponding to it. Thus the quiver consists of $|G|$ vertices labelled by characters $\chi \in G^\vee$ and out of each vertex χ emerge n arrows, each corresponding to one of the one-dimensional spaces $\mathrm{Hom}_G(x_k \otimes U_\chi, U_{\chi\rho(x_k)})$. We write $(\chi, x_k) \in Q_1$ to denote such an arrow.

2. For a concrete example, the reader can verify that the McKay quiver for $G = \frac{1}{8}(1,2,5)$ (see Example 4.1) looks like:



A good reason for contemplating the McKay quiver of G is that it is possible to establish a 1-to-1 correspondence between a subset of its *representations* and (G, R)-modules.

Definition 4.8. A `representation` of a quiver is a graded vector space $\bigoplus_{i \in Q_0} V_i$ and a collection $\{\alpha_q \colon V_{tq} \to V_{hq}\}_{q \in Q_1}$ of linear maps indexed by the arrow set of the quiver. A morphism from $(\bigoplus V_i, \{\alpha_q\})$ to $(\oplus V_i', \{\alpha_q'\})$ is a collection of linear maps $\{\theta_i : V_i \to V_i'\}_{i \in Q_0}$ forming commutative squares with the maps α_q and α_q'.

Given a G-representation V, it is traditional, when G is a general finite subgroup of GL_n, to consider representations of the McKay quiver on a graded vector space $\bigoplus V_\rho$ where $V_\rho = \mathrm{Hom}_G(\rho, V)$. It is then possible ([SI96b]) to establish a 1-to-1 correspondence between such representations and elements of $\mathrm{Hom}_G({V_{\mathrm{giv}}}^\vee \otimes V, V)$. And, in the light of the remarks after the Definition 2.2, there is a 1-to-1 correspondence between all the (G, R)-module structures on V and the elements of $\mathrm{Hom}_G({V_{\mathrm{giv}}}^\vee \otimes V, V)$ which satisfy the commutator relations (2.3).

However, in the case when G is abelian, a considerable shortcut can be taken by considering the representations directly onto graded vector space $\bigoplus V_\chi$, where V_χ is the χ-eigenspace of V. We again have the correspondence between representations of McKay quiver on $\bigoplus V_\chi$ and elements of $\mathrm{Hom}_G({V_{\mathrm{giv}}}^\vee \otimes V, V)$ and consequently the correspondence with G-constellations. Explicitly, if we have a (G, R)-structure on V, then the map $V \to V$ defined by the action of each basic monomial x_i is G-equivariant, and so splits into maps $V_\chi \to V_{\chi/\rho(x_i)}$. Each such map gives precisely the map $\alpha_{\chi,x_i} \in \mathrm{Hom}(V_\chi, V_{\chi/\rho(x_i)})$ in the corresponding representation of the quiver.

In the case $V = V_{\mathrm{reg}}$, if we make an explicit choice of a basis vector e_χ for each V_χ, this gives us bases for all $\mathrm{Hom}_G(x_i \otimes V_\chi, V_{\chi/\rho(x_i)})$. Then every representation of the McKay quiver on $\oplus V_\chi$ gains a unique map $\xi\colon Q_1 \to \mathbb{C}$ associated with it, defined by

$$\alpha_{\chi,x_i}(e_\chi) = \xi(\chi, x_i) e_{\chi/\rho(x_i)}$$

Considering a family of G-constellations $\mathcal{F}$ parametrised by an affine piece A_σ of Y, we have, as outlined in the beginning of the section,

$$\mathcal{F}(A_\sigma) \simeq \mathbb{C}[\sigma^\vee] \otimes_{\mathbb{C}} V_{\mathrm{reg}}$$

We then write the χ-eigenspace decomposition $\mathcal{F}(A_\sigma) = \bigoplus F_\chi$, and all the correspondences above work just as well with $\mathbb{C}[\sigma^\vee]$-modules as they did with complex vector spaces.

This technology presents us with a compact way to write down the R-module structure on $\mathcal{F}|_{A_\sigma}$. After a choice of bases, a representation of the McKay quiver becomes a map $\xi\colon Q_1 \to \mathbb{C}[\sigma^\vee]$ readily pictured as a McKay quiver of G with $\xi(\chi, x_i)$ written above each arrow $(\chi, x_i) \in Q_1$. In this way it is also easy to calculate explicitly the G-constellation in $\mathcal{F}$ parametrised by any point of A_σ. If a point $p \in A_\sigma$ is defined by a map $\mathrm{ev}_p\colon \mathbb{C}[\sigma^\vee] \to \mathbb{C}$, then the corresponding quiver representation is given by the map $\xi_p = \mathrm{ev}_p \circ \xi\colon Q_1 \to \mathbb{C}$.

Finally, let us consider the gnat-families (see Definition 2.4). If $\mathcal{F}$ is one such, then there exists an embedding $\iota : \mathcal{F} \to K(\mathbb{C}^n)$. Its image $\iota(\mathcal{F})$ splits into χ-eigenspaces, which are invertible sheaves, so we can take a set $\{f_\chi\} \in K(\mathbb{C}^n)$, where each f_χ is homogeneous of weight χ and a generator of the χ^{-1}-eigenspace of $\mathcal{F}$ over A_σ. The R-module structure comes for free with the inclusion of $\iota(\mathcal{F})$ into $K(\mathbb{C}^n)$ and the corresponding quiver representation is given by the map $\xi\colon Q_1 \to \mathbb{C}[\sigma^\vee]$ defined by

$$(\chi^{-1}, x_i) \mapsto \frac{x_i f_\chi}{f_{\rho(x_i)\chi}} \tag{4.6}$$

with respect to the choice of generators f_χ.

Example 4.9. Let us work through an actual example. Let $G = \frac{1}{8}(1,2,5)$ and $\sigma = \langle e_4, e_5, e_6 \rangle$. Recall from the Example 4.4 that the calculation of the dual basis in M gives us the local coordinates on $A_\sigma = \mathrm{Spec}\ \mathbb{C}[\sigma^\vee]$ as $\mathbb{C}[\sigma^\vee] = \mathbb{C}[\frac{z^2}{x^2}, \frac{xy^2}{z}, \frac{x^2}{y}]$.

Consider $\mathcal{F} = \bigoplus_{\chi_i \in G^\vee} \mathcal{O}_{A_\sigma} f_i \subset K(\mathbb{C}^n)$ where

$$\begin{array}{lll} f_0 = 1 & f_1 = x & f_2 = y \\ f_3 = xy & f_4 = \dfrac{z}{x} & f_5 = z \\ f_6 = \dfrac{yz}{x} & f_7 = yz & \end{array}$$

Now for any choice of the f_i, as long as each $f_i \in K^*_{\chi_i}(\mathbb{C}^n)$, the generic stalk $\oplus K(Y) f_i$ is the whole of $K(\mathbb{C}^n)$. The latter has a natural structure of a G-constellation, since by the Normal Basis Theorem from Galois theory ([Gar86], Theorem 19.6) we have $K(\mathbb{C}^n) = K(Y) \otimes V_{\mathrm{reg}}$. Therefore it has a corresponding quiver representation. Let $\xi' : Q_1 \to K(Y)$ be the map specifying it with respect to the $\{f_i\}$ as the choice of eigenspace bases.

We claim that $\mathcal{F}$ is closed under R-action in $K(\mathbb{C}^n)$ and hence defines a family of G-constellations parametrised by A_σ. We verify this statement in the course of calculating the map ξ', by showin that it restricts to a map $Q_1 \to \mathbb{C}[\sigma^\vee]$, which defines the quiver representation corresponding to our family.

Consider the arrow (χ_0, x). As described above, in the corresponding quiver representation the map $K(Y) f_0 \to K(Y) f_1$ is given by multiplication by x. Hence we get

$$f_0 \mapsto 1\ f_1$$

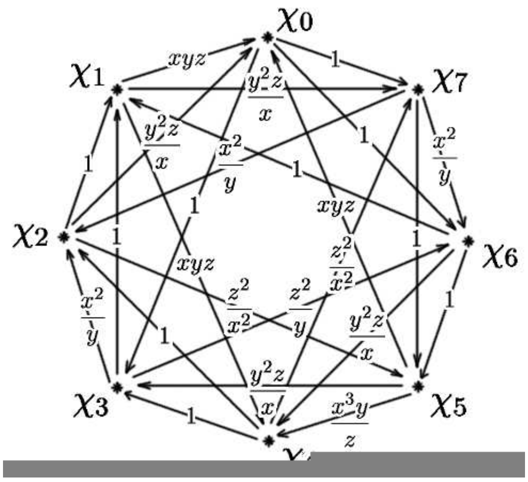
and so we label this arrow by

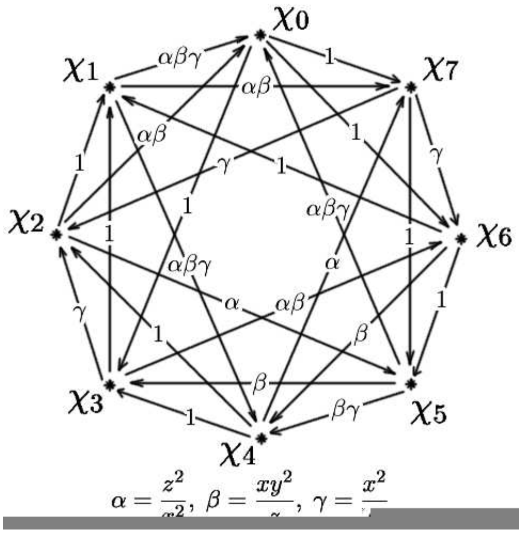
$$1 = \left(\frac{z^2}{x^2}\right)^0 \left(\frac{xy^2}{z}\right)^0 \left(\frac{x^2}{y}\right)^0$$

Similarly the arrow (χ_5, z) corresponds to the map $f_3 \mapsto xyz\ f_0$ and so we label it by

$$xyz = \left(\frac{z^2}{x^2}\right)^1 \left(\frac{xy^2}{z}\right)^1 \left(\frac{x^2}{y}\right)^1$$

Repeating this for all the arrows of the quiver we obtain:





In the diagram on the right we have written all the functions marking the arrows in terms of positive powers of the local coordinates α, β, γ on A_σ. This demonstrates that we indeed have a map

$$\xi : \quad Q_1 \to \mathbb{C}[\sigma^\vee] \quad = \mathbb{C}\left[\frac{z^2}{x^2}, \frac{xy^2}{z}, \frac{x^2}{y}\right]$$

so $\mathcal{F}$ is, as claimed, a family of G-constellations parametrised by $A_\sigma = \text{Spec}\ [\alpha, \beta, \gamma]$. The G-constellations parametrised by each point of A_σ are readily calculated by assigning specific values to α, β and γ in the diagram on the right.

Chapter 5

Reductors

5.1 Reductor Pieces

As in Chapter 4.4, let Y be a toric resolution, $\sigma \in \mathfrak{F}$ a cone in its fan and $\mathcal{F}$ a gnat-family on Y. By Proposition 2.5, there exists an embedding $\iota\colon \mathcal{F} \hookrightarrow K(\mathbb{C}^n)$. If we have a basis $\{f_\chi \mid f_\chi \in K_\chi(\mathbb{C}^n)\}$ such that

$$\iota(\mathcal{F})(A_\sigma) = \bigoplus \mathbb{C}[\sigma^\vee] f_\chi$$

then we must have

$$\frac{x_i f_\chi}{f_{\rho(x_i)\chi}} \in \mathbb{C}[\sigma^\vee] \tag{5.1}$$

for all basic monomials x_i and $\chi \in G^\vee$.

But observe that, conversely, for any set $\{f_\chi \mid f_\chi \in K_\chi(\mathbb{C}^n)\}$ for which (5.1) holds, the $\mathbb{C}[\sigma^\vee]$-submodule of $K(\mathbb{C}^n)$ generated by f_χ is closed under the natural action of R on $K(\mathbb{C}^n)$ by multiplication. It is certainly closed under the G-action, so it is a (G, R)-submodule of $K(\mathbb{C}^n)$ and a family of G-constellations parametrised by A_σ.

This observation motivates the rest of this section. But first we make a useful definition

Definition 5.1. A `reductor piece for a basic cone` $\sigma \subset L$ of the fan $\mathfrak{F}$ of the toric resolution Y is a set $\{f_\chi \mid f_\chi \in K_\chi(\mathbb{C}^n)\}$ such that for any basic monomial x_i and any $\chi \in G^\vee$ we have (5.1).

Thus, if we wanted to explicitly construct a family of G-constellations parametrised by Y, we could do it by producing a reductor piece for each cone σ in the fan $\mathfrak{F}$. Every such would give a family of G-constellations parametrised by open affine piece A_σ. However, we would need these families to 'glue together', i.e. the restrictions to $A_\sigma \cap A_{\sigma'}$ of the families generated on A_σ and $A_{\sigma'}$ must be isomorphic for any two cones $\sigma, \sigma' \in \mathfrak{F}$. The general way to guarantee this is independent of the toric technology altogether, taking us back to G-Weil divisors and to where Chapter 3 left off.

5.2 Reductor Sets

From now on Y is once again an arbitrary, not necessarily toric, resolution of $X = \mathbb{C}^n/G$.

Let $\mathcal{F}$ be a gnat-family. By Proposition 2.5, there exists an embedding $\iota\colon \mathcal{F} \hookrightarrow K(\mathbb{C}^n)$. Then $\mathcal{F}$ splits into G-eigensheaves as $\bigoplus \mathcal{F}_\chi$ and, as described in Chapter 3, each $\mathcal{F}_\chi$ defines a linear equivalence class of χ-divisors embedding it into $K(\mathbb{C}^n)$, and $\iota(\mathcal{F}_\chi)$ pinpoints a specific element of that class. Hence $\iota(\mathcal{F}) = \bigoplus_\chi \mathcal{L}(-D_\chi)$ for some unique set of G-divisors $\{D_\chi\}_{\chi \in G^\vee}$. Note that it is important here that $\mathcal{L}(-D_\chi)$ is not merely an abstract line bundle corresponding to $-D_\chi$, but is by definition a specific sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$.

Thus, in each isomorphism class of gnat-families there is at least one subsheaf of the constant sheaf $K(\mathbb{C}^n)$ on Y, which is of the form $\bigoplus \mathcal{L}(-D_\chi)$, where each D_χ is a χ-divisor on Y.

Lemma 5.2. *Let $\mathcal{F} = \bigoplus \mathcal{L}(-D_\chi)$ and $\mathcal{F}' = \bigoplus \mathcal{L}(-D'_\chi)$ be two gnat-families on Y. Then they are (G,R)-equivariantly isomorphic if and only if there exists $g \in K(Y)$ such that*

$$D'_\chi - D_\chi = \operatorname{div} g \tag{5.2}$$

for all $\chi \in G^\vee$.

Proof. The 'if' part is immediate. Observe that we have a natural isomorphism $\mathcal{L}(A) \otimes \mathcal{L}(B) \to \mathcal{L}(A+B)$ given by multiplication in $K(\mathbb{C}^n)$. Applying this to $-D_\chi - (g) = -D'_\chi$ yields isomorphism $\mathcal{F} \to \mathcal{F}'$ given by $s \mapsto s/g$.

For the ‘only if’ part, let $\phi\colon \oplus\mathcal{L}(-D_\chi) \to \oplus\mathcal{L}(-D'_\chi)$ be a (G,R)-equivariant isomorphism. Then it restricts to $\phi_\chi\colon \mathcal{L}(-D_\chi) \xrightarrow{\sim} \mathcal{L}(-D'_\chi)$ for all $\chi \in G^\vee$. Then ϕ_χ induces a map $\mathcal{L}(0) \xrightarrow{\sim} \mathcal{L}(-D'_\chi + D_\chi)$, so let $g_\chi \in K(\mathbb{C}^n)^G$ be an image of 1 under this map. Then $D'_\chi - D_\chi = (g_\chi)$ and ϕ_χ is given by $s \mapsto g_\chi s$ for any $s \in \mathcal{L}(-D_\chi)$.

It remains to show that all the g_χ are equal. Fix any $\chi \in G^\vee$ and consider any G-homogeneous $m \in R$ of weight χ. Take any $s \in \mathcal{L}(-D_{\chi_0}) \subset K(\mathbb{C}^n)$. Then $ms \in \mathcal{L}(-D_\chi)$ and by R-equivariance of ϕ

$$\phi(ms) = m\phi(s) = g_{\chi_0} ms \tag{5.3}$$

and hence $g_\chi = g_{\chi_0}$ for all $\chi \in G^\vee$. □

Corollary 5.3. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ and $\mathcal{F}' = \oplus\mathcal{L}(-D'_\chi)$ be two gnat-families on Y. Then they are equivalent if and only if there exists a χ_0-divisor N such that*

$$D'_\chi - D_\chi = N \tag{5.4}$$

for all $\chi \in G^\vee$.

Proof. Once again, the ‘if’ direction is immediate: an isomorphism $\mathcal{F}\otimes\mathcal{L}(-N) \to \mathcal{F}'$ is given by multiplication in $K(\mathbb{C}^n)$.

Conversely, if the families are equivalent then let $\mathcal{N}$ be an invertible sheaf on Y such that $\mathcal{F}' \simeq \mathcal{F} \otimes \mathcal{N}$. Choose any Weil divisor N' such that $\mathcal{N} = \mathcal{L}(-N')$. Then apply Lemma 5.2 to the isomorphic families $\oplus\mathcal{L}(-D_\chi - N')$ and $\mathcal{L}(-D'_\chi)$ to obtain $g \in K(\mathbb{C}^n)$ such that $D'_\chi - D_\chi - N' = (g)$ for all $\chi \in G^\vee$. Setting $N = N' + (g)$ finishes the proof. □

Corollary 5.4. *In every equivalence class of gnat-families there exists a unique family $\mathcal{F}$ of the form $\oplus\mathcal{L}(-D_\chi)$ with $D_{\chi_0} = 0$.*

Proof. Given an arbitrary gnat-family $\mathcal{F}$ we can find an isomorphic family of the form $\oplus\mathcal{L}(-D_\chi)$. Then setting $D'_\chi = D_\chi - D_{\chi_0}$ we obtain an equivalent family $\mathcal{L}(-D'_\chi)$ with the required properties. Finally, Corollary 5.3 shows the uniqueness. □

In the view of all of the above, we make following definitions:

Definition 5.5. Let $\{D_\chi\}_{\chi \in G^\vee}$ be a set of G-Weil divisors on Y. We call it a `prereductor set` if each D_χ is a χ-Weil divisor. We shall call it a `reductor set` if $\oplus\mathcal{L}(-D_\chi)$ with the inclusion map into $K(\mathbb{C}^n)$ is a gnat-family. We shall say the reductor set is `normalised` if $D_{\chi_0} = 0$.

5.3 Reductor Condition

We have seen that a gnat-family can be specified (up to an isomorphism) by a set of G-Weil divisors on Y which gives its embedding into $K(\mathbb{C}^n)$. Here we investigate the converse: for which prereductor sets $\{D_\chi\}$ is $\oplus\mathcal{L}(-D_\chi)$ a family of G-constellations?

We observe that $\oplus\mathcal{L}(-D_\chi)$ is always a sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$ closed under the G-action. However, for a general choice of divisors D_χ, there is no guarantee that the $\oplus\mathcal{L}(-D_\chi)$ will be closed under the R-action on $K(\mathbb{C}^n)$.

Here and below we write R_G for $R \cap K^*_G(\mathbb{C}^n)$, the G-homogeneous regular polynomials, and R_χ for $R \cap K^*_\chi(\mathbb{C}^n)$, the G-homogeneous regular polynomials of weight $\chi \in G^\vee$.

Proposition 5.6 (Reductor Condition)**.** *Let $\{D_\chi\}$ be a prereductor set. Then it is a reductor set if and only if, for any $f \in R_G$, the divisor*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \tag{5.5}$$

i.e. it is effective.

Remarks:

1. It is, of course, sufficient to check (5.5) only for f being one of the basic monomials $x_1, \dots, x_n$. This leaves us with a finite number of inequalities to check. Note also that the principal divisor (x_j) is easy to compute in toric case. It follows immediately from Proposition 4.3 that it is $\sum_{e_i \in \mathfrak{E}} e_i(x_j)E_i$. Observe that $e_i(x_j)$ is simply the jth coordinate of e_i in L.

2. Numerically, if we write each D_χ as $\sum q_{\chi,P}P$, each inequality (5.5) becomes a set of inequalities

$$q_{\chi,P} + v_P(f) - q_{\chi\rho(f),P} \geq 0 \tag{5.6}$$

for all prime divisors P on Y. The important thing to notice here is that the subsets of inequalities for each prime divisor P are all independent of one another. We can speak of $\{D_\chi\}$ satisfying or not satisfying the reductor condition at a given prime divisor P. Moreover, we can construct reductor sets $\{D_\chi\}$ by independently choosing for each prime divisor P any of the sets of numbers $\{q_{\chi,P}\}_{\chi \in G^\vee}$ which satisfy (5.6).

Proof. Take an open cover U_i on which all $\mathcal{L}(-D_\chi)$ are trivialised and write $g_{\chi,i}$ for the generator of $\mathcal{L}(-D_\chi)$ on U_i. $\{D_\chi\}$ being a reductor set is equivalent to $\oplus\mathcal{L}(-D_\chi)$ being closed under R-action on $K(\mathbb{C}^n)$. As R is a direct sum of its G-homogeneous parts, it is sufficient to check the closure under the action of just the homogeneous functions. So on each U_i, we want

$$f g_{\chi,i} \in \mathcal{O}_Y(U_i) g_{\chi\rho(f),i}$$

to hold for all $f \in R_G$, $\chi \in G^\vee$.

On the other hand, with the notation above, G-Cartier divisor $D_\chi + (f) - D_{\chi\rho(f)}$ is given on U_i by $\frac{f g_{\chi,i}}{g_{\chi\rho(f),i}}$ and it being effective is equivalent to

$$\frac{f g_{\chi,i}}{g_{\chi\rho(f),i}} \in \mathcal{O}_Y(U_i)$$

for all U_i's.

The result now follows. $\square$

We now translate the reductor condition (5.5) into toric language and investigate what it implies for the reductor pieces of the family on the open toric charts A_σ of a toric resolution Y.

Example 5.7. Let G and Y be as in previous examples. Let $\{D_\chi\}$ be a prereductor set where each $D_\chi = \sum q_{\chi,i} E_i$ is given as follows

$$\begin{aligned} D_{\chi 0} &= 0 & D_{\chi 1} &= \frac{1}{8}E_4 + \frac{2}{8}E_5 + \frac{4}{8}E_6 + \frac{5}{8}E_7 \\ D_{\chi 2} &= \frac{2}{8}E_4 + \frac{4}{8}E_5 + \frac{2}{8}E_7 & D_{\chi 3} &= \frac{3}{8}E_4 + \frac{6}{8}E_5 + \frac{4}{8}E_6 + \frac{7}{8}E_7 \\ D_{\chi 4} &= \frac{4}{8}E_4 + \frac{4}{8}E_7 & D_{\chi 5} &= \frac{5}{8}E_4 + \frac{2}{8}E_5 + \frac{4}{8}E_6 + \frac{1}{8}E_7 \\ D_{\chi 6} &= \frac{6}{8}E_4 + \frac{4}{8}E_5 + \frac{6}{8}E_7 & D_{\chi 7} &= \frac{7}{8}E_4 + \frac{6}{8}E_5 + \frac{4}{8}E_6 + \frac{3}{8}E_7 \end{aligned}$$

In the view of Proposition 4.2, the reductor condition (5.5) is equivalent to

$$q_{\chi,i} + e_i(m) - q_{\chi\rho(m),i} \geq 0 \tag{5.7}$$

for all $\chi \in G^\vee$, $e_i \in \mathfrak{E}$ and $m \in \mathbb{Z}_+^n$.

The careful reader could now verify that (5.7) holds for $m = (1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and hence $\{D_\chi\}$ is a reductor set and $\oplus\mathcal{L}(-D_\chi)$ is a family of G-constellations.

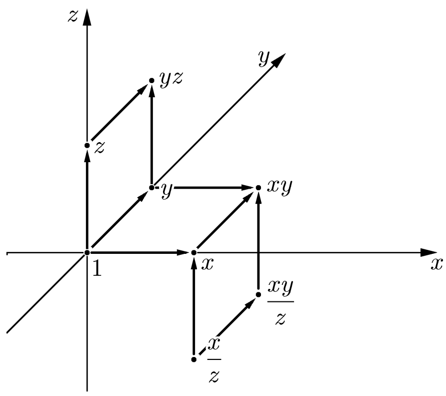
We now recall the reductor pieces introduced in Definition 5.1. Let us calculate the reductor piece $\{x^{p_\chi}\}$ specified by the generators of $\mathcal{L}(-D_\chi)$ on the affine piece $A_{\langle e_5, e_6, e_7\rangle}$. This is the same calculation of a generator of a G-Weil divisor on a given open toric chart that we saw in Example 4.4, e.g.

$$p_{\chi 1} = q_{\chi 1,5}\ \check{e}_5 + q_{\chi 1,6}\ \check{e}_6 + q_{\chi 1,7}\ \check{e}_7$$

and so

$$x^{p_{\chi 7}} = \left(\frac{y^2 z}{x}\right)^{2/8} \left(\frac{z^2}{y}\right)^{4/8} \left(\frac{x^2}{z^2}\right)^{5/8} = x$$

Repeating this for each $\chi \in G^\vee$, we obtain $\{x^{p_\chi}\} = \{1, x, y, xy, \frac{x}{z}, z, \frac{xy}{z}, yz\}$, the reductor piece pictured below as a diagram in the monomial lattice $\mathbb{Z}^n$:



The inequalities (5.7) now translate into the following form

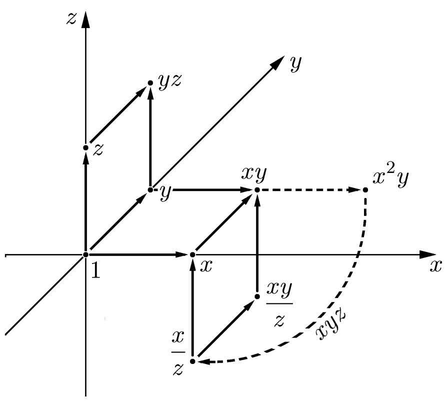
$$e_i(p_\chi + m - p_{\chi\rho(m)}) > 0 \quad (i = 5, 6, 7)$$

that is

$$\frac{x^{p_\chi} x^m}{x^{p_{\chi\rho(m)}}} \in \mathbb{C}[\sigma^\vee] \tag{5.8}$$

for every $m \in \mathbb{Z}_+^n$. This agrees with the discussion in Section 5.1, where it is precisely the condition for $\bigoplus \mathcal{O}_{A_\sigma} x^{p_\chi}$ to be a family of G-constellations parametrised by A_σ.

The reader may find the diagrams set in the monomial lattice $\mathbb{Z}^n$ convenient for checking if a given monomial set $\{x^{p_\chi}\}$ satisfies the reductor equations in the form (5.8). One merely needs to check that when adding $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$ to any p_χ, the vector reducing the result to $p_{\chi'}$ (for appropriate χ') lies within the cone $\sigma^\vee$.



5.4 Existence and symmetries

So far we have seen no indication that, over an arbitrary resolution Y of X, there exist any gnat-families.

Proposition 5.8 (Canonical family)**.** *Let Y be a resolution of $X = \mathbb{C}^n/G$ and define the set $\{D_\chi\}_{\chi\in G^\vee}$ of G-Weil divisors by $D_\chi = \sum v(P,\chi)P$, where P runs over all prime Weil divisors on Y and $v(P,\chi)$ are the fractional numbers introduced in Definition 3.3. Then the set $\{D_\chi\}_{\chi\in G^\vee}$ satisfies the reductor condition.*

We call the family $\mathcal{F} = \bigoplus \mathcal{L}(-D_\chi)$ `the canonical gnat-family on` *Y.*

Remark: For $D_\chi = \sum v(P,\chi)P$ to be a G-Weil divisor we need, in particular, for it to be a finite sum. This is implied by Corollary 3.12.

Proof. We need to show that for any $\chi \in G^\vee$, any G-homogeneous $f \in R_G$ and any prime divisor P on Y we have

$$v(P,\chi) + v_P(f) - v(P,\chi\rho(f)) \geq 0$$

First observe that the above expression must be integer valued. Also $v(P,\chi) \geq 0$ and $-v(P,\chi\rho(f)) > -1$ by definition, while $v_P(f) \geq 0$ since f^n is regular on all of Y. So we must have

$$v(P,\chi) + v_P(f) - v(P,\chi\rho(f)) > -1$$

and the result follows. □

Corollary 5.9. *Let Y be a toric resolution of X. Then the canonical gnat-family on Y is given by $\{D_\chi\}$ where*

$$D_\chi = \sum_{i\in\mathfrak{E}} v(E_i,\chi)E_i$$

Moreover, on any affine open piece A_σ, we have

$$\mathcal{F}(A_\sigma) = \mathbb{C}[\sigma \cap \mathbb{Z}^n] \tag{5.9}$$

Proof. The first statement follows trivially from the definition of the canonical family and the fact that $v(P, \chi) = 0$ whenever P is not one of the divisors E_i (Corollary 3.14).

For the second statement, without loss of generality let $\sigma = \langle e_1, \ldots, e_n \rangle$. Write $\mathcal{F}(A_\sigma) = \oplus \mathbb{C}[\sigma^\vee \cap M] x^{p_\chi}$, where x^{p_χ} are the generators of $\mathcal{L}(-D_\chi)(A_\sigma)$. Proposition 4.3 implies that for each p_χ we have $e_i(p_\chi) = v(E_i, \chi)$ for all $i \in 1, \ldots, n$. But all the numbers $v(E_i, \chi)$ are positive by definition, which implies that each p_χ lies in $\sigma^\vee$ and so $\mathcal{F}(A_\sigma) \subseteq \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$. Conversely, given any $m \in \sigma^\vee \cap \mathbb{Z}^n$

$$e_i(m - p_{\rho(m)}) = e_i(m) - v(\rho(m), E_i) \geq 0$$

as $v(E_i), \rho(m)$ is precisely the fractional part of $v_{E_i}(m) = e_i(m)$. Therefore $m - p_{\rho(m)} \in \sigma^\vee \cap M$ and so we have the inclusion in the other direction.

Geometrically, one could easily convince oneself of the truth of this statement by picturing the cone $\sigma^\vee = \{v \in \mathbb{R}^n \mid e_i(v) \geq 0\}$ in $\mathbb{Z}^n \otimes \mathbb{R}$ and observing that the set $\{p_\chi\}$ of the exponents of the reductor piece of $\mathcal{F}$ on A_σ consists precisely of all the elements of $\mathbb{Z}^n$ lying within the topmost area U of $\sigma^\vee$ given by $1 > e_i(v) \geq 0$. $\sigma^\vee \cap \mathbb{Z}^n$ is then precisely $(U \cap \mathbb{Z}^n) + (\sigma^\vee \cap M)$. We can also see why reductor condition holds: as the cone $\mathbb{R}^n_+$ lies within the cone $\sigma^\vee$, $p_\chi + m$ lies within $\sigma^\vee \cap \mathbb{Z}^n$ for any $x^m \in R$.

$\square$

Example 5.10. The reductor set $\{D_\chi\}$ given in Example 5.7 specifies the canonical family on Y. Indeed, observe that all the numbers $q_{\chi,i}$ are between 0 and 1. Equation (3.4) in Definition 3.5 of a G-Weil divisor implies they must be $v(E_i, \chi)$.

Generally, to calculate the canonical family in a toric case, one needs to choose a monomial m_χ of weight χ for each $\chi \in G$. Then, for each $e_i \in \mathfrak{E}$, one calculates the rational number $e_i(m_\chi)$ and takes its fractional part, which is precisely $v(E_i, \chi)$. The G-Weil divisors $D_\chi = \sum_i v(E_i, \chi) E_i$ are then the reductor set for the canonical family.

For instance, the numbers for the canonical family in Example 5.7 were obtained as follows: take the character $\chi_3 \in G^\vee$ and then take x^3, a monomial of weight χ_3. Calculating $e_5(3, 0, 0) = \frac{1}{8}(2 * 3 + 4 * 0 + 2 * 0) = \frac{6}{8}$, we obtain the coefficient of E_5 in D_{χ_3}. Similarly $e_7(3, 0, 0) = \frac{15}{8}$ and its fractional part $\frac{7}{8}$ is the coefficient of E_7 in D_{χ_3}.

Having established that a set of gnat-families on any resolution Y is always non-empty, we now consider symmetries which this set must possess.

Proposition 5.11 (Character Shift)**.** *Let $\{D_\chi\}$ be a reductor set. Then for any λ-Weil divisor N, the set $\{D_\chi + N\}$ also satisfies the reductor condition.*

Moreover, up to equivalence of families, the resulting gnat-family $\mathcal{F}' = \bigoplus \mathcal{L}(-D_\chi - N)$ depends only on λ and not on the choice of N. The unique normalised reductor set $\{D'_\chi\}$ specifying $\mathcal{F}'$ is given by

$$D'_{\chi\lambda} = D_\chi - D_{\lambda^{-1}} \tag{5.10}$$

Proof. The new set of divisors obviously satisfies the reductor condition:

$$(D_\chi + N) + (m) - (D_{\chi\rho(m)} + N) \geq 0$$

is immediately equivalent to the statement that $\{D_\chi\}$ satisfy the reductor condition.

For the second claim, observe that the divisor in the trivial character class is now $(D_{\lambda^{-1}} + N)$. Normalising by it we obtain

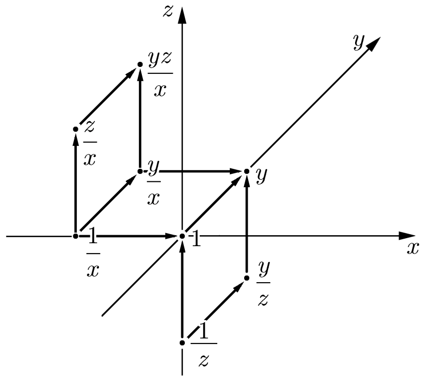
$$D_\chi + N - D_{\lambda^{-1}} - N$$

in the character class $\chi + \lambda$, which establishes the claim. □

Definition 5.12. Given a normalised reductor set $\{D_\chi\}$, we call normalised reductor set $\{D_\chi - D_{\lambda^{-1}}\}$ the λ-`shift` of $\{D_\chi\}$.

Example 5.13. On the level of reductor pieces $\{x^{p_\chi}\}$, λ-shift leaves the geometrical configuration of p_χ in the lattice $\mathbb{Z}^n$ the same, but permutes them and shifts the origin to the new location of p_{χ_0}.

For example, consider the case of the reductor piece calculated in Example 5.7. After a χ_4-shift it becomes:



Proposition 5.14 (Reflection)**.** *Let $\{D_\chi\}$ be a reductor set. Then the set $\{-D_\chi\}$ also satisfies the reductor condition.*

Proof. We need to show that

$$-D_{\chi^{-1}} + (m) - (-D_{\chi^{-1}\rho(m)^{-1}}) \geq 0$$

Rearranging we get

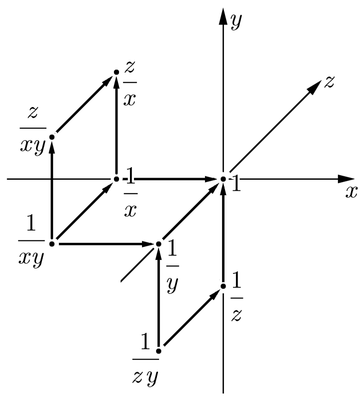
$$D_{\chi^{-1}\rho(m)^{-1}} + (m) - D_{\chi^{-1}\rho(m)^{-1}\rho(m)} \geq 0$$

which is one of the reductor equations the original set $\{D_\chi\}$ must satisfy. □

Definition 5.15. Given a reductor set $\{D_\chi\}$, we call the reductor set $\{-D_\chi\}$ the `reflection` of $\{D_\chi\}$.

Example 5.16. On the level of reductor pieces $\{x^{p_\chi}\}$, the reflection is precisely the reflection of p_χ about the origin in the lattice $\mathbb{Z}^n$.

For example, consider the case of the reductor piece calculated in Example 5.7. After a reflection it becomes:



5.5 Maximal Shifts

We now examine the individual line bundles $\mathcal{L}(-D_\chi)$ in a gnat-family and show that the reductor condition imposes a restriction on how far apart from each other they can be.

Lemma 5.17. *Let $\{D_\chi\}$ be a reductor set. Write each D_χ as $\sum q_{\chi,P}P$, where P ranges over all the prime Weil divisors on Y. Then for any $\chi_1, \chi_2 \in G^\vee$ and for any prime Weil divisor P, we necessarily have*

$$\min_{f\in R_{\chi_1/\chi_2}} v_P(f), \quad \geq \quad q_{\chi_1,P} - q_{\chi_2,P} \quad \geq \quad -\min_{f\in R_{\chi_2/\chi_1}} v_P(f) \tag{5.11}$$

where R_χ is the set of all the χ-homogeneous functions in R.

Proof. Both inequalities follow directly from the reductor condition (5.5): the right inequality by setting $\chi = \chi_1 \in G^\vee$, $\rho(f) = \frac{\chi_2}{\chi_1}$ and letting f vary within $R_{\rho(f)}$; the left inequality by setting $\chi = \chi_2$ and $\rho(f) = \frac{\chi_1}{\chi_2}$. □

This suggests the following definition:

Definition 5.18. For each character $\chi \in G^\vee$, the `maximal shift` χ`-divisor` M_χ *is defined to be*

$$M_\chi = \sum_P (\min_{f\in R_\chi} v_P(f))P \tag{5.12}$$

where P ranges over all prime Weil divisors on Y.

Observe that the fact that the sum in (5.12) is finite follows directly from Corollary 3.14.

Lemma 5.19. *The G-Weil divisor set $\{M_\chi\}$ is a normalised reductor set.*

Proof. To show that the set $\{M_\chi\}$ satisfies the reductor condition, we need to show that for every $f \in R_G$ and any prime divisor P on Y

$$v_P(m_\chi) + v_P(f) - v_P(m_{\chi\rho(f)}) \geq 0$$

where m_χ and $m_{\chi\rho(f)}$ are chosen to achieve the minimality in (5.12).

Observe that $m_\chi f$ is also a G-homogeneous element of R, therefore by the minimality of $v_P(m_{\chi\rho(f)})$ we have

$$v_P(m_\chi f) \geq v_P(m_{\chi\rho(f)})$$

as required.

To establish that $M_{\chi_0} = 0$, we observe that $v_P(1) = 0$ for any prime Weil divisor P on Y and $v_P(f) \geq 0$ for any G-homogeneous $f \in R$.

□

Observe that with Lemma 5.19 we have established another gnat-family which always exists on any resolution Y. While in some cases it coincides with the canonical family, the reader will see in Example 5.21 a case when the canonical family and the maximal shift family differ.

Putting together Lemmas 5.17 and 5.19 gives a result which shows that that the reductor set $\{M_\chi\}$ and its reflection $\{-M_\chi\}$ provide bounds on the set of all normalised reductor sets on Y.

Proposition 5.20 (Maximal Shifts)**.** *Let $\{D_\chi\}$ be a normalised reductor set. Then for any $\chi \in G^\vee$*

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}} \tag{5.13}$$

Moreover both the bounds are achieved.

Proof. To establish that (5.13) holds, set $\chi_2 = \chi_0$ in Lemma 5.17. Lemma 5.19 shows that bounds are achieved. □

Example 5.21. Let us calculate the maximal shift divisor set $\{M_\chi\}$ for the setup introduced in Example 4.1.

By the definition $M_\chi = \sum m_{\chi,P} P$ where $m_{\chi,P} = \min_{f \in R_\chi} v_P(f)$. By Corollary 3.14, the numbers $m_{\chi,P}$ are only nonzero for divisors corresponding to elements of the 1-skeleton $\mathfrak{E}$. Therefore for each $e_i \in \mathfrak{E}$, we need to find $m_{\chi,E_i} = \min e_i(p)$ where p ranges over elements of $\mathbb{Z}_+^n$ such that $\rho(p) = \chi$.

It is only necessary to consider a finite number of choices for p to establish each $m_{\chi,P}$. Observe that it suffices to take those with $0 \leq p_i \leq |G|$, as $p' = p -$

$(0, \ldots, 0, |G|, 0, \ldots, 0)$ is again element of $\mathbb{Z}^n$ with $\rho(p') = \rho(p)$ and $e_i(p') \leq e_i(p)$ for all $e_i \in \mathfrak{E}$.

For example, taking $e_5 = \frac{1}{8}(2, 4, 2)$ and considering all such p we see that:

$$
\begin{aligned}
m_{\chi_0,E_5} &= v_{E_5}(1) = e_5(0,0,0) = 0 & m_{\chi_1,E_5} &= v_{E_5}(x) = e_5(1,0,0) = \frac{2}{8} \\
m_{\chi_2,E_5} &= v_{E_5}(x^2) = e_5(2,0,0) = \frac{4}{8} & m_{\chi_3,E_5} &= v_{E_5}(x^3) = e_5(3,0,0) = \frac{6}{8} \\
m_{\chi_4,E_5} &= v_{E_5}(x^4) = e_5(4,0,0) = 1 & m_{\chi_5,E_5} &= v_{E_5}(z) = e_5(0,0,1) = \frac{2}{8} \\
m_{\chi_6,E_5} &= v_{E_5}(zx) = e_5(1,0,1) = \frac{4}{8} & m_{\chi_7,E_5} &= v_{E_5}(zx^2) = e_5(2,0,1) = \frac{6}{8}
\end{aligned}
$$

Observe that in case of χ_4 we have $m_{P,\chi} \neq v_{P,\chi}$. So the maximal shift family for this Y differs from the canonical family.

If we repeat this calculation for all elements of $\mathfrak{E}$, to obtain all numbers $m_{e_i,\chi}$, we obtain:

$$
\begin{aligned}
M_{\chi_0} &= 0, & M_{\chi_1} &= \frac{1}{8}E_4 + \frac{2}{8}E_5 + \frac{4}{8}E_6 + \frac{5}{8}E_7 \\
M_{\chi_2} &= \frac{2}{8}E_4 + \frac{4}{8}E_5 + \frac{2}{8}E_7 & M_{\chi_3} &= \frac{3}{8}E_4 + \frac{6}{8}E_5 + \frac{4}{8}E_6 + \frac{7}{8}E_7 \\
M_{\chi_4} &= \frac{4}{8}E_4 + E_5 + \frac{4}{8}E_7 & M_{\chi_5} &= \frac{5}{8}E_4 + \frac{2}{8}E_5 + \frac{4}{8}E_6 + \frac{1}{8}E_7 \\
M_{\chi_6} &= \frac{6}{8}E_4 + \frac{4}{8}E_5 + \frac{6}{8}E_7 & M_{\chi_7} &= \frac{7}{8}E_4 + \frac{6}{8}E_5 + \frac{4}{8}E_6 + \frac{3}{8}E_7
\end{aligned}
$$

Compare it to the reductor set of the canonical family given in Example 5.7.

If we now want to calculate all the normalised reductor sets (and hence all the normalised gnat-families), we simply need to check each of the finite number of prereductor sets between $\{M_\chi\}$ and its reflection $\{-M_\chi\}$ and pick out those which satisfy the reductor condition (5.5).

Recall now the remark after Proposition 5.6 about checking reductor condition independently at each prime divisor in Y. Here it means that for any reductor set $\{\sum_i q_{\chi,i} E_i\}_{\chi \in G^\vee}$, the numbers $\{q_{\chi,i}\}_{\chi \in G^\vee}$ satisfy or fail the reductor condition inequalities independently for each $e_i \in \mathfrak{E}$. This can be seen from the fact that each of the inequalities (5.7) features numbers $q_{\chi,i}$ all for the same i.

In particular it means that to list all the possible normalised reductor sets on Y, it is sufficient to list for each $e_i \in \mathfrak{E}$ all the sets $\{q_{\chi,i}\}_{\chi \in G^\vee}$ satisfying the inequalities (5.7). Then all the normalised reductor sets on Y are given by all the possible choices of one of these sets $\{q_{\chi,i}\}_{\chi\in G^\vee}$ for each $e_i \in \mathfrak{E}$. Note that the choice for each e_i is independent of all others. For our particular Y, we give these lists below.

$$e_1, e_2, e_3 : \quad \frac{1}{8}\begin{pmatrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e_4 : \quad \frac{1}{8}\begin{pmatrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & -1 \\ 0 & 1 & 2 & 3 & 4 & 5 & -2 & -1 \\ 0 & 1 & 2 & 3 & 4 & -3 & -2 & -1 \\ 0 & 1 & 2 & 3 & -4 & -3 & -2 & -1 \\ 0 & 1 & 2 & -5 & -4 & -3 & -2 & -1 \\ 0 & 1 & -6 & -5 & -4 & -3 & -2 & -1 \\ 0 & -7 & -6 & -5 & -4 & -3 & -2 & -1 \end{pmatrix}$$

$$e_5 : \quad \frac{1}{8}\begin{pmatrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \\ 0 & 2 & 4 & 6 & 8 & 2 & 4 & 6 \\ 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & -2 & 0 & 2 & 4 & 6 \\ 0 & 2 & 4 & 6 & 0 & 2 & 4 & -2 \\ 0 & 2 & 4 & -2 & 0 & 2 & 4 & -2 \\ 0 & 2 & -4 & -2 & 0 & 2 & 4 & -2 \\ 0 & 2 & 4 & -2 & 0 & 2 & -4 & -2 \\ 0 & 2 & -4 & -2 & 0 & 2 & -4 & -2 \\ 0 & -6 & -4 & -2 & 0 & 2 & -4 & -2 \\ 0 & 2 & -4 & -2 & 0 & -6 & -4 & -2 \\ 0 & -6 & -4 & -2 & 0 & -6 & -4 & -2 \\ 0 & -6 & -4 & -2 & -8 & -6 & -4 & -2 \end{pmatrix}$$

$$e_6 : \quad \frac{1}{8}\begin{array}{c} \begin{matrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \end{matrix} \\ \begin{pmatrix} 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & -4 & 0 & -4 & 0 & -4 & 0 & -4 \end{pmatrix} \end{array}$$

$$e_7 : \quad \frac{1}{8}\begin{array}{c} \begin{matrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 \end{matrix} \\ \begin{pmatrix} 0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\ 0 & 5 & 2 & -1 & 4 & 1 & 6 & 3 \\ 0 & 5 & 2 & -1 & 4 & 1 & -2 & 3 \\ 0 & -3 & 2 & -1 & 4 & 1 & -2 & 3 \\ 0 & -3 & 2 & -1 & -4 & 1 & -2 & 3 \\ 0 & -3 & 2 & -1 & -4 & 1 & -2 & -5 \\ 0 & -3 & -6 & -1 & -4 & -7 & -2 & -5 \end{pmatrix} \end{array}$$

For one particular resolution Y, the family provided by the maximal shift divisors is already well known.

Proposition 5.22. *Let $Y = G\text{-Hilb}\,\mathbb{C}^n$, the moduli space of G-clusters in $\mathbb{C}^n$. If Y is smooth, then $\bigoplus \mathcal{L}(-M_\chi)$ is the universal family $\mathcal{F}$ of G-clusters parametrised by Y, up to the usual equivalence of families.*

Proof. Firstly $\mathcal{F}$ is a gnat-family, as over any set $U \subset X$ such that G acts freely on $q^{-1}(U)$ we have $\pi_*\mathcal{F}|_U \simeq q_*\mathcal{O}_{\mathbb{C}^n}|_U$. Hence write $\mathcal{F}$ as $\oplus\mathcal{L}(-D_\chi)$ for some reductor set $\{D_\chi\}$. Take an open cover $\{U_i\}$ of Y and consider the generators $\{f_{\chi,i}\}$ of D_χ on each U_i. Working up to equivalence, we can consider $\{D_\chi\}$ to be normalised and so $f_{\chi_0,i} = 1$ for all U_i.

Now any G-cluster Z is given by some invariant ideal $I \subset R$ and so the corresponding G-constellation $H^0(\mathcal{O}_Z)$ is given by R/I. In particular note that R/I is generated by R-action on the generator of χ_0-eigenspace. Therefore any $f_{\chi,i}$ is generated from $f_{\chi_0,i} = 1$ by R-action, which means that all $f_{\chi,i}$ lie in R.

But this means that for any prime Weil divisor P on Y we have

$$v_P(f_{\chi,i}) \geq \min_{f \in R_\chi} v_P(f)$$

and therefore $D_\chi \geq M_\chi$. Now Corollary 5.20 forces the equality. $\square$

5.6 Mark Haiman's Families

It has been pointed out to us by Mark Haiman, who discovered it in the course of researching into the existence of families of θ-stable G-constellations over projective crepant resolutions Y of X in the case $G \subset \mathrm{SL}_n(\mathbb{C})$, that over every such resolution there exist the following two families of G-constellations:

Proposition 5.23 (Mark Haiman)**.** *Let G be a finite abelian subgroup of $\mathrm{SL}_n(\mathbb{C})$ and Y a crepant resolution of $X = \mathbb{C}^n/G$.*

Take the fibre product $Y \otimes_X \mathbb{C}^n$ and take its normalisation Y'. Then the pushdown to Y of $\mathcal{O}_{Y'}$ is a (first) family of G-constellations parametrised by Y.

On the other hand, let $Y_1 \subset Y$ be the union of all the codimension 0 and 1 orbits of the torus T in Y. Observe that Y_1 is independent of the choice of Y since Y is crepant. Moreover, since Y_1 has codimension 1 in Y, any line bundle, and hence any family of G-constellations parametrised by Y_1 extends uniquely to the whole of Y.

*Therefore Y_1 parametrises a unique family $\mathcal{F}_1$ of G-clusters. It is the restriction of the universal family of G-clusters over G-*Hilb*, and it is unique because the universal family of G-clusters is unique. And now, for arbitrary Y, unique extension of $\mathcal{F}_1$ from Y_1 to Y defines a (second) family of G-constellations parametrised Y.*

We now demonstrate that Haiman's first family is the canonical gnat-family defined in Proposition 5.8.

Let $\sigma \subset L \otimes \mathbb{R}$ be a basic cone.

Lemma 5.24. *Define a homomorphism $\alpha \colon \mathbb{C}[\sigma^\vee \cap M] \otimes_{R^G} R \to K(\mathbb{C}^n)$ by $a \otimes b \mapsto ab$.*

Then the kernel of α is the nilradical of $\mathbb{C}[\sigma^\vee \cap M] \otimes_{R^G} R$.

Proof. $K(\mathbb{C}^n)$ is a field, hence $\operatorname{nil} \mathbb{C}[\sigma^\vee \cap M] \otimes_{R^G} R \subseteq \ker \alpha$.

Now take $p = \sum a_i \otimes b_i \in \ker \alpha$. Regroup the monomials in b_i according to their character, writing $p = \sum p_\chi$, with $p_\chi = \sum a_j \otimes b_j$ for each $\chi \in G^\vee$, where we only sum over those j for which $\rho(b_j) = \chi$.

As $K(\mathbb{C}^n) = K(\mathbb{C}^n)^G \otimes V_{\mathrm{reg}}$ and α is G-equivariant, $\alpha(p_\chi) = 0$ for all $\chi \in G^\vee$. It thus suffices to prove each p_χ is nilpotent.

Consider $p_\chi^{|G|} = (\sum a_j \otimes b_j)^{|G|}$. Expand it into $\sum c_k \otimes d_k$ and note that each d_k is a product of $|G|$ elements b_j and hence lies in R^G. So

$$p_\chi^{|G|} = (\sum c_k d_k) \otimes 1 = 0$$

since $\sum c_k d_k = \alpha(p_\chi^{|G|}) = 0$. □

Lemma 5.25. *The image of* α *in* $K(\mathbb{C}^n)$ *is* $\mathbb{C}[\sigma^\vee \cap M + \mathbb{Z}_+^n]$ *where* $\sigma^\vee \cap M + \mathbb{Z}_+^n$ *is considered as an abelian semigroup in* $\mathbb{Z}^n$*.*

Proof. Any $p \in \mathbb{C}[\sigma^\vee] \otimes_{R^G} R$ can be written as $\sum \lambda_i x^{m_i} \otimes x^{n_i}$ where $m_i \in \sigma^\vee \cap M$, $n_i \in \mathbb{Z}_+$ and $\lambda_i \in \mathbb{C}$. Then

$$\alpha(p) = \sum \lambda_i x^{m_i + n_i} \in \mathbb{C}[\sigma^\vee \cap M + \mathbb{Z}_+^n]$$

Conversely, any polynomial of the form $\sum \lambda_i x^{m_i+n_i}$, with $m_i \in \sigma^\vee \cap M$, $n_i \in \mathbb{Z}_+$ and $\lambda_i \in \mathbb{C}$, is the image of $\sum \lambda_i x^{m_i} \otimes x^{n_i}$ under α. □

Corollary 5.26. *Let* Y *be a toric resolution of* X *and* σ *a cone in its fan. Then the reduced fibre product* $A_\sigma \otimes_X \mathbb{C}^n$ *is* Spec $\mathbb{C}[\sigma^\vee \cap M + \mathbb{Z}_+^n]$.

Proposition 5.27. $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$ *is the integral closure of* $\mathbb{C}[\sigma^\vee + \mathbb{Z}_+^n]$.

Proof. Both rings are integral domains which share a field of fractions and we have the following chain of extensions

$$\mathbb{C}[\sigma^\vee + \mathbb{Z}_+^n] \subseteq \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n] \subseteq K(\mathbb{C}^n) \tag{5.14}$$

Step 1 We claim that $\mathbb{C}[\sigma \cap \mathbb{Z}^n]$ lies within the integral closure of $\mathbb{C}[\sigma^\vee + \mathbb{Z}_+^n]$ in $K(\mathbb{C}^n)$. Indeed, take any $m \in \sigma^\vee \cap \mathbb{Z}^n$. Then $|G|m$ is G-invariant and so $|G|m \in \sigma^\vee \cap M$. Hence $(x^m)^{|G|} \in \mathbb{C}[\sigma^\vee \cap M + \mathbb{Z}_+^n]$.

Step 2 We claim that $\mathbb{C}[\sigma^\vee]$ is an integrally closed domain. This follows from the general fact that every toric affine variety is normal. We recall the proof: let $e_i \in L$ be the generators of σ in L. Then $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n] = \cap_i \mathbb{C}[\langle e_i \rangle^\vee \cap \mathbb{Z}^n]$. And each $\mathbb{C}[\langle e_i \rangle^\vee \cap \mathbb{Z}^n]$ is isomorphic to $\mathbb{C}[x_1, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$, which is normal.

□

Corollary 5.28. *Over any toric resolution Y of X, the canonical gnat-family $\mathcal{F}$ is isomorphic to the the pushdown $\mathcal{F}'$ to Y of the structure sheaf of the normalisation of the reduced fibre product $Y \times_X \mathbb{C}^n$.*

Proof. Take a cone σ in the fan of Y. Then it follows from Corollary 5.26 and Proposition 5.27 that $\mathcal{F}'(A_\sigma) \simeq \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$. On the other hand, Corollary 5.9 states that $\mathcal{F}(A_\sigma) = \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$. Therefore $\mathcal{F}$ and $\mathcal{F}'$ are equivalent families. Moreover, as the invariant part of either is $\mathcal{O}_Y$, they are isomorphic. □

We now prove that Haiman's second family is the maximal shift gnat-family across Y. This follows immediately from Proposition 5.22 and the following lemma:

Lemma 5.29. *Let Y and Y' be two toric resolutions of X with common set 1-skeleton $\mathfrak{E}$. Let $\mathcal{F}$ and $\mathcal{F}'$ be gnat-families on Y and Y', respectively. If $\mathcal{F}|_{\mathcal{Y}_1} \simeq \mathcal{F}'|_{\mathcal{Y}_1}$, then there exists a set of numbers $\{q_{\chi,i}\}_{\chi \in G^\vee, e_i \in \mathfrak{E}}$ such that $\mathcal{F} = \oplus \mathcal{L}(-\sum q_{\chi,i} E_i)$ and $\mathcal{F} = \oplus \mathcal{L}(-\sum q_{\chi,i} E'_i)$, where E_i and E'_i are the exceptional divisors corresponding to $e_i \in \mathfrak{E}$ in Y and Y', respectively.*

Proof. This follows from the fact that $E_i|_{Y_1} = E'_i|_{Y_1}$ for any $e_i \in \mathfrak{E}$. Indeed, the E_i are irreducible, so they contain a unique codimension 1 orbit of torus, which is, by the definition of Y_1, precisely $E_i|_{Y_1}$. □

5.7 Summary

Finally, we combine the results achieved thus far into a classification theorem.

Theorem 5.30 (Classification)**.** *Let G be a finite abelian subgroup of $\mathrm{GL}_n(\mathbb{C})$, X the quotient of $\mathbb{C}^n$ by the action of G and Y a resolution of X. Then every gnat-family on Y, up to isomorphism, is of the form $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$, where each D_χ is a χ-Weil divisor and the set $\{D_\chi\}$ satisfies the inequalities*

$$D_\chi + (f) - D_{\chi\rho(f)} \geq 0 \quad \forall\, \chi \in G^\vee, G\text{-homogeneous } f \in R$$

Here $\rho(f) \in G^\vee$ is the homogeneous weight of f. Conversely for any such set $\{D_\chi\}$, $\bigoplus \mathcal{L}(-D_\chi)$ is a gnat-family.

Moreover, each equivalence class of gnat-families has precisely one family with $D_{\chi_0} = 0$. The divisor set $\{D_\chi\}$ corresponding to such a family satisfies the inequalities

$$M_\chi \geq D_\chi \geq -M_{\chi^{-1}}$$

where $\{M_\chi\}$ is a fixed divisor set depending only on G and Y. In particular, the number of equivalence classes of families is finite.

Proof. Proposition 5.6 establishes the correspondence of isomorphism classes of gnat-families and reductor sets. Corollary 5.4 lifts the correspondence to the level of equivalence classes and normalised reductor sets. Corollary 5.20 gives the bounds on the set of all normalised reductor sets, and as each M_χ is a finite sum by Corollary 3.14, this set is finite. $\square$

Chapter 6

Towards orthonormal families for $G \subset \mathrm{SL}_3(\mathbb{C})$

6.1 Motivation

Let us consider the case that G is a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$. Bridgeland, King and Reid [BKR01] prove that $Y = G\text{-Hilb}\,\mathbb{C}^3$ is a smooth, crepant resolution of $\mathbb{C}^3/G$ in the process of establishing an equivalence of categories between the bounded derived categories $D(Y)$ of coherent sheaves on Y and and $D^G(\mathbb{C}^3)$ of G-equivariant coherent sheaves on $\mathbb{C}^n$.

We now give an outline of the method [BKR01] uses to establish the equivalence of derived categories, as it would apply in our situation, where we assume from the start that Y is a crepant resolution of $\mathbb{C}^3/G$. For further details and for the explanation of the derived category terminology see [BKR01], [Bri99], [BO95].

Given a crepant resolution Y of $\mathbb{C}^3/G$, consider the following commutative square:

$$\begin{tikzcd} & Y \times \mathbb{C}^3 \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_{\mathbb{C}^3}"] & \\ Y \arrow[dr, "\pi"'] & & \mathbb{C}^3 \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

Given any gnat-family $\mathcal{F}$ on Y, we consider the universal G-constellation $\mathcal{U}_{\mathcal{F}}$ on $Y \times \mathbb{C}^3$, introduced in Section 2.1. For any $p \in Y$ we have

$$(\pi_{\mathbb{C}^3})_*(\mathcal{U}_{\mathcal{F}}|_{p \times \mathbb{C}^3}) = (\tilde{\mathcal{F}_{|p}})$$

Recall that the fibre $\mathcal{F}_{|p}$ is the (G, R)-module obtained by taking the pullback of $\mathcal{F}$ over the point-scheme inclusion $p \hookrightarrow Y$. We write $(\tilde{\mathcal{F}_{|p}})$ for the associated sheaf on $\mathbb{C}^3$.

We define the *Fourier–Mukai transform* functor $D(Y) \to D^G(\mathbb{C}^3)$ by

$$\Phi(-) : \mathbf{R}\pi_{\mathbb{C}^3,*}(\mathcal{U}_{\mathcal{F}} \otimes \pi_{\mathrm{Y}}^*(- \otimes \chi_0))$$

We then proceed to prove that Φ is an equivalence of categories. At the heart of the proof is the following criterion, established by Bridgeland [Bri99], for an exact functor between two triangulated categories to be a equivalence of categories.

Theorem ([Bri99], Theorems 2.3)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be triangulated categories and $F\colon \mathfrak{A} \to \mathfrak{B}$ an exact functor with right and left adjoints. Let Ω be a spanning class for $\mathfrak{A}$. Then F is fully faithful if and only for all elements ω_1, ω_2 of Ω and for all integers i, the homomorphism*

$$F\colon \ \mathrm{Hom}^i_{\mathfrak{A}}(\omega_1, \omega_2) \to \mathrm{Hom}^i_{\mathfrak{B}}(F\omega_1, F\omega_2) \tag{6.1}$$

is an isomorphism.

Theorem ([Bri99], Theorem 3.3)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be triangulated categories, with $\mathfrak{B}$ indecomposable and not every object in $\mathfrak{A}$ isomorphic to 0. Let $F\colon \mathfrak{A} \to \mathfrak{B}$ be a fully faithful functor. Then F is an equivalence of categories if and only if F has a left adjoint G and a right adjoint H such that ,*

$$Hb \simeq 0 \Rightarrow Gb \simeq 0 \quad \textit{for any object } b \in \mathfrak{B} \tag{6.2}$$

The indecomposability of $D^G(\mathbb{C}^3)$ is a consequence of the fact that G acts faithfully on $\mathbb{C}^3$ ([BKR01], Lemma 4.2).

Since the canonical classes of $\mathbb{C}^3$ and Y are trivial, Grothendieck duality can be applied to show that

$$\Psi(-) : [\mathbf{R}\pi_{\mathrm{Y}}(\mathcal{U}_{\mathcal{F}}^{\vee} \overset{\mathbf{L}}{\otimes} \pi_{\mathbb{C}^3}^{*}(-))[3]]^{\mathrm{G}}$$

is both left and right adjoint to Φ. This also takes care of (6.2).

It remains to prove (6.1). A *spanning class* is a subclass Ω of objects of a triangulated category $\mathfrak{A}$ such that for any object $a \in \mathfrak{A}$

$$\mathrm{Hom}_{\mathfrak{A}}^{i}(a, \omega) = 0 \quad \forall\, \omega \in \Omega \quad \forall i \in \mathbb{Z} \Rightarrow a \simeq 0$$

and

$$\mathrm{Hom}_{\mathfrak{A}}^{i}(\omega, a) = 0 \quad \forall\, \omega \in \Omega \quad \forall i \in \mathbb{Z} \Rightarrow a \simeq 0$$

In our case, because Y is nonsingular, we can take Ω to be $\{\mathcal{O}_y \mid y \in Y\}$.

It remains to establish that Φ is fully faithful. Observe that for any $y \in Y$, $\Phi\mathcal{O}_y = (\tilde{\mathcal{F}_{|y}})$. We need

$$\Phi : \mathrm{Ext}_{Y}^{i}(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) \to G\text{-}\,\mathrm{Ext}_{\mathbb{C}^3}^{i}((\tilde{\mathcal{F}_{|y_1}}), (\tilde{\mathcal{F}_{|y_2}})) \tag{6.3}$$

to be an isomorphism for all i and all $y_1, y_2 \in Y$.

It is well known what the groups on left-hand side of (6.3) look like:

$$\mathrm{Ext}_{Y}^{i}(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = \begin{cases} 0, \text{ if } y_1 \neq y_2, \\ \bigwedge^{i} T_{Y,y_1} \text{ for } y_1 = y_2, i \in 0, \ldots, 3, \\ 0 \text{ for } y_1 = y_2, i \notin 0, \ldots, 3 \end{cases} \tag{6.4}$$

where T_{Y,y_1} is the tangent space at y_1 in Y.

We now see that for (6.3) to be isomorphisms, it is necessary for $\mathcal{F}$ to have the following two properties:

Definition 6.1. A family $\mathcal{F}$ of G-constellations parametrised by Y is `simple` if for any $y \in Y$, $\mathcal{F}_{|y}$ is a simple (G, R)-module, i.e.

$$\mathrm{Hom}_{G,R}(\mathcal{F}_{|y}, \mathcal{F}_{|y}) = \mathbb{C} \tag{6.5}$$

It is `orthogonal` if for any y_1, y_2 distinct points of Y, we have:

$$\mathrm{Hom}_{G,R}(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2}) = 0 \tag{6.6}$$

It is `orthonormal` if it is both orthogonal and simple.

When Y is $G-\mathrm{Hilb}(\mathbb{C}^3)$, the family of all G-clusters is an orthonormal family because for any G-cluster Z, $H^0(\mathcal{O}_Z)$ is generated by 1 as a (G,R) module, so any nonzero homomorphism between two G-clusters is necessarily an isomorphism, fully determined by the image of $1 \in H^0(\mathcal{O}_Z)$. And by G-equivariance, 1 has to map to the χ_0-eigenspace of $H^0(\mathcal{O}_Z)$, which is isomorphic to $\mathbb{C}$.

We now indicate why we expect that it is not only necessary, but also sufficient for $\mathcal{F}$ to be an orthonormal family in order for Φ to be fully faithful, and hence an equivalence of categories.

We first deal with the case $y_1 \neq y_2$: here we need to show that the groups on the right hand side of (6.3) vanish. Let $\mathcal{Q}$ be the object of $D(Y \times Y)$ which gives the functor $\Psi\Phi\colon Y \to Y$ as a Fourier–Mukai transform ([Muk81], Proposition 1.3). We have

$$\mathrm{Hom}^i_{D(Y\times Y)}(\mathcal{Q}, \mathcal{O}_{y_1,y_2}) = \mathrm{Hom}^i_{D_Y}(\Phi\Psi\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = G\text{-}\mathrm{Ext}^i_{\mathbb{C}^3}((\tilde{\mathcal{F}_{|y_1}}), (\tilde{\mathcal{F}_{|y_2}}))$$

Suppose now $\pi(y_1) \neq \pi(y_2)$. Lemma 2.6 implies that supports of $\mathcal{F}_{|y_1}$ and $\mathcal{F}_{|y_2}$ are disjoint and so $G\text{-}\mathrm{Ext}^i_{\mathbb{C}^3}(\tilde{\mathcal{F}_{|y_1}}, \tilde{\mathcal{F}_{|y_2}})$ vanishes for all i. Also, this proves that the support of $\mathcal{Q}$ lies within $Y \times_X Y$.

When $y_1 \neq y_2$, (6.6) together with Serre duality on $\mathbb{C}^3$ implies that the groups on the right hand side in (6.3) vanish for $i \neq 1, 2$. Also, this shows that the restriction of $\mathcal{Q}$ to $Y \times_X Y \setminus \Delta$ has homological dimension of 1, where Δ is the diagonal,

To show that the groups on the right hand side in (6.3) also vanish for $i = 1, 2$, we apply the intersection theorem from commutative algebra (for detailed argument see [BKR01], Section 6, Steps 3-4) which implies that, for non-zero objects of the derived category of a scheme, their homological dimension is greater or equal to the codimension of their support. Since the codimension of $Y \times_X Y$ in $Y \times Y$ is ≥ 2, $\mathcal{Q}$ restricted to $Y \times Y \setminus \Delta$ must be zero, which is equivalent to $G\text{-}\mathrm{Ext}^i_{\mathbb{C}^3}(\tilde{\mathcal{F}_{|y_1}}, \tilde{\mathcal{F}_{|y_2}})$ vanishing for all i when $y_1 \neq y_2$.

It remains to deal with the case $y_1 = y_2$. For this we would like to apply an analogue of the following theorem of Bondal and Orlov ([BO95], Theorem 1.1 or [Bri99], Theorem 5.1)

Theorem. *Let M and X be smooth algebraic varieties and $E \in D(M \times X)$. Then the Fourier–Mukai transform $\Phi_E \colon D(M) \to D(X)$ is a full and faithful functor if and only if*

1. $\mathrm{Hom}^i_X(\Phi(\mathcal{O}_{t_1}), \Phi(\mathcal{O}_{t_2})) = 0$ *for every i when $t_1 \neq t_2$.*

2. $\mathrm{Hom}^i_X(\Phi(\mathcal{O}_{t_1}), \Phi(\mathcal{O}_{t_1})) = 0$ *for $i \notin [0, \dim M]$ when $t_1 = t_2$.*

3. $\mathrm{Hom}^0_X(\Phi(\mathcal{O}_{t_1}), \Phi(\mathcal{O}_{t_1})) = \mathbb{C}$ *when $t_1 = t_2$.*

for any pair of points t_1, t_2 in M.

We expect that there should hold a G-equivariant version of this theorem, with a finite group G acting on X, and hence $M \times X$, and with $E \in D^G(M \times X)$ and $\Phi_E : D(M)- > D^G(X)$. If so, then we may complete the argument as follows:

1. We have already shown that $G\text{-}\mathrm{Ext}^i_{\mathbb{C}^3}(\tilde{\mathcal{F}_{|y_1}}, \tilde{\mathcal{F}_{|y_2}})$ vanishes when $y_1 \neq y_2$

2. When $y_1 = y_2$ but $i \neq 0, \ldots, 3$, then $G\text{-}\mathrm{Ext}^i_{\mathbb{C}^3}(\tilde{\mathcal{F}_{|y_1}}, \tilde{\mathcal{F}_{|y_1}})$ vanishes by dimension considerations.

3. The statement that $G\text{-}\mathrm{Hom}_{\mathbb{C}^3}(\tilde{\mathcal{F}_{|y_1}}, \tilde{\mathcal{F}_{|y_2}}) = \mathbb{C}$ whenever $y_1 = y_2$ is precisely the requirement that $\mathcal{F}$ is a simple family.

Thus Φ is fully faithful and hence an equivalence of categories. In other words, the theorem tells us that the remaining (the case of $y_1 = y_2$ and $i = 1, 2, 3$) isomorphisms in (6.3) follow once all the others are known.

Thus what we expect is that for any crepant resolution Y of $\mathbb{C}^3/G$ the Fourier–Mukai transform using a gnat-family $\mathcal{F}$ gives an equivalence of categories if and only if $\mathcal{F}$ is orthonormal.

Craw and Ishii proved in [CI02] that every projective crepant resolution Y of $\mathbb{C}^3/G$ can be realised as a moduli space M_θ of θ-stable G-constellations. It is straightforward to show that a homomorphism between any pair of θ-stable

G-constellations is either 0 or an isomorphism. This implies that the tautological family on M_θ, which parametrises all θ-stable G-constellations is necessarily orthonormal. Therefore any projective crepant resolution has an orthonormal gnat-family on it. In this case, because Y is realised as a moduli space, it is possible to adapt the argument of [BKR01] and show that there is an associated derived equivalence $\Phi\colon D(Y) \to D^G(\mathbb{C}^3)$. Thus the remains a following question:

Question 6.2. *Let G be a finite abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$. Let Y be any crepant resolution of $\mathbb{C}^3/G$. Does there exists an orthonormal gnat-family across Y and there is an equivalence of derived categories*

$$D(Y) \xrightarrow{\sim} D^G(\mathbb{C}^3)?$$

As a step towards answering this question, we investigate the problem of finding simple gnat families on crepant resolutions.

6.2 Simplicity of G-constellations

Now let $Y \to X$ be a toric resolution of $X = \mathbb{C}^3/G$, given by fan $\mathfrak{F}$ with set $\mathfrak{E}$ of the generators of the cones, and let $\mathcal{F}$ be gnat-family on Y. We are interested in studying the simplicity of G-constellations in $\mathcal{F}$. Recall the correspondence between G-constellations and representations of McKay quiver of G into V_{reg} introduced in 4.4. The following definition was first introduced in [CI02], section 10.2:

Definition 6.3. Let V be a G-constellation and let $\{\alpha_q\}_{q\in Q_1}$, where Q_1 is the arrow set of the McKay quiver, be the corresponding quiver representation. Define Γ_V to be the graph whose vertex set is the vertex set Q_0 of the McKay quiver of G and whose edges are those arrows $q \in Q_1$, forgetting the orientation, for which α_q is not a zero map.

Proposition 6.4.

$$\mathrm{Hom}_{G,R}(V,V) = \oplus_{\Gamma_{V,i}}\mathbb{C} \tag{6.7}$$

where $\Gamma_{V,i}$ are the connected components of Γ_V.

Proof. We can write $\mathrm{Hom}_{G,R}(V,V)$ as

$$\{\alpha \in \mathrm{Hom}_G(V,V) \mid m.\alpha(v) = \alpha(m.v) \quad \forall m \in R,\ v \in V\} \tag{6.8}$$

We have $\mathrm{Hom}_G(V,V) = \oplus_{\chi \in G^\vee} \mathrm{Hom}_G(V_\chi, V_\chi) = \oplus_{\chi \in G^\vee}\mathbb{C}$. Rewrite (6.8) as

$$\{(\alpha_\chi) \in \oplus_{\chi \in G^\vee}\mathbb{C} \mid m.(\alpha_\chi v) = \alpha_{\chi\rho(m)} m.v \quad \forall m \in R, \chi \in G^\vee, v \in V_\chi\} \tag{6.9}$$

Observe that for any $m \in R$, $\chi \in G^\vee$ and $v \in V_\chi \setminus \{0\}$, we have $m.v = 0$ if and only if χ and $\chi\rho(m)$ are in different connected components of Γ_V. And $m.v \neq 0$ implies $\alpha_\chi = \alpha_{\chi\rho(m)}$ in (6.9). Therefore:

$$\begin{aligned} \mathrm{Hom}_{G,R}(V,V) \qquad &= \{(\alpha_\chi) \in \oplus_{\chi \in G^\vee}\mathbb{C} \mid \alpha_\chi = \alpha_{\chi'} \text{ if } \exists\, \Gamma_{V,i} \ni \chi, \chi'\} \\ &= \oplus_{\Gamma_{V,i}} \mathbb{C} \end{aligned}$$

□

Corollary 6.5. *A G-constellation is simple if and only if Γ_V is connected.*

Now consider gnat-family $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$. Recall that its χ-eigensheaf $\mathcal{F}_\chi$ is $\mathcal{L}(-D_{\chi^{-1}})$, since G acts on a monomial of weight χ by a character χ^{-1}.

In a corresponding representation of the McKay quiver, the arrow $(\chi, j) \in Q_1$ is represented by action of x_j on $\mathcal{F}_\chi$, which we can think of as a global section β of $\mathcal{O}_Y$-module

$$\mathcal{H}om_{G,\mathcal{O}_Y}(x_j \otimes \mathcal{F}_\chi, \mathcal{F}_{\chi\rho^{-1}(x_j)}) \tag{6.10}$$

defined by $x_j \otimes s \mapsto x_j.s$ for any section s of $\mathcal{F}_\chi$. By $x_j \otimes \mathcal{F}_\chi$ we mean, similar to Section 4.4, the sheaf $\mathcal{O}_Y x_j \otimes_{\mathcal{O}_Y} \mathcal{F}_\chi$ and note that $\mathcal{O}_Y x_j$, the free rank 1 sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^3)$, is precisely $\mathcal{L}(-(x_j))$.

Now consider $\mathcal{O}_Y$-module

$$\mathcal{L}(D_{\chi^{-1}} + (x_j) - D_{\chi^{-1}\rho(x_j)}) \tag{6.11}$$

Divisor $D_{\chi^{-1}} + (x_j) - D_{\chi^{-1}\rho(x_j)}$ is an effective Weil divisor (Reductor condition 5.5) and so (6.11) has a global section β' given by $1 \in K(\mathbb{C}^3)$, which vanishes precisely on $D_{\chi^{-1}} + (x_j) - D_{\chi^{-1}\rho(x_j)}$.

There exists a canonical isomorphism from (6.10) to (6.11), observe that it maps β to β'.

Definition 6.6. Given a gnat-family $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ and an arrow $q = (\chi, j) \in Q_1$ of the McKay quiver, define $B_{\mathcal{F},q}$, the `divisor of zeroes of` q `in` $\mathcal{F}$ to be an (effective) divisor

$$B_{\mathcal{F},q} = D_{\chi^{-1}} + (x_j) - D_{\chi^{-1}\rho(x_j)} \tag{6.12}$$

If a prime Weil divisor P belongs to $B_{\mathcal{F},q}$ we shall say that `in` $\mathcal{F}$`,` q `vanishes along` P.

Proposition 6.7. *Let $\mathcal{F}$ be a gnat-family, p be any point in Y and $q \in Q_1$ be an arrow of the McKay quiver. Then $q \notin \Gamma_{\mathcal{F}|_p}$ if and only if $p \in B_{\mathcal{F},q}$.*

Proof. Follows from the discussion prior to definition 6.6. The map α_q in the representation of McKay quiver into $\mathcal{F}|_p$ is a zero map if and only if the image of global section β in $Hom_{G,\mathcal{O}_Y}(x_j \otimes \mathcal{F}_\chi, \mathcal{F}_{\chi\rho(x_j)})_{|p}$ is zero. Which happens if and only if image of $1 \in K(\mathbb{C}^3)$ in $\mathcal{L}(B_{\mathcal{F},q})_{|p}$ is zero, and that is equivalent to $p \in B_{\mathcal{F},q}$. $\square$

Proposition 6.8. *Let $\mathcal{F}$ be a gnat-family. Let $\mathcal{F}'$ be an equivalent family. Then for all $p \in Y$, we have:*

$$\Gamma_{\mathcal{F}|_p} = \Gamma_{\mathcal{F}'|_p}$$

Proof. Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ then Corollary 5.3 implies that $\mathcal{F}' = \oplus\mathcal{L}(-D_\chi + N)$ for some Weil divisor N on Y. This implies $B_{\mathcal{F},q} = B_{\mathcal{F}',q}$ for each $q \in Q_1$, and now the claim follows from Proposition 6.7. $\square$

Proposition 6.8 implies, in particular, that when studying the simplicity of gnat-families it is sufficient to restrict ourselves to the normalised ones. Recall now that if $\oplus\mathcal{F}(-D_\chi)$ is a normalised gnat-family, then Proposition 5.20 and Corollary 3.14 together imply that each D_χ is of form $\sum q_{\chi,i}E_i$. So we have the following lemma:

Lemma 6.9. *Let $\mathcal{F} = \oplus\mathcal{L}(-\sum q_{\chi,i}E_i)$ be a normalised gnat-family. Then for any $(\chi, a) \in Q_1$, an arrow in the McKay quiver, we have*

$$B_{\mathcal{F},(\chi,a)} = \sum_{e_i \in \mathfrak{E}} (q_{\chi^{-1},i} + e_i(x_a) - q_{\chi^{-1}\rho(x_a),i})E_i \tag{6.13}$$

Proof. Follows from (6.12) and the fact that $(x_a) = \sum_P v_P(x_a) = \sum_{e_i \in \mathfrak{E}} e_i(x_a)$, using Corollary 3.14 and Proposition 4.2. $\square$

6.3 Orbits of T in Y

Let us recall some more of the toric magic. The orbits of quotient torus T in Y are in one-to one correspondence with interiors of cones in fan Σ of Y: any orbit S of T in Y defines the cone $\sigma^\vee$ in M of the Laurent monomials, which are regular on S. The points in the interior of the dual cone σ in L are precisely the points $f \in L$, for which the limit point of the corresponding one-parameter subgroup of T, as embedded into its own open orbit in Y, exists and lies on S. We shall write S_σ for the orbit corresponding to the interior of cone $\sigma \in \Sigma$.

Explicitly: for any basic cone $\sigma' \in \mathfrak{F}$, on the affine piece $A_{\sigma'}$, orbit S_σ is given by

$$\{p \in A_{\sigma'} \mid e_i \in \sigma \;\Leftrightarrow\; \check{e}_i(p) = 0\}$$

for the local coordinates $\check{e}_i$, which are the basis dual to the generators e_i of σ'.

On the other hand, for any cone $\sigma \in \Sigma$ we have the affine subvariety of Y, which consists precisely of those orbits $S_{\sigma'}$ of T for which $\sigma \subseteq \sigma'$. We shall denote it by E_σ. It is closed, because, in any affine piece A_μ, it is given by the vanishing of the ideal in $\mathbb{C}[\mu^\vee \cap M]$ generated by those Laurent monomials x^m, for which m lies in the interior of $\sigma^\vee$.

Explicitly: for any basic cone $\sigma' \in \mathfrak{F}$, on the affine piece $A_{\sigma'}$, $E_\sigma \cap A_{\sigma'}$ is given by

$$\{p \in A_{\sigma'} \mid e_i \in \sigma \;\Rightarrow\; \check{e}_i(p) = 0\}$$

for the local coordinates $\check{e}_i$.

With this notation, the divisors E_i on Y are precisely $E_{<e_i>}$.

Proposition 6.9 shows that the vanishing of any map in the representation of the McKay quiver lies along the exceptional divisors or the pullbacks of coordinate hyper-planes in Y. This allows for the following result:

Proposition 6.10. *Let $p, p' \in Y$ be two points lying on the same orbit of the quotient torus T. Then $\Gamma_{\mathcal{F}_p} = \Gamma_{\mathcal{F}_{p'}}$.*

Proof. Since each E_i is a union of orbits of T, we have $p \in E_i$ if and only if $p' \in E_i$ for all $e_i \in \mathfrak{E}$. Now Lemma 6.9 now implies that $p \in B_{\mathcal{F},q}$ if and only if $p' \in B_{\mathcal{F},q}$, for any arrow $q \in Q_1$ of the McKay quiver. Proposition 6.7 now implies the claim. $\square$

Definition 6.11. Let σ be a cone in the fan of Y. Define $\Gamma_{\mathcal{F},\sigma}$ to be $\Gamma_{\mathcal{F},p}$ for any $p \in S_\sigma$.

Proposition 6.12. *Let σ be a cone in the fan of Y and q be an arrow in the McKay quiver of G. Then $q \in \Gamma_{\mathcal{F},\sigma}$ if and only if $B_q|_{A_\sigma} = 0$.*

Proof. Observe that

$$A_\sigma = \bigcup_{\sigma' \subseteq \sigma} S_{\sigma'}$$

Therefore, for any $e_i \in \mathfrak{E}$, $A_\sigma \cap E_i = 0$ is equivalent to $e_i \notin \sigma$, which is equivalent to $S_\sigma \cap E_i = \emptyset$. Therefore $B_q|_{A_\sigma} = 0$ is equivalent $S_\sigma \cap B_q = \emptyset$ and now Proposition 6.7 implies the claim. $\square$

Observe that there exist just the one 0-dimensional cone in Σ - the zero cone. The corresponding orbit, S_0, is the unique open orbit and T acts faithfully on it. S_0 consists of points $p \in Y$ such that $p \notin E_i$ for all $e_i \in \mathfrak{E}$.

Corollary 6.13. *For any gnat-family $\mathcal{F}$, $\Gamma_{\mathcal{F},0}$ is the full McKay quiver of G. In particular, it is connected.*

Proof. If $p \notin E_i$ for all $e_i \in \mathfrak{E}$, then $p \notin B_{\mathcal{F},q}$ for all $q \in Q_1$, the arrows of McKay quiver. Hence, by Proposition 6.12, every $q \in Q_1$ contributes an edge to $\Gamma_{\mathcal{F},0}$. $\square$

We see now that to study the simplicity of G-constellations in a gnat-family $\mathcal{F}$ we need to study the graph $\Gamma_{\mathcal{F},\sigma}$ for each of the finite number of the orbits of T in Y.

6.4 Embedding of the McKay quiver into a real 2-torus

The fact that $G \subseteq \mathrm{SL}_3(\mathbb{C})$ allows us to embed its McKay quiver in a two dimensional real torus. Following is the construction introduced by Craw and Ishii in [CI02]. Recall the following exact sequence, which we introduced in Section 4.1:

$$0 \to M \to \mathbb{Z}^3 \xrightarrow{\rho} G^\vee \to 0 \tag{6.14}$$

Note that as $G \subseteq \mathrm{SL}_3(\mathbb{C})$ we have $(1,1,1) \in M$, i.e. $x_1x_2x_3$ is an invariant monomial. Take $H = \mathbb{Z}^3/\mathbb{Z}(1,1,1)$ and $M' = M/\mathbb{Z}(1,1,1)$. Then (6.14) induces

$$0 \to M' \to H \to G^\vee \to 0 \tag{6.15}$$

For every Laurent monomial $p = \prod x_i^{m_i}$ for some $(m_i) \in \mathbb{Z}^3$ we shall write $[p]$ for the class of (m_i) in H, e.g. $[x_1x_2^2]$ for the class of $(1,2,0)$.

Definition 6.14. The universal cover U of the McKay quiver of G is the quiver whose vertex set are the elements of H and whose arrow set is

$$\{(h, h + [x_i]) \mid h \in H, i \in 1,2,3\}$$

The McKay quiver of G is a quotient of U by the action of M' in the following sense: we identify the class of a vertex h in the quotient with vertex $\rho(h)^{-1}$ of the McKay quiver (recall that G acts by character χ^{-1} on the monomial of weight χ), and identify the class of an arrow $(h, h+[x_i])$ with arrow $(\rho(h)^{-1}, i)$ of the McKay quiver.

We have a natural 'embedding' of U into $H \otimes \mathbb{R}$, where the arrow $(h, h+[x_i])$ is identified with the line segment $\{h + \lambda[x_i] \mid \lambda \in (0,1)\}$. Observe that $H \otimes \mathbb{R}$, with the natural (weak) topology, is a topological vector space isomorphic to $\mathbb{R}^2$.

Definition 6.15. Write $C(h,\sigma)$ for a cycle in U formed by the arrows

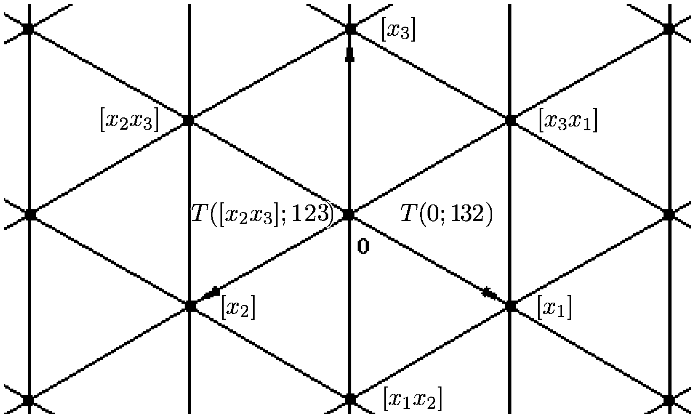
$$\{(h, h+[x_{\sigma(1)}]),\ (h+[x_{\sigma(1)}], h+[x_{\sigma(1)}]+[x_{\sigma(2)}]),\ (h+[x_{\sigma(1)}]+[x_{\sigma(2)}], h)\} \tag{6.16}$$

for some $h \in H$ and $\sigma \in S_3$.

For illustrative purposes, we fix a specific isomorphism:

$$\phi_H : \ H \otimes \mathbb{R} \to \mathbb{R}^2 : \quad \begin{cases} [x_1] \mapsto (\frac{\sqrt{3}}{2}, -\frac{1}{2}) \\ [x_3] \mapsto (0,1) \end{cases}$$

and observe that the image of U under ϕ_H is a tessellation of $\mathbb{R}^2$ into regular triangles with boundaries $C(h;\sigma)$:



Consider now quotient $T_G = H\otimes\mathbb{R}/M'$. It is a two dimensional real torus and observe that U/M', the McKay quiver of G tessellates this torus into triangles, whose boundaries are the images of the $C(h,\sigma)$.

Definition 6.16. Write $C(\chi;\sigma)$ for a cycle in the McKay quiver formed by the arrows

$$\{(\chi,\sigma(1)),\ (\chi\rho([x_{\sigma(1)}])^{-1},\sigma(2)),\ (\chi\rho([x_{\sigma(1)}x_{\sigma(2)}])^{-1},\sigma(3))\} \tag{6.17}$$

for some $\chi\in G^\vee$ and $\sigma\in S_3$.

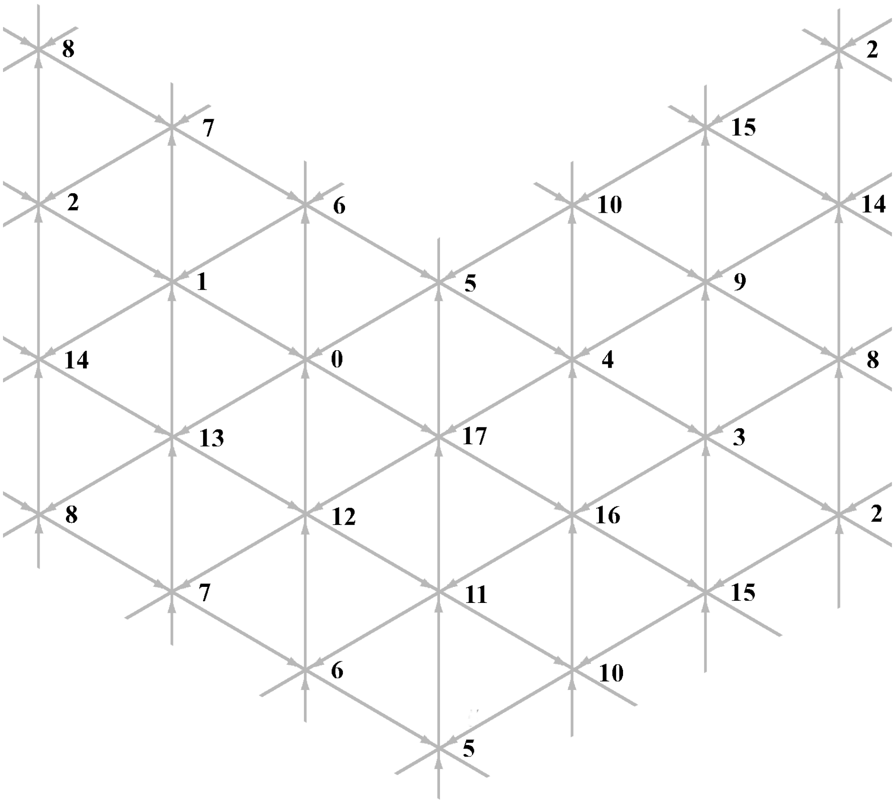
Then $C(\chi;\sigma)$ is the image in the McKay quiver of all $C(h,\sigma)$ in U such that $\rho(h)=\chi^{-1}$. We see there are altogether $2|G|$ distinct cycles $C(\chi,\sigma)$: considering $C(\chi,123)$ and $C(\chi,132)$, for χ ranging across $G^\vee$, counts them all. Hence the McKay quiver tessellates T_G into $2|G|$ regular triangles.

We shall usually depict T_G on the diagrams by drawing a fundamental domain, i.e. a region of $H\otimes\mathbb{R}$ which maps 1-to-1 to T_G on the interior. In our case this would be a fragment of U consisting of $2|G|$ triangles $C(h,\sigma)$, each of which maps to distinct $C(\chi,\sigma)$.

Example 6.17. Most of the examples in this chapter will be given with G being the group $\frac{1}{18}(1,5,12)$. i.e. the group of 18th roots of unity embedded into $\mathrm{SL}_3(\mathbb{C})$ by

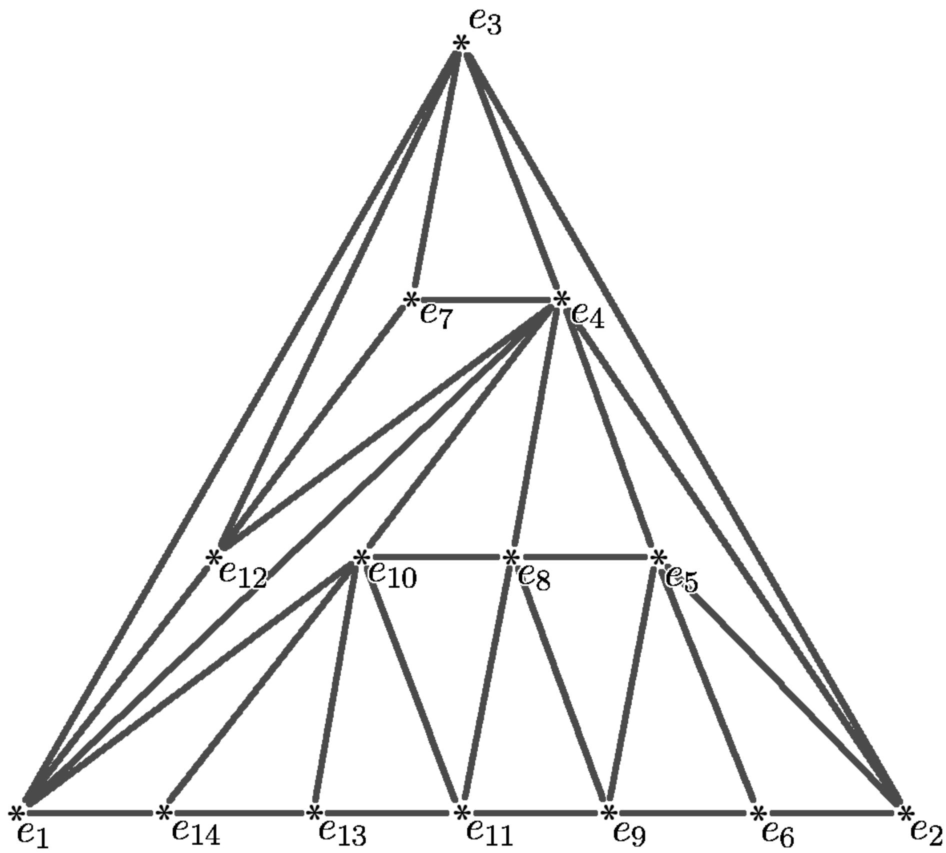
$$\xi\mapsto\begin{pmatrix}\xi^1 & & \\ & \xi^5 & \\ & & \xi^{12}\end{pmatrix}$$

So the tessellation of T_G by the McKay quiver of G looks like:



where we denote by i the vertex χ_i of the McKay quiver in order not to clutter the diagram.

The resolution Y, which we shall use in subsequent examples, is the one whose fan triangulates junior simplex Δ as follows:



where

$$\begin{array}{lll}
e_1 = \frac{1}{18}(18,0,0) & e_6 = \frac{1}{18}(3,15,0) & e_{11} = \frac{1}{18}(9,9,0) \\
e_2 = \frac{1}{18}(0,18,0) & e_7 = \frac{1}{18}(4,2,12) & e_{12} = \frac{1}{18}(11,1,6) \\
e_3 = \frac{1}{18}(0,0,18) & e_8 = \frac{1}{18}(5,7,6) & e_{13} = \frac{1}{18}(12,6,0) \\
e_4 = \frac{1}{18}(1,5,12) & e_9 = \frac{1}{18}(6,12,0) & e_{14} = \frac{1}{18}(15,3,0) \\
e_5 = \frac{1}{18}(2,10,6) & e_{10} = \frac{1}{18}(8,4,6) &
\end{array}$$

The family we shall normally consider will be the maximal shift family $\mathcal{F}_{\max} = \oplus\mathcal{L}(-M_\chi)$. Maximal shift divisors $M_\chi = \sum m_{\chi,i}E_i$ for these G and Y, can be calculated, as in Example 5.21, to be:

	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}	E_{14}
χ_0	0	0	0	0	0	0	0	0	0	0	0
χ_1	$\frac{1}{18}$	$\frac{2}{18}$	$\frac{3}{18}$	$\frac{4}{18}$	$\frac{5}{18}$	$\frac{6}{18}$	$\frac{8}{18}$	$\frac{9}{18}$	$\frac{11}{18}$	$\frac{12}{18}$	$\frac{15}{18}$
χ_2	$\frac{2}{18}$	$\frac{4}{18}$	$\frac{6}{18}$	$\frac{8}{18}$	$\frac{10}{18}$	$\frac{12}{18}$	$\frac{16}{18}$	1	$\frac{4}{18}$	$1\frac{6}{18}$	$\frac{12}{18}$
χ_3	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{9}{18}$	$\frac{12}{18}$	$\frac{15}{18}$	1	$1\frac{6}{18}$	$1\frac{9}{18}$	$\frac{15}{18}$	1	$\frac{9}{18}$
χ_4	$\frac{4}{18}$	$\frac{8}{18}$	$\frac{12}{18}$	$\frac{16}{18}$	$1\frac{2}{18}$	$1\frac{6}{18}$	$\frac{14}{18}$	1	$\frac{8}{18}$	$\frac{12}{18}$	$\frac{9}{18}$
χ_5	$\frac{5}{18}$	$\frac{10}{18}$	$\frac{15}{18}$	$\frac{2}{18}$	$\frac{7}{18}$	$\frac{12}{18}$	$\frac{4}{18}$	$\frac{9}{18}$	$\frac{1}{18}$	$\frac{6}{18}$	$\frac{3}{18}$
χ_6	$\frac{6}{18}$	$\frac{12}{18}$	0	$\frac{6}{18}$	$\frac{12}{18}$	0	$\frac{12}{18}$	0	$\frac{12}{18}$	0	0
χ_7	$\frac{7}{18}$	$\frac{14}{18}$	$\frac{3}{18}$	$\frac{10}{18}$	$\frac{17}{18}$	$\frac{6}{18}$	$1\frac{2}{18}$	$\frac{9}{18}$	$\frac{5}{18}$	$\frac{12}{18}$	$\frac{15}{18}$
χ_8	$\frac{8}{18}$	$\frac{16}{18}$	$\frac{6}{18}$	$\frac{14}{18}$	$1\frac{4}{18}$	$\frac{12}{18}$	$1\frac{10}{18}$	1	$\frac{16}{18}$	$1\frac{6}{18}$	$\frac{12}{18}$
χ_9	$\frac{9}{18}$	1	$\frac{9}{18}$	1	$1\frac{9}{18}$	1	1	$\frac{9}{18}$	$\frac{9}{18}$	1	$\frac{9}{18}$
χ_{10}	$\frac{10}{18}$	$1\frac{2}{18}$	$\frac{12}{18}$	$\frac{4}{18}$	$\frac{14}{18}$	$1\frac{6}{18}$	$\frac{8}{18}$	1	$\frac{2}{18}$	$\frac{12}{18}$	$\frac{6}{18}$
χ_{11}	$\frac{11}{18}$	$1\frac{4}{18}$	$\frac{15}{18}$	$\frac{8}{18}$	$1\frac{1}{18}$	$\frac{12}{18}$	$\frac{16}{18}$	$\frac{9}{18}$	$\frac{13}{18}$	$\frac{6}{18}$	$\frac{3}{18}$
χ_{12}	$\frac{12}{18}$	$\frac{6}{18}$	0	$\frac{12}{18}$	$\frac{6}{18}$	0	$\frac{6}{18}$	0	$\frac{6}{18}$	0	0
χ_{13}	$\frac{13}{18}$	$\frac{8}{18}$	$\frac{3}{18}$	$\frac{16}{18}$	$\frac{11}{18}$	$\frac{6}{18}$	$\frac{14}{18}$	$\frac{9}{18}$	$\frac{17}{18}$	$\frac{12}{18}$	$\frac{15}{18}$
χ_{14}	$\frac{14}{18}$	$\frac{10}{18}$	$\frac{6}{18}$	$1\frac{2}{18}$	$\frac{16}{18}$	$\frac{12}{18}$	$1\frac{4}{18}$	1	$\frac{10}{18}$	$1\frac{6}{18}$	$\frac{12}{18}$
χ_{15}	$\frac{15}{18}$	$\frac{12}{18}$	$\frac{9}{18}$	$\frac{6}{18}$	$1\frac{3}{18}$	1	$\frac{12}{18}$	$1\frac{9}{18}$	$\frac{3}{18}$	1	$\frac{9}{18}$
χ_{16}	$\frac{16}{18}$	$\frac{14}{18}$	$\frac{12}{18}$	$\frac{10}{18}$	$1\frac{8}{18}$	$1\frac{6}{18}$	$1\frac{2}{18}$	1	$\frac{14}{18}$	$\frac{12}{18}$	$\frac{6}{18}$
χ_{17}	$\frac{17}{18}$	$\frac{16}{18}$	$\frac{15}{18}$	$\frac{14}{18}$	$\frac{13}{18}$	$\frac{12}{18}$	$\frac{10}{18}$	$\frac{9}{18}$	$\frac{7}{18}$	$\frac{6}{18}$	$\frac{3}{18}$

6.5 Cell Complexes

To describe the structures arising in the subgraphs Γ_σ, introduced in Definitions 6.11 and 6.3, we shall need the language of cell complexes: both abstract set-theoretical ones and the topological CW-complexes. This section gathers together the formal definitions involved and constructs abstract cell complexes, which mimic topologies of universal cover quiver U, as embedded into plane $H \otimes \mathbb{R}$, and of the McKay quiver of G, as embedded into torus T_G.

Definition 6.18. An `abstract cell complex` is a triple $(\mathcal{A}, <, \dim)$, where $\mathcal{A}$ is a set of objects called `cells`, '$<$' is an irreflexive, anti-symmetric and transitive relation on the elements of $\mathcal{A}$ called `bounding relation`, and 'dim' is a map $\mathcal{A} \to \mathbb{N}$, called the `dimension function`, satisfying $a_1 < a_2 \Rightarrow \dim(a_1) < \dim(a_2)$.

Given $a \in \mathcal{A}$, the `boundary` of a is a set $\{a' \in \mathcal{A} \mid a' < a\}$.

The complex is called n-dimensional if $n = max_{a \in \mathcal{A}} \dim(a)$. We say that a is an m-cell if $\dim(a) = m$, writing A_m for the collection of all m-cells.

Definition 6.19. We define the relation 'a `is connected to` b' on cells of an abstract cell complex $\mathcal{A}$, to be the unique equivalence relation extending the relation $a < b$.

Given a subset B of cells of $\mathcal{A}$, we call the equivalence classes on B, under the relation of being connected, `the connected components` of B. We say that B is connected if it has it has only one connected component.

Definition 6.20. We define a 2-dimensional cell complex $(\mathcal{U}, <, \dim)$ as follows. The set $\mathcal{U}_0$ of 0-cells is H, the vertices of the quiver U. The set of $\mathcal{U}_1$ of 1-cells is the set of the arrows (h, i) of the quiver U. The set of 2-cells is the set of abstract objects

$$\{T(h, \sigma) \mid h \in H; \sigma \in S_3\}$$

in which we identify $T(h, \sigma)$ and $T(h', \sigma')$ if $C(h, \sigma) = C(h', \sigma')$ are the same cycle in U. We call $T(h, \sigma)$ a $\mathcal{U}$`-triangle`. Finally we set $\mathcal{U} = \cup \mathcal{U}_i$.

We define the bounding relation $<$ as the unique irreflexive, anti-symmetric and transitive relation on $\mathcal{U}$ satisfying for every $q \in \mathcal{U}_1$ the following conditions:

1. $hq < q$

2. $tq < q$

3. $q < T(h,\sigma)$ if and only if $q \in C(h,\sigma)$

In similar way, we define a 2-dimensional cell complex $\mathcal{T}$, only with McKay quiver of G instead of U. Observe that we have quotient map $\mathcal{U} \to \mathcal{T}$, with fibre at every cell being a coset of M'.

Observe that now we can consider the graph Γ_V, defined in Definition 6.3, as a 1-dimensional subcomplex of cell complex $\mathcal{T}$, with the edges and vertices of Γ_V being the 1-cells and 0-cells in $\mathcal{T}$.

Compared to a topological CW-complex (see, for example, [Hat01], the Appendix), an abstract cell complex lacks the information on exactly how the cells are attached to each other. On an algebraic level, in a CW-complex X, this information is reflected in the boundary maps δ of the chain complex of singular homology groups

$$\ldots \xrightarrow{\delta} H_k(X_k, X_{k-1}) \xrightarrow{\delta} H_{k-1}(X_{k-1}, X_{k-2}) \to \ldots \tag{6.18}$$

where X_k is the k-skeleton of X, see ([Hat01], Section 2.2, p. 137). The idea is that $H_k(X_k, X_{k-1})$ is a free abelian group, which splits as a direct sum of one $H_n(S_n) = \mathbb{Z}$ for each k-cell in X. For each k-cell a, a choice of a map e_a, homeomorphic on interiors, from the standard k-simplex Δ_k into a defines a class $[e_a]$ in $H_k(X_k, X_{k-1})$, which is a generator of the $\mathbb{Z}$-component of $H_k(X_k, X_{k-1})$, corresponding to a . It is important that there is no intrinsic such choice - as it is equivalent to choosing an orientation on the cell a. Now, given a map e_a, we can use the gluing map from the boundary δa into X_{k-1} to produce a map from each face of Δ_k into X_{k-1}. These let us construct the cycle $\delta\Delta_k$ in $H_{k-1}(X_{k-1})$ and the image of this cycle in $H_{k-1}(X_{k-1}, X_{k-2})$ is exactly the image of $[e_a]$ under the complex boundary map δ. For each $(k-1)$-cell b, the projection of $\delta[e_a]$ onto the $\mathbb{Z}$-component of $H_{k-1}(X_{k-1}, X_{k-2})$ counts, for each time b occurs in the boundary of a, the orientation on b induced from the orientation given on a given by e_a.

Definition 6.21. Given an abstract cell complex $\mathcal{A}$, an `associated chain complex` $\mathcal{C}^{\mathcal{A}}_{\bullet}$ is a collection of free abelian groups $C^{\mathcal{A}}_k = \oplus_{a \in A_k} \mathbb{Z} e_a$ and boundary maps

$$\ldots \xrightarrow{\delta} C^{\mathcal{A}}_k \xrightarrow{\delta} C^{\mathcal{A}}_{k-1} \xrightarrow{\delta} \ldots$$

satisfying following conditions:

1. $\delta^2 = 0$

2. Let a be a k-cell and let $\delta(e_a) = \sum_{b \in A_{k-1}} \lambda_b e_b$. Then, for any $(k-1)$-cell b, $\lambda_b \neq 0$ implies $b < a$.

Similar to CW-complexes, we consider elements e_a and $-e_a$ as corresponding to the two possible choice of "orientation" on cell c. We can then think of the differential maps δ as abstract versions of gluing maps of a CW-complex, in that they tell us us how a choice of "orientation" on a cell induces a choice of "orientation" on the cells in its boundary.

Therefore we shall say that a k-cell $b \in \mathcal{A}_k$ belongs to a k-chain $n = \sum_{a \in \mathcal{A}_k} \lambda_a e_a$ in $\mathcal{C}_\bullet^{\mathcal{A}}$ if the coefficient λ_b of e_b in n is non-zero. By `multiplicity` of b in n we shall mean the absolute value of λ_b. Finally, we define the `orientation` of b in n to be $+e_b$, if λ_b is positive, and $-e_b$, if λ_b is negative.

Given a k-chain $n = \sum_{a \in \mathcal{A}_k} \lambda_a e_a$ in $\mathcal{C}_\bullet^{\mathcal{A}}$, we say that a chain $n' = \sum_{a \in \mathcal{A}_k} \lambda'_a e_a$ is a `subchain` of n if for every k-cell b we have

$$0 \leq \lambda'_b \leq \lambda_b \quad \text{or} \quad \lambda_b \leq \lambda'_b \leq 0 \tag{6.19}$$

In other words, cells, which lie in n', must also lie in n and have in n the same orientation and greater or equal multiplicity.

Given a chain $n \in \mathcal{C}_\bullet^{\mathcal{A}}$, we define a `support` $\tilde{n}$ of n in $\mathcal{A}$ to be subcomplex of A defined by saying that cell a lies in $\tilde{n}$ if and only if a lies in n or there exists b, such that b lies in n and $a < b$.

We say that a chain n is connected if and only if its support $\tilde{n}$ is connected. We say that two chains n and n' are disjoint, if and only if their supports are disjoint.

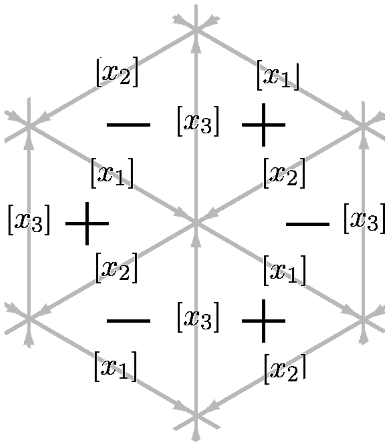
We now proceed to define an associate chain complex for the cell complex $\mathcal{U}$ (and, respectively, for $\mathcal{T}$) constructed in Definition 6.20. We make our choices, based on wanting e_a to correspond, for 1-cells, to the orientation of the respective arrow in the quiver U and, for 2-cells, to the 'anti-clockwise' orientation of the respective triangular domain in $\mathbb{R}^2$, under the embedding ϕ_H.

Definition 6.22. Let $T(h, \sigma)$ be an element of $\mathcal{U}_2$. We say that it is a 'plus' (respectively 'minus') triangle, if $\epsilon(\sigma)$, the sign of σ as a permutation in S_3, is $+1$

(respectively -1).

And similarly for the cycles $C(h,\sigma)$ in the quiver U.

Example 6.23. If we consider the quiver U as embedding into $\mathbb{R}^2$ by ϕ, then the cycles $C(h,\sigma)$ get labelled as follows:



Definition 6.24. For the complex $\mathcal{U}$ from Definition 6.20 we define an associated chain complex $C^{\mathcal{U}}_{\bullet}$ by setting the boundary maps to be:

1. $\delta: C^{\mathcal{U}}_1 \to C^{\mathcal{U}}_0$ maps

$$e_q \mapsto e_{hq} - e_{tq} \tag{6.20}$$

for each $q \in \mathcal{U}_1$,

2. $\delta: C^{\mathcal{U}}_2 \to C^{\mathcal{U}}_1$ maps

$$e_{T(h,\sigma)} \mapsto \epsilon(\sigma) \sum_{q \in C(h,\sigma)} e_q \tag{6.21}$$

for each $T(h,\sigma) \in \mathcal{U}_2$.

In the same way, we define chain complex $C^{\mathcal{T}}_{\bullet}$ associated with cell complex $\mathcal{T}$.

We can now naturally define a `geometrical realisation` of a pair $(\mathcal{A}, \mathcal{C}^{\mathcal{A}}_{\bullet})$ to be a topological space X, with a CW-structure, equipped with:

1. A 1-to-1 correspondence α between k-cells of $\mathcal{A}$ and k-cells of X, which induces equivalence between relations $<$ on $\mathcal{A}$ and 'lie in the boundary of' on X.

2. A chain isomorphism α'

$$\begin{tikzcd}
\cdots \arrow[r] & C_k^{\mathcal{A}} \arrow[r, "\delta"] \arrow[d, "\alpha'", "\sim"'] & C_{k-1}^{\mathcal{A}} \arrow[r] \arrow[d, "\alpha'", "\sim"'] & \cdots \\
\cdots \arrow[r] & H_k(X_k, X_{k-1}) \arrow[r, "\delta"] & H_{k-1}(X_{k-1}, X_{k-2}) \arrow[r] & \cdots
\end{tikzcd}$$

which for each k-cell $a \in A_k$ takes $e_a \in \mathcal{C}_\bullet^{\mathcal{A}}$ to one of the two generators of the $\mathbb{Z}$-component of $H_k(X_k, X_{k-1})$, corresponding to $\alpha(a)$.

The embedding ϕ_H of $H \otimes R$ in $\mathbb{R}^2$ and the induced embedding of U into $\mathbb{R}^2$ as a regular triangular lattice, allows us to define the following geometrical realisations of $(\mathcal{U}, C_\bullet^{\mathcal{U}})$ and $(\mathcal{T}, C_\bullet^{\mathcal{T}})$:

Definition 6.25. Define $\phi\mathcal{U}$ to be a CW-complex on $\mathbb{R}^2$, whose set of 0-cells consists of the images ϕh of all vertices of U under ϕ, whose set of 1-cells consists of an open segment $\phi(h,i) = \{h + \lambda x_i \mid \lambda \in (0,1)\}$ for each arrow (h,i) of U, whose set of 2-cells consists of a 2-cell $\phi T(\chi,\sigma)$, for each 2-cell $T(\sigma,\chi)$ in $\mathcal{U}$, which we define to be open triangular domain enclosed by the cells ϕa, where a ranges over all cells in $\delta T(\chi,\sigma)$. The gluing maps of this complex are simply the identity maps in $\mathbb{R}^2$.

And similarly for the CW-complex $\phi\mathcal{T}$ on torus T_G.

A theorem in algebraic topology([Hat01], Theorem 2.35) tells us that, for a CW-complex X, homology groups of cellular chain complex (6.18) are canonically isomorphic to singular homology groups of X. Hence giving a geometrical realisation of a pair $(\mathcal{A}, \mathcal{C}_\bullet^{\mathcal{A}})$ as a CW-structure on a space X gives an isomorphism from $H_\bullet(X,\mathbb{Z})$ to $H_\bullet(\mathcal{A},\mathbb{Z})$. In particular, this implies that $H_\bullet(\mathcal{U},\mathbb{Z})$ computes the homology of $\mathbb{R}^2$ and $H_\bullet(\mathcal{T},\mathbb{Z})$ computes the homology of the torus T_G.

Given an abstract n-dimensional cell complex pair $(\mathcal{A}, \mathcal{C}_\bullet^{\mathcal{A}})$ we define `dual pair` to consist of:

1. Chain complex $\mathcal{A}^\vee$, whose set of i-cells consists of cell $a^\vee$ for every $(n-i)$-cell a in $\mathcal{A}$ and whose boundary relation is the inversion of the boundary relation on $\mathcal{A}$.

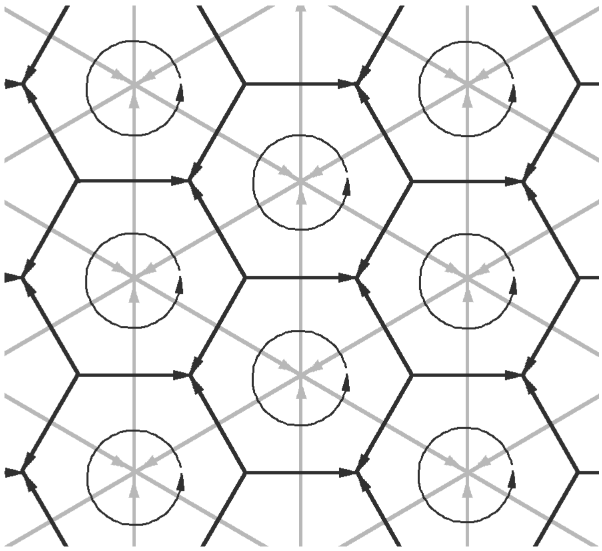
2. The associated chain complex, which we identify with the cochain complex $\mathrm{Hom}(\mathcal{C}_\bullet^{\mathcal{A}}, \mathbb{Z})$ via identifying generator $e_{a^\vee}$ with the co-chain $e_a^\vee$.

Below we describe a CW-structure on $\mathbb{R}^2$ (and, respectively, for the quotient torus T_G), which is a geometrical realisation of dual $\mathcal{U}^\vee$ of complex $\mathcal{U}$ (resp. dual $\mathcal{T}^\vee$ of $\mathcal{T}$) described in Definition 6.25.

Definition 6.26. Define $\phi\mathcal{U}^\vee$ to be a CW-structure on $\mathbb{R}^2$ who 0-cells $\phi T(h,\sigma)^\vee$ are the centres of the triangles $\phi T(h,\sigma)$; whose 1-cells $\phi(h,i)^\vee$ are the open intervals connecting the centres of the two triangles $\phi T(h,\sigma)$, for which $(h,i) \in C(h,\sigma)$; whose 2-cells $h^\vee$ are the hexagonal-shaped domains enclosed by the 1-cells $\phi q^\vee$, where q are those arrows of U for which either $hq = h$ or $tq = h$.

Similarly, we define $\phi\mathcal{T}^\vee$ on T_G.

Below we give a picture of the fragment of CW-complex $\phi\mathcal{U}^\vee$, where we additionally depict the choices of orientations on its 1-cells and its 2-cells which induce chain isomorphism α', required to make it a geometrical realisation of $(\mathcal{A}^\vee, \mathrm{Hom}(\mathcal{C}^{\mathcal{A}}_\bullet, \mathbb{Z}))$:



Finally, observe that since $\mathbb{R}^2$ is a topological universal cover of T_G, and so lift of any path is unique up to a choice of lift of a basepoint, there is a natural 1-to-1 correspondence between elements of $H^1(T_G, \mathbb{Z})$ and translations in the fibre of the quotient map $\mathbb{R}^2 \to T_G$, that is, in the lattice M'. This establishes the following two isomorphisms, which we shall later make a use of:

$$H_1(\mathcal{T}, \mathbb{Z}) \xrightarrow{\sim} M' = M/\mathbb{Z}(1,1,1) \tag{6.22}$$

$$(M')^\vee = L \cap (1,1,1)^\perp \xrightarrow{\sim} H^1(\mathcal{T}, \mathbb{Z}) \tag{6.23}$$

Observe that $L \cap (1,1,1)^{\perp}$, which we shall denote by L', is the lattice which contains differences between any two vectors in the junior simplex $\Delta \subset L$, in particular, for any $\langle e_i, e_j \rangle$, a cone in the fan Σ of Y, it contains $e_i - e_j$, the side of a corresponding basic triangle in the triangulation of Δ.

6.6 Edge-paths

In this section we describe formalities of the language of edge-paths, which we shall require in latter sections to describe 1-cochains in $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Z})$, whose supports in $\mathcal{T}^{\vee}$ have topological structure of a continuous image of $[0,1]$.

An edge-path (see [Spa66], Section 3.6) in a simplicial or CW-complex is defined as a sequence of oriented 1-cells, such that for each cell, but the last one, its end is the origin of the next cell in the sequence.

We have a notion of an oriented 1-cell already: for any cell c in an abstract cell complex $\mathcal{A}$, associated chain complex $\mathcal{C}_{\bullet}^{\mathcal{A}}$ contains elements e_c and $-e_c$ (see Definition 6.21), which in a geometrical realization of $\mathcal{A}$ get identified with the two possible choices of orientations on c.

Definition 6.27. An `oriented cell` in $\mathcal{A}$ is a pair (a, o_a), where a is a cell of A and o_a is one of the elements $+e_a$ or $-e_a$ of $\mathcal{C}_{\bullet}^{\mathcal{A}}$.

The complexes $\mathcal{U}$ and $\mathcal{T}$, and their duals $\mathcal{U}^{\vee}$ and $\mathcal{T}^{\vee}$, all have geometrical realisations as CW-complexes. Based on that, we can make sense of a notion of an origin, and an end, of an oriented 1-cell:

Definition 6.28. Let (q, o_q) be an oriented 1-cell in one of the complexes $\mathcal{U}$, $\mathcal{T}$, $\mathcal{U}^{\vee}$ or $\mathcal{T}^{\vee}$.

Then a 0-cell p is an `origin` of (q, o_q) if the coefficient of e_p in δo_q is -1 and an `end` of (q, o_q), if the coefficient of e_p in δo_q is $+1$.

Observe that equation (6.20) in Definition 6.24 implies that an oriented 1-cell in $\mathcal{U}$, $\mathcal{T}$, $\mathcal{U}^{\vee}$ or $\mathcal{T}^{\vee}$ always has just the one origin and just the one end, as desired. Moreover, (6.20) implies that, in $\mathcal{U}$ (resp. $\mathcal{T}$), the origin and the end of an oriented 1-cell $(q, +e_q)$ are precisely the tail and the head of an underlying arrow q of the universal cover quiver U (resp. McKay quiver of G).

Definition 6.29. A `formal edgepath` in one of the complexes $\mathcal{U}$, $\mathcal{T}$, $\mathcal{U}^\vee$ or $\mathcal{T}^\vee$ is an ordered sequence

$$(q_1, o_{q_1}), \dots, (q_k, o_{q_k})$$

of oriented 1-cells, such that, for any $i \in \{1, \dots, k-1\}$, the end of (q_i, o_{q_i}) is the origin of $(q_{i+1}, o_{q_{i+1}})$.

We say that a formal edge-path $(q_1, o_{q_1}), \dots, (q_k, o_{q_k})$ is `closed`, if the end of (q_k, o_{q_k}) coincides with the origin of (q_1, o_{q_1}), and that it is `open`, otherwise.

We say that a formal edge-path $(q_1, o_{q_1}), \dots, (q_k, o_{q_k})$ is `non self-intersecting` if no 0-cell is an origin of more than (q_i, o_{q_i}) or an end of more than one (q_i, o_{q_i}).

Definition 6.30. Let $\mathcal{A}$ be one of the complexes $\mathcal{U}$, $\mathcal{T}$, $\mathcal{U}^\vee$, $\mathcal{T}^\vee$. We say that a chain n in $\mathcal{C}_1^{\mathcal{A}}$ is an open (resp. closed) edge-path, if n is an image of an open (resp. closed) formal edge-path in $\mathcal{A}$ under the map:

$$(q_1, o_{q_1}), \dots, (q_k, o_{q_k}) \mapsto \sum_{i \in \{1, \dots, k\}} o_{q_k} \tag{6.24}$$

If n is a cochain in $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{A}}, \mathbb{Z})$, we consider it as a 1-chain in a chain complex of dual complex $\mathcal{A}^\vee$ and make the same definition.

If $n \in \mathbb{C}_1^{\mathcal{A}}$ is an edgepath, and $(q_1, o_{q_1}), \dots, (q_k, o_{q_k})$ a choice of its pre-image under (6.24), then observe that:

$$\delta(n) = e_{c_1} - e_{c_2} \tag{6.25}$$

where c_1 is the origin of q_1, while c_2 is the end of q_k.

Definition 6.31. Let $\mathcal{A}$ be one of complexes $\mathcal{U}$, $\mathcal{T}$, $\mathcal{U}^\vee$, $\mathcal{T}^\vee$. Let $n \in \mathcal{C}_1^{\mathcal{A}}$ be an edgepath and c be a 0-cell in the support of n.

Then we say that c is the `startpoint` of n if the coefficient of e_c in δn is -1, that c is the `endpoint` of n if the coefficient is $+1$, and that c is an `internal point` of n if the coefficient is 0.

Thus, as (6.25) demonstrates, an open edgepath has a well-defined startpoint and a well-defined endpoint, while in a closed edgepath all 0-cells are internal.

We conclude the section with two technical lemmas:

Lemma 6.32. *Let $\mathcal{A}$ be one of the complexes $\mathcal{U}$, $\mathcal{T}$, $\mathcal{U}^\vee$, $\mathcal{T}^\vee$. Let n be a chain in $\mathcal{C}_1^{\mathcal{A}}$ such that:*

1. *n is connected*
2. *Maximum multiplicity of any 1-cell in n is 1.*
3. *No 0-cell in the support of n has more than two 1-cells of n attached to it.*
4. *Any 0-cell in the support of n, which has two 1-cells attached to it, doesn't lie in δn.*

Then n is a non self-intersecting edge-path.

Proof. We construct pre-image of n under (6.24) by the following algorithm: we start with a formal edge-path, which consists of an arbitrarily chosen 1-cell in n together with the orientation it has in n. Suppose, at some step of the algorithm, we have already defined a formal edge-path:

$$(q_1, o_{q_1}), \ldots, (q_k, o_{q_k}) \tag{6.26}$$

where each q_k is a distinct 1-cell of n and o_{q_k} is its orientation in n. Observe that, by assumption 2, coefficient of each e_{q_k} in n is precisely its coefficient in o_{q_k}. So, if (6.26) contains every 1-cell in n, then its image under (6.24) is precisely n.

Let p_0 denote the origin of (q_1, o_{q_1}) and p_{k+1} denote the end of (q_1, o_{q_1}).

If p_0 has another 1-cell q_0 of n, distinct from q_1, attached to it, let o_{q_0} be orientation of q_0 in n. By assumption 3, q_0 and q_1 are the only cells of n attached to p_0, therefore $e^\vee_{p_0}(\delta(o_{q_0} + o_{q_1})) = e^\vee_{p_0}(\delta n)$. By assumption 4, $e^\vee_{p_0}(\delta n)$ is zero, therefore $e^\vee_{p_0}(\delta o_{q_0}) = -e^\vee_{p_0}(\delta o_{q_1})$. Therefore, p_0 is an end of (q_0, o_{q_0}), and we can enlarge (6.26) to $(q_0, o_{q_0}), (q_1, o_{q_1}), \ldots, (q_k, o_{q_k})$.

In the same fashion, if p_{k+1} has another 1-cell q_{k+1} of n attached to it, we enlarge (6.26) to $(q_1, o_{q_1}), \ldots, (q_k, o_{q_k}), (q_{k+1}, o_{q_{k+1}})$, where $o_{q_{k+1}}$ is the orientation of q_{k+1} in n.

Finally, if q_1 is the only 1-cell of n attached to p_0, and q_k the only 1-cell of n attached to p_{k+1}, then by assumption 3, no other 1-cell of n can be connected to $q_1, \ldots, q_k$. Since n is assumed to be connected, we conclude that $q_1, \ldots, q_k$ are all the 1-cells in n, and therefore n is the image of the formal edge-path (6.26) under the map (6.24). $\square$

Lemma 6.33. *Let $\mathcal{A}$ be one of the complexes $\mathcal{T}$ or $\mathcal{T}^\vee$. Let $n \in \mathcal{C}_1^{\mathcal{A}}$ be a closed, connected, non self-intersecting edge-path. Then its homology class $[n]$ is either 0 or primitive, i.e. not a proper multiple of any other element of $H_1(\mathcal{A})$.*

Proof. To prove the lemma we pass to the homology on torus T_G, a geometrical realization of both $\mathcal{T}$ and $\mathcal{T}^\vee$, and demonstrate the claim of the lemma for a singular homology class of a loop $n' : [0,1] \to T_G$, defined by any formal edge-path, whose image under (6.24) is n.

Let N denote an open neighbourhood which retracts onto the image of n' in T_G. And consider the long exact sequence for relative homology:

$$\cdots \to H_1(N) \to H_1(T_G) \to H_1(T_G, N) \to \ldots \tag{6.27}$$

Now observe that n' factors through $n'' : [0,1] \to N$, that the homology class $[n''] \in H_1(N)$ generates $H_1(N)$ and that $\delta_3([n'']) = [n']$. Therefore to show that $[n']$ is zero or primitive, it suffice to show that $H_1(T_G, N)$ is torsion-free.

By one of the duality theorems ([Hat01], Theorem 3.46)

$$H_1(T_G, N) \simeq H^1(T_G - N)$$

By the Universal Coefficient Theorem ([Hat01], Theorem 3.2), the torsion subgroup of $H^1(T_G - N)$ is isomorphic to $\mathrm{Ext}^1(H_0(T_G - N), \mathbb{Z})$. The result follows. $\square$

6.7 Codimension 1 Orbits

Let e_i be an element of $\mathfrak{E}$. Then $S_{\langle e_i \rangle}$ is the codimension 1 orbit of T which consists of the 'general' points of E_i, i.e. the points which do not lie on any other exceptional divisor. The following was first observed by Craw and Ishii in [CI02].

Proposition 6.34. *For any of the $2|G|$ triangles $C(\chi, \sigma)$ in the McKay quiver of G, any $e_i \in \mathfrak{E}$ and any gnat-family $\mathcal{F}$, exactly one of the arrows making up $C(\chi, \sigma)$ fails to contribute an edge to $\Gamma_{\mathcal{F}, \langle e_i \rangle}$.*

Proof. Choose any triangle $C(\chi, \sigma)$. From (6.17) and (6.12) it follows that

$$\sum_{i \in \{1,2,3\}} B_{\mathcal{F},(\chi,\sigma(i))} = (x_1) + (x_2) + (x_3) = \sum_{e_i \in \mathfrak{E}} E_i \tag{6.28}$$

On the other hand, we know that every $B_{\mathcal{F},q} = \sum b_{q,j} E_j$ is an effective divisor. This, together with (6.28) implies that each $b_{q,j} \in \{0,1\}$ and that exactly one of $b_{(\chi,\sigma(1)),j}$, $b_{(\chi,\sigma(2)),j}$ and $b_{(\chi,\sigma(3)),j}$ is equal to 1 for each $e_j \in \mathfrak{E}$. This proves the claim. $\square$

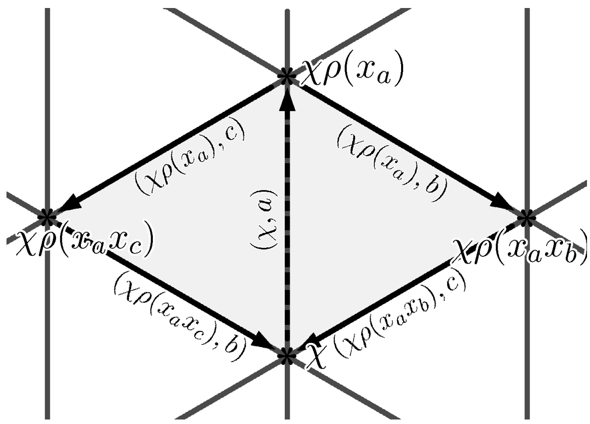
Corollary 6.35. *For any gnat-family $\mathcal{F}$ and any $e_i \in \mathfrak{E}$, $\Gamma_{\mathcal{F},\langle e_i \rangle}$ is connected.*

Proof. Let $r = (\chi, a)$ be an arrow of McKay quiver which doesn't contribute an edge to $\Gamma_{\langle e_i \rangle}$. So $B_r|_{A_{\langle e_i \rangle}} = E_i$. Let $C(\chi, \sigma)$ be either of the two triangles, which contains r. But then (6.28) restricted to $A_{\langle e_i \rangle}$ gives

$$\sum_{q \in C(\chi,\sigma)} B_q|_{A_{\langle e_i \rangle}} = E_i$$

which implies that the other two arrows in $C(\chi, \sigma)$ restrict to zero on $A_{\langle e_i \rangle}$ and so, by Proposition 6.12, do each contribute an edge to $\Gamma_{\langle e_i \rangle}$. These two edges then connect the tail tq and the head hq of q. Thus the connectivity relation on the vertices of McKay quiver in $\Gamma_{\langle e_i \rangle}$ is the same as on the full McKay quiver, and the result follows. $\square$

Observe that for each arrow $q \in Q_1$, there are exactly two triangles in $\mathcal{T}_2$, which contain q in their boundary: if $q = (\chi, a)$, they are $T(\chi, abc)$ and $T(\chi, acb)$ where $\{a, b, c\} = \{1, 2, 3\}$. Proposition 6.34 implies that, if $q \notin \Gamma_{\langle e_i \rangle}$ for some $e_i \in \mathfrak{E}$, then the remaining four edges comprising the boundaries of these two triangles, do belong to $\Gamma_{\langle e_i \rangle}$. Correspondingly, in the quotient torus T_G the closure of the union of the cells $\phi T(\chi, abc)$ and $\phi T(\chi, acb)$ is a rhombus-shaped domain, whose boundary lies in $\phi\Gamma_{\langle e_i \rangle}$ and whose interior doesn't:



Definition 6.36. Given an arrow $q = (\chi, a) \in Q_1$ of the McKay quiver of G, we write $D(q)$ for the chain

$$e_{T(\chi,abc)} + e_{T(\chi,acb)}$$

in $\mathcal{C}_2^{\mathcal{T}}$ and say that it is an a-`oriented diamond`.

Definition 6.37. For any $e_i \in \mathfrak{E}$, an e_i-`diamond` is a diamond $D(q)$ if q doesn't lie in $\Gamma_{\langle e_i \rangle}$ and every 1-cell in $\delta D(q)$ does.

Observe that, by Proposition 6.34, $q \notin \Gamma_{\langle e_i \rangle}$ if and only if every cell in $\sigma D(q)$ does lie $\Gamma_{\langle e_i \rangle}$.

Thus, Proposition 6.34 together with Proposition 6.12 imply:

Corollary 6.38. *For any $e_i \in \mathfrak{E}$ and any arrow q of the McKay quiver, $D(q)$ is an e_i-diamond if and only if $B_q|_{A_{\langle e_i \rangle}} = E_i$, i.e. E_i occurs in B_q with non-zero coefficient.*

Proposition 6.34 therefore implies that every triangle $T(\chi; \sigma)$ lies in an unique e_i-diamond $D(q)$. Thus we see that there are $|G|$ e_i-diamonds and that $\mathcal{T}$ is a disjoint union of 2-cells comprising all e_i-diamonds.

Also, observe that a 1-cell q lies in the boundary of some diamond if and only if it lies $\Gamma_{\langle e_i \rangle}$. Therefore, in the quotient torus T_G, the image of $\Gamma_{\langle e_i \rangle}$ is a tessellation of T_G into $|G|$ rhombus-shaped domains.

Example 6.39. Let the setup be as in Example 6.17. In this example, we shall calculate $\Gamma_{\langle e_{10} \rangle}$. First, for each $q \in Q_1$ we calculate the divisor B_q, as in (6.12). Then we check whether E_{10} belongs to B_q or not.

Recall that $(x_a) = \sum e_i(x_a)E_i$, so

$$\begin{aligned}
(x_1) = \quad & E_1 + \frac{1}{18}E_4 + \frac{2}{18}E_5 + \frac{3}{18}E_6 + \frac{4}{18}E_7 + \frac{5}{18}E_8 + \frac{6}{18}E_9 + \\
& \frac{8}{18}E_{10} + \frac{9}{18}E_{11} + \frac{11}{18}E_{12} + \frac{12}{18}E_{13} + \frac{15}{18}E_{14} \\
(x_2) = \quad & E_2 + \frac{5}{18}E_4 + \frac{10}{18}E_5 + \frac{15}{18}E_6 + \frac{2}{18}E_7 + \frac{7}{18}E_8 + \frac{12}{18}E_9 + \frac{4}{18}E_{10} \\
& + \frac{9}{18}E_{11} + \frac{1}{18}E_{12} + \frac{6}{18}E_{13} + \frac{3}{18}E_{14} \\
(x_3) = \quad & E_3 + \frac{12}{18}E_4 + \frac{6}{18}E_5 + \frac{12}{18}E_7 + \frac{6}{18}E_8 + \frac{6}{18}E_{10} + \frac{6}{18}E_{12}
\end{aligned}$$

So, for example,

$$B_{(\chi_{16},1)} = D_{\chi_2} + (x_1) - D_{\chi_3} = E_{13} + E_{14}$$

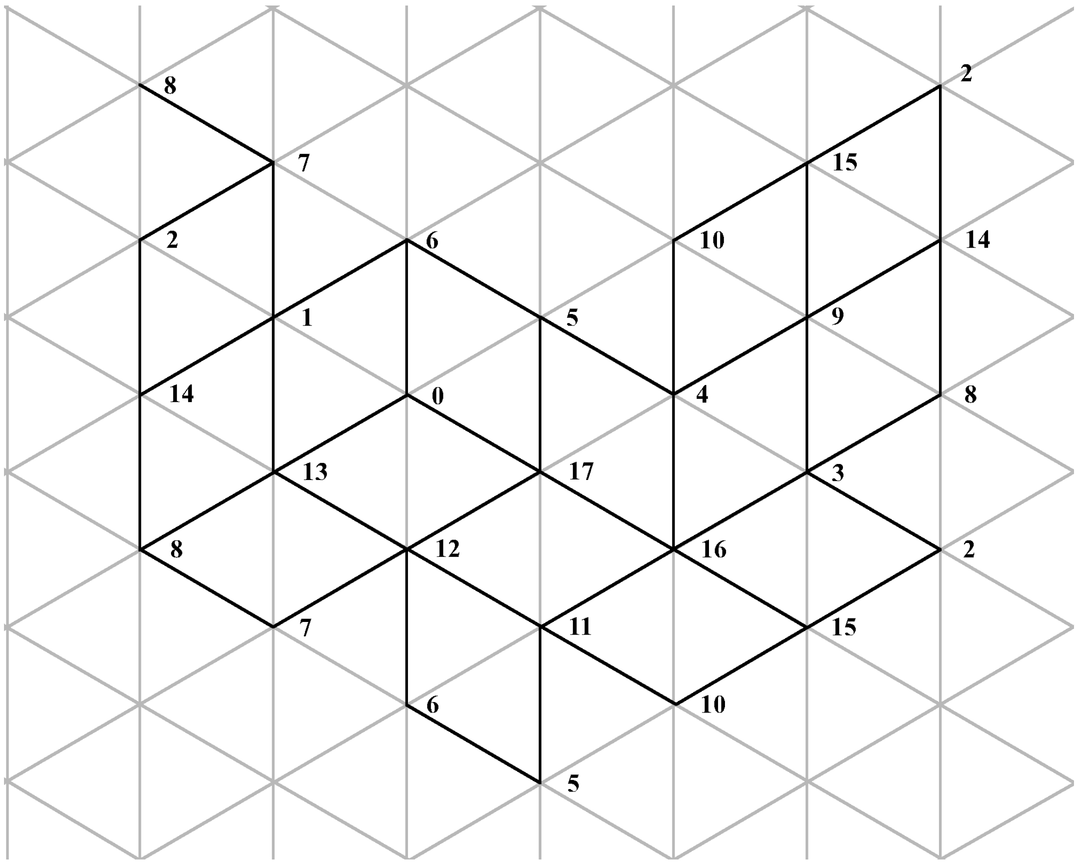
and so by Proposition 6.7 we have $(\chi_{16}, 1) \in \Gamma_{\langle e_{10} \rangle}$.

On the other hand,

$$B_{(\chi_{15},3)} = D_{\chi_3} + (x_3) - D_{\chi_{15}} = E_7 + E_{10} + E_{12}$$

and so $(\chi_{15}, 3) \notin \Gamma_{\langle e_{10} \rangle}$.

Continuing this, we eventually obtain $\Gamma_{\langle e_{10} \rangle}$ being:



Let us now consider the numbers of x_1, x_2 and x_3 oriented e_{10}-diamonds, respectively, in $\Gamma_{\langle e_{10} \rangle}$. Counting them on the diagram in the Example, we can see that there are 8 x_1-oriented e_{10}-diamonds, 4 x_2-oriented e_{10}-diamonds and 6 x_3-oriented e_{10}-diamonds. And $e_{10} = \frac{1}{18}(8, 4, 6)$. This is not a coincidence.

Proposition 6.40. *For any gnat-family $\mathcal{F}$ and any $e_i \in \mathfrak{E}$, let n_a be the number of x_a-oriented e_i-diamonds in T_G. Then*

$$n_a = |G| e_i(x_a) \tag{6.29}$$

Proof. Let $a \in \{1, 2, 3\}$. Consider the multiplicity with which E_i occurs in

$$\sum_{\chi \in G^\vee} B_{(\chi,a)} \tag{6.30}$$

On one hand, Proposition 6.34 implies that E_i occurs in each $B_{(\chi,a)}$ with multiplicity of either 0 or 1. Also Proposition 6.7 implies that $q \in \Gamma_{\langle e_i \rangle}$ if and only if multiplicity of E_i in B_q is 0. So we see that multiplicity of E_i in (6.30) is the number of x_a oriented arrows of the McKay quiver, which do not belong to $\Gamma_{\langle e_i \rangle}$, and that is precisely n_a.

On the other hand, from definition of B_q, we can re-write (6.30) as

$$\sum_{\chi \in G^\vee} D_\chi + (x_a) - D_{\chi\rho(x_a)}$$

which, since $\sum D_\chi = \sum D_{\chi\rho(x_a)}$, is simply $|G|(x_a)$. Thus we see that

$$n_a = |G| v_{E_i}(x_a) = |G| e_i(x_a)$$

as required. $\square$

Observe, that, as a lift to $\mathrm{Hom}(\mathbb{Z}^3, \mathbb{Q})$, $e_i = \frac{1}{|G|}(e_i(x_1), e_i(x_2), e_i(x_3))$, so Proposition 6.40 implies the observation that

$$e_i = \frac{1}{|G|}(n_1, n_2, n_3)$$

.

6.8 Homological digression

We know that the cohomology ring of torus, with multiplication given by the cup product, is an exterior algebra of the first cohomology group. Therefore, using natural isomorphism (6.23):

$$L' \xrightarrow{\sim} H^1(\mathcal{T}, \mathbb{Z})$$

established in the end of the Section 6.5, we can choose to identify the classes in $H^\bullet(T, \mathbb{Z})$ with elements of L' and $\Lambda^2 L'$. For example, given a cocycle $c \in \mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}), \mathbb{Z})$, we shall speak of a class of c in L' to mean the pre-image of the cohomology class of c under (6.23).

In this section we shall establish how, given numerically a pair of cocycles c and c' in $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$, to calculate their classes in $L' \otimes \mathbb{Q}$ as well as the cup product of their classes in $\Lambda^2 L' \otimes \mathbb{Q}$.

To do that, we explicitly compute the natural isomorphism (6.22), which (6.23) is a dual of.

Lemma 6.41. *Consider natural homomorphism*

$$\mathcal{C}_1^{\mathcal{T}} \to \mathbb{Z}^3 : \quad e_{(\chi,i)} \mapsto [x_i] \tag{6.31}$$

where $[x_i]$ denotes exponent of x_i in $\mathbb{Z}^3$.

Then it descends to a map

$$H_1(\mathcal{T}, \mathbb{Z}) \to M' \tag{6.32}$$

and this map is the natural isomorphism (6.22).

Proof. To show that (6.31) descends to a map from $H^1(\mathcal{T}, \mathbb{Z})$ to M' we need to show it maps cycles into M and boundaries into $\mathbb{Z}(1,1,1)$.

That (6.31) maps cycles to invariant monomials follows from the fact that it clearly maps any closed edge-path to an invariant monomial, and any cycle is a sum of closed edge-paths.

That (6.31) maps boundaries into $\mathbb{Z}(1,1,1)$ follows from the fact that if we take any triangle $T(\chi, \sigma)$ then

$$\epsilon(\sigma) \sum_{q \in T(\chi,\sigma)} e_q \mapsto \epsilon(\sigma)[x_{\sigma(1)}] + [x_{\sigma(2)}] + [x_{\sigma(3)}] = \epsilon(\sigma)(1,1,1)$$

Finally, to see that resulting map from $H_1(\mathcal{T}, \mathbb{Z})$ to M' agrees with the natural isomorphism (6.22), consider any closed edge-path c in $\mathcal{C}_1^{\mathcal{T}}$. Given a 0-cell b, which lies in c, and a choice b' of its lift to $\mathcal{C}_0^{\mathcal{U}}$, there exists a unique edge-path c' lifting c to $\mathcal{C}_1^{\mathcal{U}}$, such that b' is its startpoint. It is clear from definition of (6.31), that the endpoint of c' is a translate of b' by the image of c under (6.31). This gives the requisite statement for the classes of closed edge-paths, and hence for the whole of H_1. □

Corollary 6.42. *Let $\nu : \mathbb{Q}^3 \to \mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$ be a homomorphism defined by setting $\nu(l)$ to be the map*

$$e_{(\chi,i)} \mapsto l([x_i])$$

for all arrows $q = (\chi, i)$ in the McKay quiver.

Then the restriction of ν to L' descends to a map $L' \to \mathrm{Hom}(H_1(\mathcal{T}, \mathbb{Z}), \mathbb{Z})$, which is precisely the isomorphism (6.22)*.*

Proof. This follows from Lemma 6.41 and the fact that ν restricted to L and composed with map $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q}) \to \mathrm{Hom}(Z\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$ is precisely the $\mathrm{Hom}(\bullet, \mathbb{Z})$ of a restriction of (6.31) to map $Z\mathcal{C}_1^{\mathcal{T}} \to M$, where $Z\mathcal{C}_1^{\mathcal{T}}$ denotes the subgroup of $\mathcal{C}_1^{\mathcal{T}}$ consisting of all 1-cycles. □

Effectively, map ν singles out in each cohomology class a representative cocycle.

Definition 6.43. Let $l \in \mathbb{Q}^3$. We say that cochain $\nu(l)$ in Corollary 6.42 is a `constant` l`-cochain`.

The word 'constant' is chosen so as to refer to the fact that values of $\nu(l)$ on the side of a triangle T are the same for all T in $\mathcal{T}_2$. Conversely, let c be an arbitrary cochain in $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$, observe that for any triangle $T \in \mathcal{T}_2$, there exists an unique $l \in L$ such that $\nu(l)$ agrees with c on e_q for all three sides $q \in T$.

Definition 6.44. Let c be a cochain in $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$. Define a local approximation map $\psi(c) : \mathcal{T}_2 \to \mathbb{Q}^3$ by:

$$\psi(c)(T)([x_i]) = c(e_{\chi,i}) \tag{6.33}$$

where (χ, i) is the unique x_i-oriented arrow in T.

Observe that if c filters through $\mathbb{Z} \hookrightarrow \mathbb{Q}$, then $\psi(c)$ filters through $L \hookrightarrow \mathbb{Q}^3$. And if c is a cocycle, then for any $T \in \mathcal{T}_2$ we have $\sum_{q \in T} c(e_q) = 0$ and so $\psi(c)$ filters through inclusion $L' \hookrightarrow L$.

The local approximation map of constant l-cochain $\nu(l)$ is, naturally, the constant map which maps every triangle in $\mathcal{T}_2$ to l. For an arbitrary cochain c, as it turns out, taking average of its local approximations across $\mathcal{T}_2$ calculates its cohomology class:

Proposition 6.45. *Let c be a cocycle in* $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$*. Write $[c] \in L' \otimes \mathbb{Q}$ for the cohomology class of c. Then*

$$[c] = \frac{1}{2|G|} \sum_{T \in \mathcal{T}_2} \psi(c)(T) \tag{6.34}$$

NB: Observe, that in a special case of $c = \nu(l)$ for some $l \in L' \otimes \mathbb{Q}$, (6.34) is trivially true since $[\nu(l)] = l$ and $\psi(\nu(l))(T) = l$ for every $T \in \mathcal{T}_2$.

Proof. Since $[\nu([c])] = c$, $\nu([c]) - c$ must be a co-boundary. Since (6.34) is additive in c and trivially true for $c = \nu(l)$, it suffices to show it holds whenever c is a co-boundary.

Let $c = \delta b$ for some $b \in \mathrm{Hom}(\mathcal{C}_0^{\mathcal{T}}, \mathbb{Q})$. Defining equation (6.33) of $\psi(c)(T)$ implies that evaluating RHS in (6.34) at any $[x_i] \in \mathbb{Z}^3$, we get

$$\frac{1}{2|G|} c(\sum_{\chi \in \mathcal{T}_0} 2e_{\chi,i}) \tag{6.35}$$

Observe that $\sum_{\chi \in \mathcal{T}_0} 2e_{\chi,i}$ is a cocycle in $\mathcal{T}_1$, hence

$$\delta b(\sum_{\chi \in \mathcal{T}_0} 2e_{\chi,i}) = b(\delta \sum_{\chi \in \mathcal{T}_0} 2e_{\chi,i}) = 0$$

Since choice of basic monomial x_i was arbitrary, RHS in (6.34) evaluates to 0 on all of $\mathbb{Z}^3$ and hence must be 0 itself, as required. $\square$

More surprisingly, for an arbitrary pair of cocycles, taking average of the wedge products of their local approximations across $\mathcal{T}_2$ calculates the cup product of their cohomology classes in $\Lambda^2 L' \otimes \mathbb{Q}$:

Proposition 6.46. *Let c_1 and c_2 be cocycles in* $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$*. Let $[c_1]$ and $[c_2]$*

denote their cohomology classes in $L' \otimes \mathbb{Q}$.

$$[c_1] \wedge [c_2] = \frac{1}{2|G|} \sum_{T \in \mathcal{T}_2} \psi(c_1)(T) \wedge \psi(c_2)(T) \tag{6.36}$$

Proof. If $c_1 = \nu(l)$ for some $l \in L' \otimes \mathbb{Q}$, then (6.36) becomes $l \wedge (\sum \psi(c_2)(T))$ and Proposition 6.45 gives the result. Therefore, as before, we can assume that c_1 is co-boundary δb for some $b \in \operatorname{Hom}(\mathcal{C}_0^{\mathcal{T}}, \mathbb{Q})$.

Define a cochain $d \in \operatorname{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$ by $d(e_q) = (b(e_{hq}) + b(e_{tq}))c_2(e_q)$. Then it is straightforward calculation to verify that, for any $T \in \mathcal{T}_2$,

$$\psi(\delta b)(T) \wedge \psi(c_2)(T) = \delta d(e_T)(1, 0, -1) \wedge (0, 1, -1)$$

and hence (6.36) becomes $d(\delta(\sum_{\mathcal{T}_2} e_T))(1, 0, -1) \wedge (0, 1, -1)$, which is zero since $\sum_{\mathcal{T}_2} e_T$ is a cycle. $\square$

We now seek to apply this to our situation at hand. Given a gnat-family $\mathcal{F} = \oplus \mathcal{L}(-D_\chi)$, the valuation at E_i of each D_χ , i.e. the coefficient of E_i in D_χ, defines a cochain in $\operatorname{Hom}(\mathcal{C}_0^{\mathcal{T}}, \mathbb{Q})$.

Definition 6.47. Let $\mathcal{F} = \oplus \mathcal{L}(-\sum q_{\chi,i} E_i)$ be a normalised gnat-family. We define a cochain $\mu_{\mathcal{F},e_i}$ to be an element of $\operatorname{Hom}(\mathcal{C}_0^{\mathcal{T}}, \mathbb{Q})$ given by

$$e_\chi \mapsto q_{\chi^{-1},i}$$

Now, Lemma 6.9 implies that, for every arrow q of McKay quiver, the 1-cochain $\nu(e_i) - \delta\mu_{\mathcal{F},e_i}$ maps e_q to a coefficient of E_i in $B_{\mathcal{F},q}$. Therefore it is actually an integer 1-cochain, and moreover for any arrow $q \in Q_1$ we have $e_q \mapsto 0$ if and only if $q \in \Gamma_{\langle e_i \rangle}$ and $e_q \mapsto 1$ if and only if $q \notin \Gamma_{\langle e_i \rangle}$. Thus we can think of it as a characteristic map for e_i-diamonds: it maps e_q to 0, if $D(q)$ is not an e_i-diamond, and to 1 if it is.

Definition 6.48. We define e_i`-diamond cochain` $d_{\mathcal{F},e_i}$ to be an element of $\operatorname{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Z})$ given by

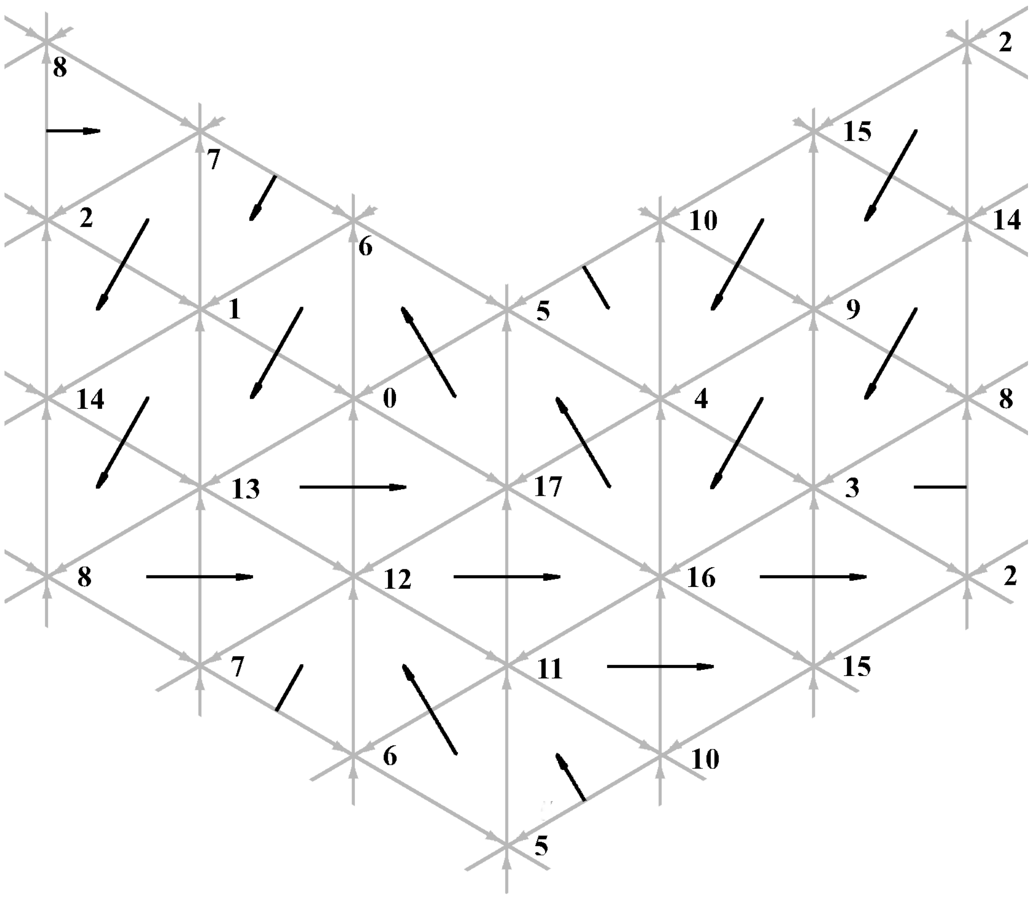
$$d_{\mathcal{F},e_i} = \nu(e_i) - \delta\mu_{\mathcal{F},e_i} \tag{6.37}$$

In other words,

$$d_{\mathcal{F},e_i} = \sum_{q \notin \Gamma_{\langle e_i \rangle}} e_q^\vee \tag{6.38}$$

In the dual cell-complex on the quotient torus T_G, its image is a disjoint union of line segments joining the centres of the two triangles in each e_i-diamond. Each of those line segments is an image of a 1-cell $q^\vee$, which we think of as corresponding to the diamond $D(q)$.

Example 6.49. Let the setup be as in Example 6.39. Then the image of the e_{10}-diamond cochain $d_{\mathcal{F},e_{10}}$ in the dual complex $\mathcal{T}^\vee$ on T_G looks like:



The arrowhead on each cell $q^\vee$ denotes its orientation in the cochain: its direction agrees with $e_q^\vee$ if the orientation is $e_q^\vee$, and goes in the opposite direction if the orientation is $-e_q^\vee$. In further diagrams, whenever the multiplicity of $q^\vee$ in cochain is greater than 1, we shall denote its value by writing it on the centre of the cell $q^\vee$.

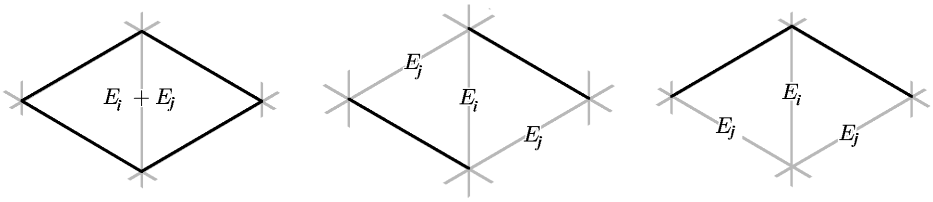
6.9 Codimension 2 Orbits

For each two-dimensional cone $\sigma = \langle e_i, e_j \rangle$ in the fan Σ of Y, where $e_i, e_j \in \mathfrak{E}$, we have a codimension 2 torus orbit S_σ, which consists of the general points on the intersection of the exceptional divisors E_i and E_j, i.e. points which do not also lie on some third exceptional divisor E_k.

Consider now vector $e_i - e_j \in L'$. On one hand, as demonstrated in Section 6.8, $e_i - e_j$ can be viewed as a cohomology class in $H^1(\mathcal{T})$, and therefore in $H^1(T_G)$.

On one hand, $e_i - e_j$ is precisely $\sigma \cap \Delta$, i.e. the edge, in the triangulation of the junior simplex Δ by the fan of Y, which connects e_i and e_j. In toric picture, we think of this edge as representing the curve E_σ, and of its interior as representing the orbit S_σ. As it turns out, for each family $\mathcal{F}$ we can describe the way, in which graph $\Gamma_{\mathcal{F},\sigma}$ is disconnected, by a closed curve in torus T_G. We first give an intuitive sketch:

Example 6.50. Let $\mathcal{F}$ be a gnat-family across Y. Let $q \in Q_1$ be an arrow of the McKay quiver, and assume further that $E_i \in B_{\mathcal{F},q}$, i.e. $D(q)$ is an e_i-diamond. There are three possible cases:



Case I **Case II** **Case III**

On the diagram, we marked each arrow by the restriction, to A_σ, of its divisor of zeroes, except the arrows for which this restriction is zero. Such arrows belong to Γ_σ and are drawn in bold instead.

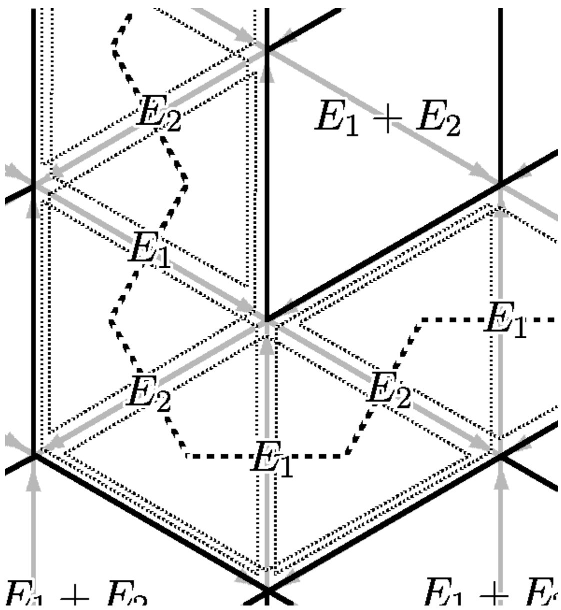
Case I First case is that $D(q)$ is also an e_j-diamond, i.e. $B_q|_{A_\sigma} = E_i + E_j$. In this case, the four arrows belonging to the boundary $\sigma D(q)$ must all lie in Γ_σ, by the Proposition 6.34.

Case II Suppose $D(q)$ is not also an e_j-diamond, i.e. $B_q|_{A_\sigma} = E_i$. Then Proposition 6.34 implies that out of the four arrows comprising the boundary σD_q, exactly two have the restriction of their divisor of zeroes in $\mathcal{F}$ to A_σ being E_j. The Case II is when these two arrows are not adjacent to each other, i.e. have no common 0-cells in their boundaries.

Case III This is the last remaining possibility: $D(q)$ is not an e_j-diamond and the two arrows in the boundary of $D(q)$, whose divisors of zeroes in $\mathcal{F}$ restricted to A_σ is E_j, are adjacent to each other.

Observe that it is Cases II and III, which contribute to disconnectivity of Γ_σ, as in them we see vertices of $D(q)$ become disconnected from one another.

Moreover, suppose $D(q)$ is a Case II or III e_i-diamond. Let q' be one of the two arrows in the boundary σD_q, which vanishes along E_j. As e_i-diamonds tessellate T_G, there exists exactly one more e_i-diamond, whose boundary contains q'. Denote this e_i-diamond by $D(q'')$ and observe that it must also be Case II or III. Applying the same reasoning to $D(q'')$ as we did to $D(q)$, we see that Case II and III e_i-diamonds form a closed band, whose boundary arrows all belong to Γ_q and whose internal arrows have their divisor of zeroes in $\mathcal{F}$ restricted to A_σ being E_i and E_j, consecutively. In other words, this band consists of interlocked e_i and e_j diamonds:



This band is precisely what disconnects Γ_σ: the rest is made up out of Case I e_i-diamonds, in each of which all four vertices are connected by edges of Γ_q. Consider a curve passing through the middle of the band, as depicted on the figure. This curve is closed, and therefore defines a homology class in T_G. It makes sense to ask whether it is somehow related to the cohomology class in $H^1(T_G)$ defined by $e_i - e_j$.

We now make the ideas sketched out in the Example 6.50 precise.

Definition 6.51. Let $\mathcal{F}$ be a gnat-family across Y and $\sigma = \langle e_i, e_j \rangle$ be a two-dimensional cone in the fan of Y. Define $S_{\mathcal{F},\sigma,e_i}$ to be the 1-cochain in $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Z})$ defined by

$$S_{\mathcal{F},\sigma,e_i} = d_{\mathcal{F},e_i} - d_{\mathcal{F},e_j} \tag{6.39}$$

By the σ-Strand we mean the subcomplex of $\mathcal{T}^\vee$ which is the support of $S_{\mathcal{F},\sigma,e_i}$ (or, equivalently, of $S_{\mathcal{F},\sigma,e_j}$).

Observe that (6.37) implies that:

$$d_{\mathcal{F},e_i} - d_{\mathcal{F},e_j} = \delta(\mu_{\mathcal{F},e_i} - \mu_{\mathcal{F},e_j}) - \nu(e_i - e_j) \tag{6.40}$$

Since $e_i - e_j$ lies in L', $\nu(e_i - e_j)$ is a cocycle, and hence so is $S_{\mathcal{F},\sigma,e_i}$.

Next, we characterise the cells which belong to σ-Strand:

Lemma 6.52. *Let $\mathcal{F}$ be a gnat-family across Y, $\sigma = \langle e_i, e_j \rangle$ be a two-dimensional cone in the fan of Y and q be an arrow in the McKay quiver. Denote by μ the coefficient of $e_q^\vee$ in $S_{\mathcal{F},\sigma,e_i}$. Then μ depends on $B_q|_{A_\sigma}$ as follows:*

$B_q\vert_{A_\sigma}$	μ
E_i	1
E_j	-1
0 *or* $E_i + E_j$	0

Proof. The coefficients of E_i and E_j in B_q are each either 0 or 1, as demonstrated by (6.28). Since $E_i|_{A_\sigma} \neq 0$ only if $e_i \in \sigma$, there are four possible values of $B_q|_{A_\sigma}$: E_i, E_j, $E_i + E_j$ and 0.

Lemma 6.9 implies that coefficient of $e_q^\vee$ in d_{e_i} is the coefficient of E_i in B_q. Therefore, for each of the four cases above, we can compute the coefficient of $e_q^\vee$ in $S_{\mathcal{F},\sigma,e_i} = d_{e_i} - d_{e_j}$, which yields the table in the claim. E.g. if $B_q|_{A_\sigma} = E_i$, then $d_{e_i}(e_q) = 1$ and $d_{e_j}(e_q) = 0$, and therefore coefficient of $e_q^\vee$ in $S_{\mathcal{F},\sigma,e_i}$ is 1. □

In other words, $q^\vee$ lies in σ-Strand if and only if $D(q)$ is an e_i-diamond, or an e_j-diamond, but not both. Compare this with the construction in Example 6.50.

Lemma 6.53. *Let $\mathcal{F}$ be a gnat-family across Y, $\sigma = \langle e_i, e_j \rangle$ be a two-dimensional cone in the fan of Y and T be a triangle in $\mathcal{T}_2$. Then the $T^\vee \in \sigma$-Strand if and only if the restrictions, to A_σ, of divisors of zeroes of sides of T are E_i, E_j and* 0.

Proof. First assume that $T^\vee$ belongs to σ-Strand. Then T must has a side q, such that $q^\vee \in \sigma$-Strand. Then by Lemma 6.52, $B_q|_{A_\sigma}$ is E_i or E_j. Without loss of generality, let it be E_i. Then, by Proposition 6.34, for one of the remaining two sides restriction to A_σ of its divisor of zeroes is E_j and for the other is 0.

Now assume that the restrictions, to A_σ, of divisors of zeroes of sides of T are E_i, E_j and 0. Denote by q_i, q_j and q_0, respectively, these sides of T. By Lemma 6.52, $q_i^\vee$ and $q_j^\vee$ belong to σ-Strand (note that $q_0^\vee$ doesn't!), and hence so does $T^\vee$. □

This immediately tells us the following about the shape of σ-Strand:

Corollary 6.54. *Let $\mathcal{F}$ be a gnat-family across Y and $\sigma = \langle e_i, e_j \rangle$ be a two-dimensional cone in the fan of Y. Then connected components of $S_{\mathcal{F},\sigma,e_i}$ are closed, non self-intersecting edge-paths.*

Proof. Let n be a connected component of $S_{\mathcal{F},\sigma,e_i}$. By Lemma 6.52, no 1-cell in n has multiplicity greater than 1. By Lemma 6.53, every 0-cell, which lies in the support of n, has exactly two 1-cells of n attached to it. Since $S_{\mathcal{F},\sigma,e_i}$ is a cocycle, so is n, i.e. $\delta(n) = 0$. Thus all the conditions of Lemma 6.32 are satisfied, and therefore n is a non self-intersecting edge-path. □

Observe that the image of σ-Strand in T_G is precisely the curve constructed in the end of the Example 6.50. It was claimed that it contains all the information about connectivity of Γ_σ. We demonstrate this with the following proposition:

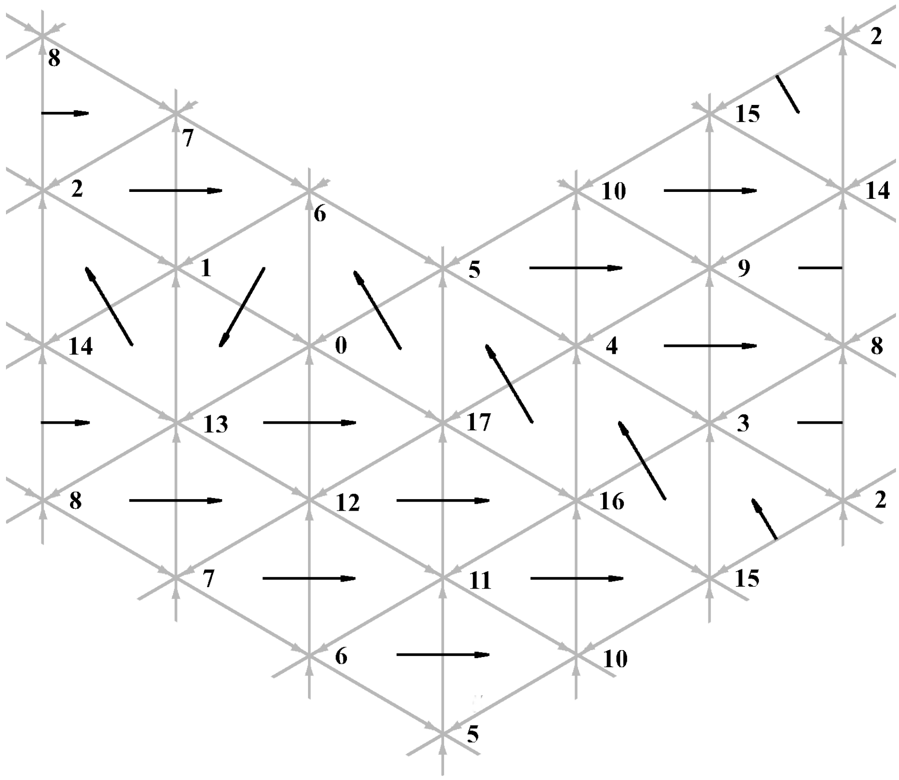
Proposition 6.55. *Let $\mathcal{F}$ be a gnat-family across Y and $\sigma = \langle e_i, e_j \rangle$ be a two-dimensional cone in the fan of Y. Then Γ_σ is connected if and only if $\mathcal{T}^\vee \setminus$ σ-Strand is connected.*

Proof. Firstly, observe that $\mathcal{T}^\vee \setminus \sigma$-Strand is connected if and only if its dual, $\mathcal{T} \setminus \sigma\text{-Strand}^\vee$ is. Note that, since σ-Strand is defined as a support of a 1-cochain, it contains no 2-cells. Hence $\mathcal{T} \setminus \sigma\text{-Strand}^\vee$ contains full 0-skeleton of $\mathcal{T}$. Therefore $\mathcal{T} \setminus \sigma\text{-Strand}^\vee$ is connected if and only if all its 0-cells are in the same connectedness class. Same is true for Γ_σ, which contains the full 0-skeleton of $\mathcal{T}$.

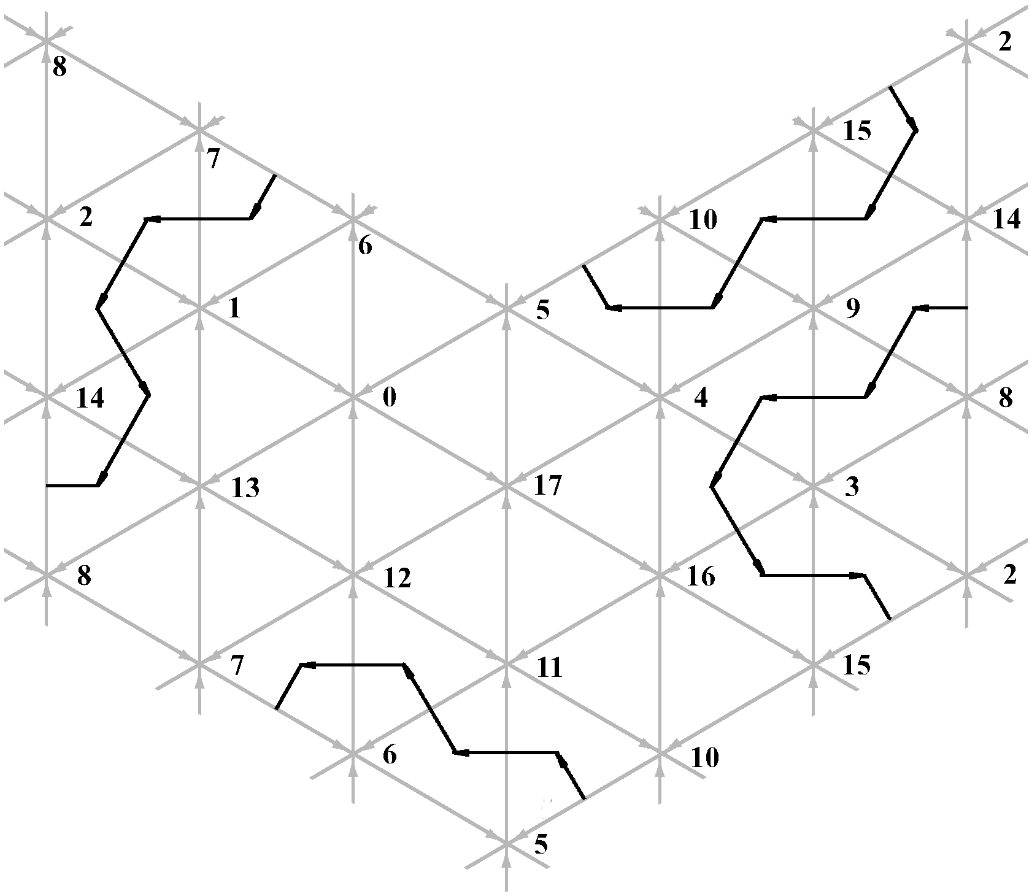
Observe that Γ_σ is a subset of 1-skeleton of $\mathcal{T} \setminus \sigma\text{-Strand}^\vee$, and Corollary 6.52 implies that its complement consists precisely of those arrows q, for which $D(q)$ is both e_i and e_j-diamond. Let q be such arrow. We shall demonstrate that, in Γ_σ, hq is still connected to tq, and therefore the connectivity relation on 0-cells in Γ_σ is the same as in $\mathcal{T}_1 \setminus \sigma\text{-Strand}^\vee$. Indeed, $D(q)$ is an e_i and e_j diamond, therefore edges in $\sigma D(q)$ all belong to Γ_σ and connect hq to tq. $\square$

Example 6.56. Let the setup be as in Example 6.39. Let σ be the cone $\langle e_{10}, e_4 \rangle$ in the fan of Y. We shall now calculate $S_{\mathcal{F},\sigma,e_{10}}$ and the corresponding σ-Strand.

Repeating the calculations in Examples 6.39 and 6.49, we obtain $d_{\mathcal{F},e_4}$ to be:



Since $S_{\mathcal{F},\sigma,e_{10}} = d_{\mathcal{F},e_{10}} - d_{\mathcal{F},e_4}$, we can see that the diagram of its image in T_G is a combination of the diagram of $d_{\mathcal{F},e_{10}}$ in the Example 6.49 and the diagram of $d_{\mathcal{F},e_4}$ above, but with arrows of the latter reversed, and with same arrows in the opposite direction cancelling each other out:



We can ask, what is the cohomology class of $S_{\mathcal{F},\sigma,e_{10}}$ in L'?

Proposition 6.57. *Let $\mathcal{F}$ be a gnat-family across Y and $\sigma = \langle e_i, e_j\rangle$ be a two-dimensional cone in the fan of Y. Then, writing $[S_{\mathcal{F},\sigma,e_i}]$ for the cohomology class of $S_{\mathcal{F},\sigma,e_i}$ in L', we have*

$$[S_{\mathcal{F},\sigma,e_i}] = e_i - e_j$$

Proof. We see from (6.40) that $S_{\mathcal{F},\sigma,e_i}$ and $\nu(e_i - e_j)$ differ by a co-boundary in $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Q})$. Hence $[S_{\mathcal{F},\sigma,e_i}] = [\nu(e_i - e_j)] = e_i - e_j$, as required. □

Proposition 6.58. *Let $\mathcal{F}$ be a gnat-family across Y and $\sigma = \langle e_i, e_j\rangle$ be a two-dimensional cone in the fan of Y. Let $s_1, \ldots, s_k$ be the disjoint connected components of $S_{\mathcal{F},\sigma,e_i}$. Then for any s_i, its cohomology class is either 0 or $\pm(e_i - e_j)$.*

Proof. For any $1 \le i < j \le k$, we know that s_i and s_j are disjoint. Therefore for any $T \in \mathcal{T}_2$, the dual 0-cell $T^\vee$ lies in at most one of $\tilde{s}_i$ or $\tilde{s}_j$. Therefore, for any $T \in \mathcal{T}_2$, one of $\psi(s_i)(T)$ or $\psi(s_j)(T)$ is zero. By Lemma 6.46, this implies $[s_i] \wedge [s_j] = 0$.

Hence, there exists $f_1 \in L'$ such that each $[s_i] = \lambda_i f_1$, for some integer λ_i.

Then as $\sum [s_i] = [S_{\mathcal{F},\sigma,e_i}]$, Proposition 6.57 shows that $e_i - e_j = (\sum \lambda_i) f_1$. But, since Y is smooth, in its fan Σ each 3-cone σ is basic, i.e. the generators of σ are a basis for L. Consequently, since 2-cone $\langle e_i, e_j \rangle$ is a face of a basic 3-cone in Σ, $e_i - e_j$ is not a scalar multiple of any element of L'. This establishes $f_1 = \pm(e_i - e_j)$.

Finally, by Corollary 6.54, each s_i is a closed, non self-intersecting edgepath, and hence Lemma 6.33 implies that each $[s_i]$ is either zero or primitive in L'. Therefore each λ_i is 0 or ± 1, as required. □

Corollary 6.59. *$S_{\mathcal{F},\sigma,e_i}$ decomposes into a sum of disjoint subcycles a and c, such that a is connected, and $[c] = 0$.*

Proof. Let $s_1, \dots, s_k$ be connected components of $S_{\mathcal{F},\sigma,e_i}$. Proposition 6.58 implies that for each $l \in 1, \dots, k$, $[s_l]$ is either 0 or $\pm(e_i - e_j)$. Therefore, there exists $l \in 1, \dots, k$ such that $[s_l] = e_i - e_j$, as

$$\sum_{m=1}^{k} [s_m] = [S_{\mathcal{F},\sigma,e_i}] = (e_i - e_j)$$

Set $a = s_l$ and $c = \sum_{m \neq l} s_m$, and observe that $[c] = [S_{\mathcal{F},\sigma,e_i}] - [s_l] = 0$ as required. □

The reason we concern ourselves with components of σ-Strand, whose cohomology class is zero, is that we can actually contract them. That is to say, we can produce a new family $\mathcal{F}'$, whose σ-Strand would be that of $\mathcal{F}$ minus any given contractible component.

Proposition 6.60. *Let $\mathcal{F} = \oplus \mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j \rangle$ be a two dimensional cone in the fan of Y. Let c be any subcycle of $S_{\mathcal{F},\sigma,e_i}$, such that its cohomology class $[c]$ is zero.*

Then there exists a family $\mathcal{F}' = \oplus \mathcal{L}(-D'_\chi)$ such that

$$S_{\mathcal{F}',\sigma,e_i} = S_{\mathcal{F},\sigma,e_i} - c \tag{6.41}$$

Furthermore, $\mathcal{F}'$ can be chosen in either such a way that $D'_\chi - D_\chi$ is always effective (or always anti-effective), or in such a way that $D'_\chi - D_\chi$ is always an integer multiple of E_i (or always an integer multiple of E_j).

Finally, if $\mathcal{F}$ was normalised family, then we can have $\mathcal{F}'$ also to be normalised.

Proof. If $[c] = 0$, there exists a cochain $s \in \mathrm{Hom}(\mathcal{C}_0^{\mathcal{T}}, \mathbb{Z})$ such that $\delta(s) = c$. Let $s = \sum \lambda_\chi e_\chi^\vee$. Without loss of generality, we can assume that $\lambda_{\chi_0} = 0$, since $(-\lambda_{\chi_0}) \sum_{\chi \in G^\vee} e_\chi^\vee$ is a cocycle, and so adding it to s doesn't change δs.

We claim that the set of G-Weil divisors defined by

$$D'_{\chi^{-1}} = D_{\chi^{-1}} + \lambda_\chi E_i \tag{6.42}$$

satisfies reductor condition (5.5), and therefore $\mathcal{F}' = \oplus \mathcal{L}(-D'_\chi)$ is a gnat-family across Y.

Indeed, observe first that (5.5) is equivalent to divisor of zeroes $B_{\mathcal{F}',q}$ being effective for every arrow q of the McKay quiver. And defining equation (6.12) of $B_{\mathcal{F},q}$, together with (6.42), implies that

$$B_{\mathcal{F}',q} = B_{\mathcal{F},q} - (\lambda_{hq} - \lambda_{tq}) E_i \tag{6.43}$$

Since $B_{\mathcal{F},q}$ is effective, we immediately see that, if $(\lambda_{hq} - \lambda_{tq}) \leq 0$, $B_{\mathcal{F}',q}$ is also effective. Suppose $(\lambda_{hq} - \lambda_{tq}) > 0$. Since $c = \delta(s)$, the coefficient with which $e_q^\vee$ appears in c is $(\lambda_{hq} - \lambda_{tq})$. As c is a subcycle of $S_{\mathcal{F},\sigma,e_i}$, we must have $(\lambda_{hq} - \lambda_{tq}) \leq S_{\mathcal{F},\sigma,e_i}(e_q)$. Since multiplicity of any 1-cell in $S_{\mathcal{F},\sigma,e_i}$ is 0 or 1, $(\lambda_{hq} - \lambda_{tq}) > 0$ implies $(\lambda_{hq} - \lambda_{tq}) = S_{\mathcal{F},\sigma,e_i}(e_q) = 1$. By Lemma 6.52, $S_{\mathcal{F},\sigma,e_i}(e_q) = 1$ implies $B_{\mathcal{F},q}|_{A_\sigma} = E_i$, i.e. coefficient of E_i in $B_{\mathcal{F},q}$ is 1. Substituting $(\lambda_{hq} - \lambda_{tq}) = 1$ into (6.43) we obtain $B_{\mathcal{F}',q} = B_{\mathcal{F},q} - E_i$. Since $B_{\mathcal{F},q}$ is effective and the coefficient of E_i in it equals to 1, $B_{\mathcal{F}',q}$ is also effective. Thus $B_{\mathcal{F}',q}$ is effective for all arrows q in the McKay quiver, as required.

This demonstrates that we can choose $\mathcal{F}'$ so that $D'_\chi - D_\chi$ is always an integer multiple of E_i. To make it always be an integer multiple of E_j, set

$$D'_{\chi^{-1}} = D_{\chi^{-1}} - \lambda_\chi E_j \tag{6.44}$$

This time, we obtain

$$B_{\mathcal{F}',q} = B_{\mathcal{F},q} + (\lambda_{hq} - \lambda_{tq})E_j \tag{6.45}$$

Only now it is the case of $(\lambda_{hq} - \lambda_{tq}) = -1$, that we have to check. We use Lemma 6.52 again, which yields $B_{\mathcal{F},q}|_{A_\sigma} = E_j$, and therefore $B_{\mathcal{F}',q}$ is still effective.

To get $D'_\chi - D_\chi$ to always be effective, we set

$$D'_{\chi^{-1}} = \begin{cases} D_{\chi^{-1}} + \lambda_\chi E_i & \text{if } \lambda_\chi \geq 0 \\ D_{\chi^{-1}} - \lambda_\chi E_j & \text{if } \lambda_\chi < 0 \end{cases} \tag{6.46}$$

Since $(\lambda_{hq} - \lambda_{tq})$ is either 0 or ± 1, λ_{hq} and λ_{tq} must both be non-negative or both be non-positive. This, together with (6.46), implies that, for each arrow q, at least one of (6.43) or (6.45) holds. And that, as we saw already, implies that $B_{\mathcal{F}',q}$ is effective for all q, as required.

Case of $D'_\chi - D_\chi$ being always anti-effective is dealt with identically, by setting:

$$D'_{\chi^{-1}} = \begin{cases} D_{\chi^{-1}} - \lambda_\chi E_j & \text{if } \lambda_\chi \geq 0 \\ D_{\chi^{-1}} + \lambda_\chi E_i & \text{if } \lambda_\chi < 0 \end{cases} \tag{6.47}$$

To show (6.41) we recall Definition 6.47, which implies that $\mu_{\mathcal{F},e_i}$ maps each e_χ to the coefficient of E_i in $D_{\chi^{-1}}$. In the view of that, (6.42), (6.44), (6.46) and (6.47) each imply that

$$\mu_{\mathcal{F}',e_i} - \mu_{\mathcal{F}',e_j} = \mu_{\mathcal{F},e_i} - \mu_{\mathcal{F},e_j} + s$$

and therefore

$$\begin{aligned} S_{\mathcal{F}',\sigma,e_i} &= d_{\mathcal{F}',e_i} - d_{\mathcal{F}',e_j} \\ &= (\nu(e_i) - \nu(e_j)) - \delta(\mu_{\mathcal{F},e_i} + s - \mu_{\mathcal{F},e_j}) \\ &= S_{\mathcal{F},\sigma,e_i} - c \end{aligned}$$

as $\delta(s) = c$.

Note that since $\lambda_{\chi_0} = 0$, we have $D'_{\chi_0} = D_{\chi_0}$ and thus if $\mathcal{F}$ was normalised, then so is $\mathcal{F}'$. $\square$

Thus, by modifying family $\mathcal{F}$, we can get rid of all the contractible components of σ-Strand, so that only a single connected component, whose cohomology class is $(e_i - e_j)$, remains. And as we shall now see, if σ-Strand is connected then so is $\mathcal{T}^\vee \setminus \sigma$-Strand:

Proposition 6.61. *Let $\mathcal{F}$ be a gnat-family across Y and $\sigma = \langle e_i, e_j \rangle$ be a two-dimensional cone in the fan of Y. Then the number of connected components of* σ-Strand *equals to the number of connected components in* $\mathcal{T}^\vee \setminus \sigma$-Strand.

Proof. We argue by passing to the curve, defined by σ-Strand on the torus T_G, which is a geometrical realisation of the dual cell complex $\mathcal{T}^\vee$.

More generally, let S be any non self-intersecting closed curve on T_G, with a non-zero homology class. By Propositions 6.54 and 6.57 σ-Strand satisfies these conditions.

Let $\dot{S}$ be an open neighbourhood of S, which retracts onto S. Consider the long exact sequence for relative homology:

$$0 \to H_2(T_G) \xrightarrow{\delta_1} H_2(T_G, \dot{S}) \xrightarrow{\delta_2} H_1(\dot{S}) \xrightarrow{\delta_3} H_1(T_G) \to \dots \tag{6.48}$$

Since T_G is connected and orientable, δ_1 is an injection. By a duality theorem ([Hat01], Theorem 3.46) we have $H_2(T_G, \dot{S}) \simeq H^0(T_G - \dot{S})$. Thus $H_2(T_G, \dot{S})$ is a free $\mathbb{Z}$-module, whose rank is the total number of connected components in $T_G - S$.

Since each connected component of S is homeomorphic to a circle, $H_1(\dot{S})$ is a free $\mathbb{Z}$-module, whose rank is the total number of connected components of S. The long exact sequence (6.48) yields

$$\operatorname{rk} H_2(T_G, \dot{S}) - 1 = \operatorname{rk} H_1(\dot{S}) - \operatorname{rk} \operatorname{Im} \delta_3$$

Since the homology class of S in T_G is non-zero, $\operatorname{rk} \operatorname{Im} \delta_3 > 0$. On the other hand, since S is non self-intersecting, the homology classes, in $H_1(T_G)$, of its connected components are all scalar multiples of each other and so $\operatorname{rk} \operatorname{Im} \delta_3 = 1$. This implies that $\operatorname{rk} H_2(T_G, \dot{S}) = \operatorname{rk} H_1(\dot{S})$, as required. $\square$

Corollary 6.62. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j\rangle$ be a two-dimensional cone in the fan of Y.*

There exists a one-step modification of $\mathcal{F}$, which produces a gnat-family $\mathcal{F}' = \oplus\mathcal{L}(-D'_\chi)$ such that $\mathcal{F}'|_{A_\sigma}$ is a simple family. Moreover, it can be ensured that $D'_\chi \geq D_\chi$ for all $\chi \in G^\vee$.

Proof. First, we apply Corollary 6.59 to identify a contractible component C of σ-Strand, whose complement is a single connected component. Then apply Proposition 6.60 to contract C away, obtaining a family $\mathcal{F}'$, whose σ-Strand is connected. Therefore, restriction of $\mathcal{F}'$ to A_σ is simple by Propositions 6.61 and 6.55. Observe that Proposition 6.60 allows us to ensure that $D'_\chi \geq D_\chi$ for all $\chi \in G^\vee$. □

So far we have shown that, for any given codimension 2 orbit S_σ, there exists a gnat-family whose restriction to S_σ is simple. But there is no apriori reason for there to exist a gnat-family, which is simultaneously simple across all codimension 2 orbits. Modifying a family to get rid of contractible component of σ-Strand, may create a contractible component in σ'-Strand for some other two-dimensional cone σ'.

Proposition 6.63. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j\rangle$ be a two-dimensional cone in the fan of Y.*

Then there exists an algorithm, which keeps modifying the family $\mathcal{F}$ in such a way, that eventually it produces a family $\mathcal{F}'$ which is simple across the whole of Y.

Proof. The algorithm runs as follows: if there exists a three-dimensional cone σ in the fan of Y, such that $\mathcal{F}|_{A_\sigma}$ is not simple, then we can apply Corollary 6.62 to modify $\mathcal{F}$ so that new family $\oplus\mathcal{L}(-D'_\chi)$ is simple restricted to A_σ and so that $D'_\chi \geq D'_\chi$ for all $\chi \in G^\vee$. Observe that there also must exist $\chi \in G^\vee$ such that $D'_\chi > D_\chi$, as $\oplus\mathcal{L}(-D'_\chi)$ and $\oplus\mathcal{L}(-D_\chi)$ can not be the same family. Thus, at every step of the algorithm, we strictly increase one of the divisors D_χ in $\mathcal{F}$.

Now recall that, by Proposition 5.20, there exists a set of maximal shift G-Weil divisors M_χ (see Definition 5.18), such that for any normalised gnat-family $\oplus\mathcal{L}(-D'_\chi)$

$$D_\chi \leq M_\chi$$

Therefore the algorithm must eventually terminate, producing a family $\mathcal{F}'$, such that $\mathcal{F}'|_{A_\sigma}$ is simple for every three-dimensional cone σ in the fan of Y. $\square$

Observe that, in particular, the family $\oplus(-M_\chi)$ itself has to be a terminal point for the algorithm in Proposition 6.63.

Corollary 6.64. *Let $\mathcal{F}_{max}$ be maximal shift family $\oplus\mathcal{L}(-M_\chi)$ on Y. Then $\mathcal{F}|_{A_\sigma}$ is a simple family.*

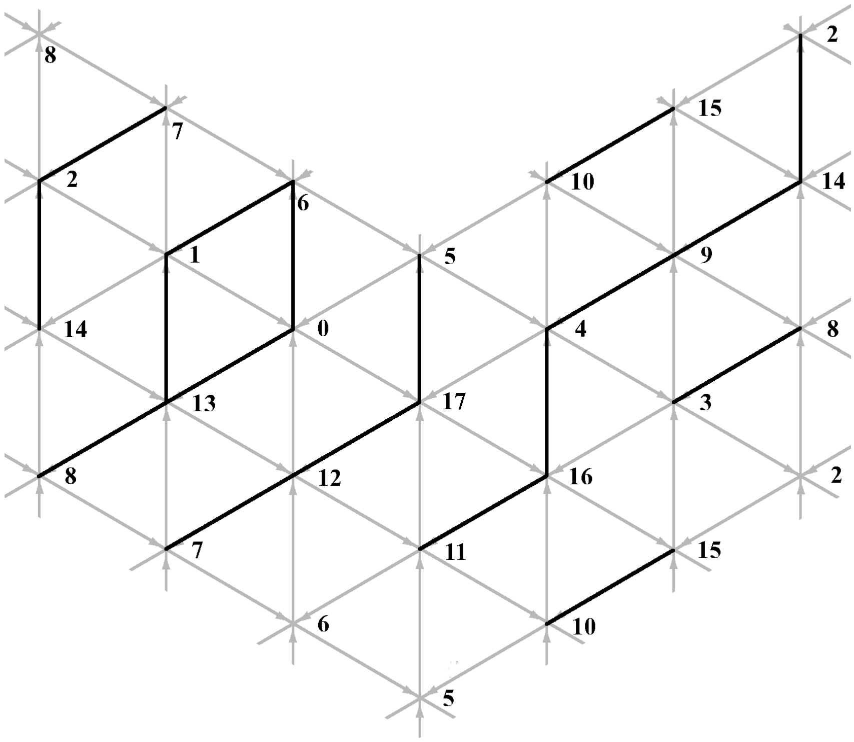
6.10 Codimension 3 Orbits

For each three-dimensional cone $\sigma = \langle e_i, e_j, e_k\rangle$ in the fan Σ of Y, where $e_i, e_j, e_k \in \mathfrak{E}$, we have a codimension 3 torus orbit S_σ, consisting of a single toric fixed point, which is the intersection of the exceptional divisors E_i, E_j and E_k.

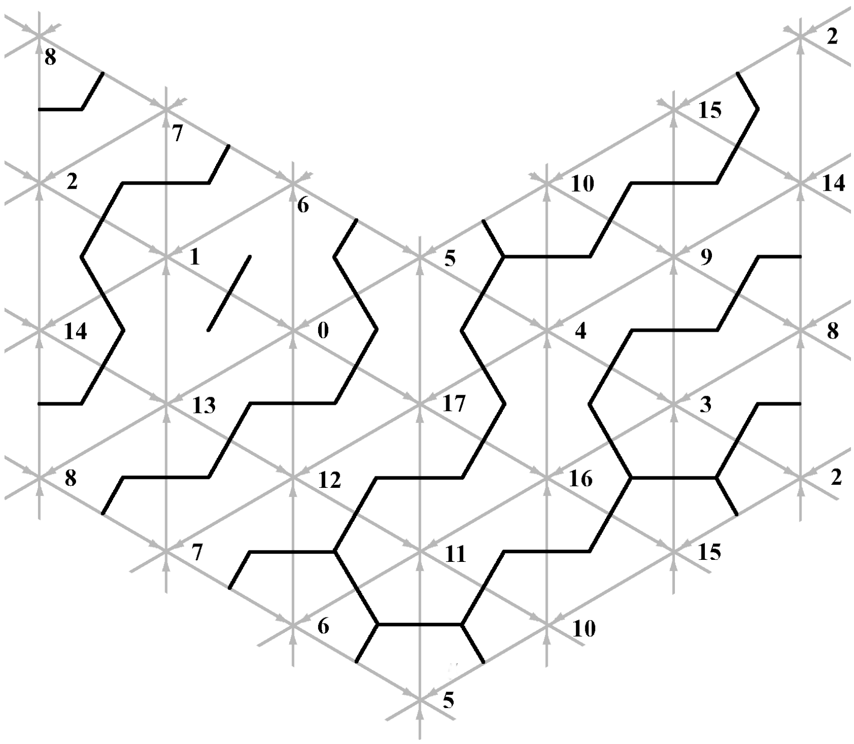
Unfortunately, as the following example demonstrates, it is perfectly possible for a gnat-family to be simple everywhere but at these toric fixed points of Y and yet fail to be simple at one of them:

Example 6.65. Let the general setup be as in Example 6.17. We shall take our family $\mathcal{F}$ to be the maximal shift family $\mathcal{F}_{\max}$ and calculate Γ_σ for $\sigma = \langle e_1, e_4, e_{10}\rangle$.

Using the method demonstrated in Example 6.39, we first compute B_q for each arrow q of the McKay quiver. Then we look at each restriction $B_q|_{A_\sigma}$ as, according to Proposition 6.12, $q \in \Gamma_\sigma$ if and only if $B_q|_{A_\sigma} = 0$. In the end, we obtain Γ_σ to be:



Instead of Γ_σ, we could consider the complex $\mathcal{N} = \mathcal{T}^\vee \setminus \Gamma_\sigma^\vee$ and check whether it disconnects $\mathcal{T}^\vee$. In our case, $\mathcal{N}$ looks like:

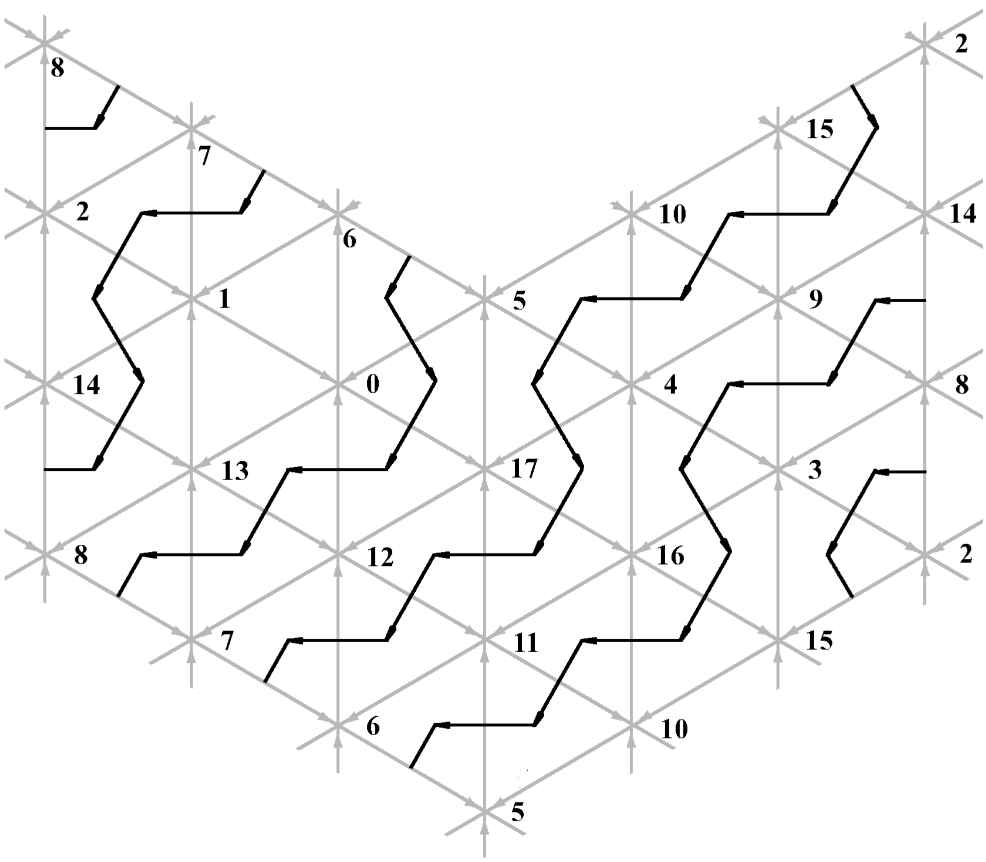


Either way, observe that we have three disjoint regions:

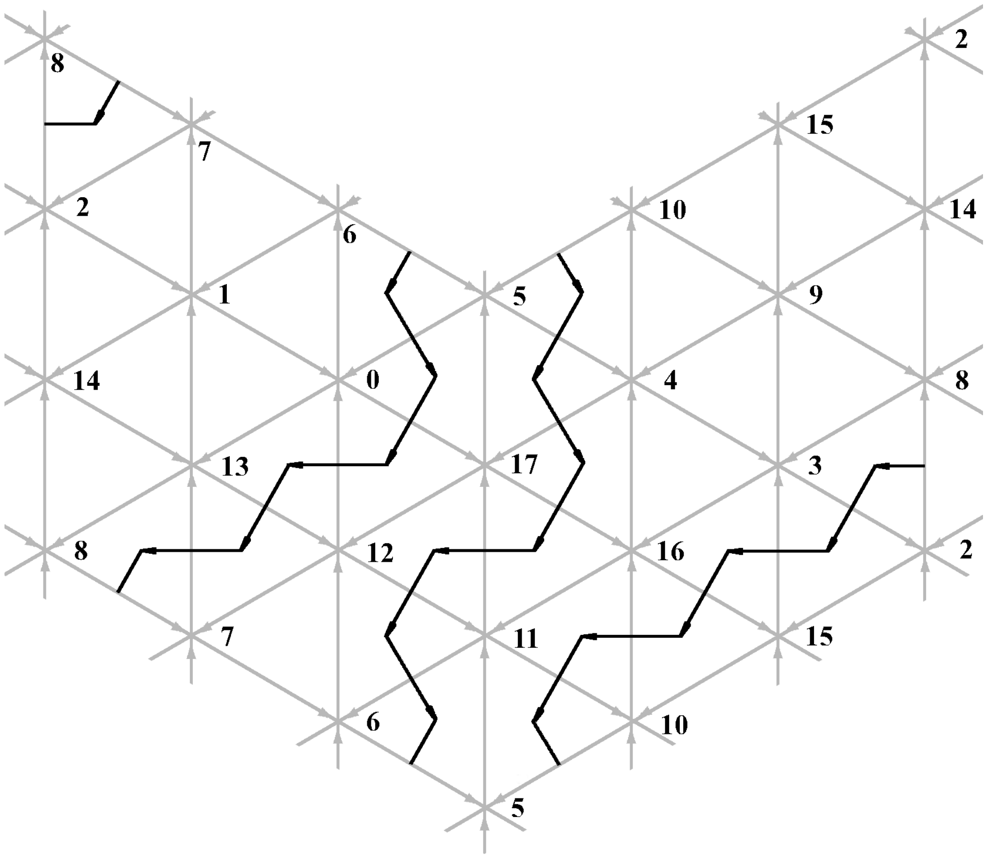
$$\begin{aligned} R_1 : &\quad \{\chi_0, \chi_1, \chi_6, \chi_{13}, \chi_8, \chi_3\} \\ R_2 : &\quad \{\chi_{10}, \chi_{15}\} \\ R_3 : &\quad \{\chi_{11}, \chi_{16}, \chi_4, \chi_9, \chi_{14}, \chi_2, \chi_7, \chi_{12}, \chi_{17}, \chi_5\} \end{aligned}$$

If we denote by V the G-constellation parametrised in $\mathcal{F}_{\max}$ by toric fixed point S_σ, then $\oplus_{\chi \in R_j} V_\chi$, for $j = 1, 2, 3$, are the three simple summands of V.

However, we know from Corollary 6.13, Proposition 6.35 and Corollary 6.64, that $\Gamma_{\mathcal{F}_{\max},\sigma}$ is connected for all the codimension 0, 1 and 2 orbits S_σ. Therefore, $\mathcal{F}_{\max}$ is simple everywhere, away from the toric fixed points of Y. In particular, the strands for the three 2-dimensional faces of σ consist each of a single connected component and do not disconnect T_G. We have already seen the diagram for $S_{\mathcal{F},\langle e_4, e_{10}\rangle, e_{10}}$, when we calculated it in the Example 6.56. If we repeat this calculation for the other two faces of σ, we obtain $S_{\mathcal{F},\langle e_4, e_1\rangle, e_1}$ to be:



and $S_{\mathcal{F},\langle e_{10},e_1\rangle,e_1}$ to be:

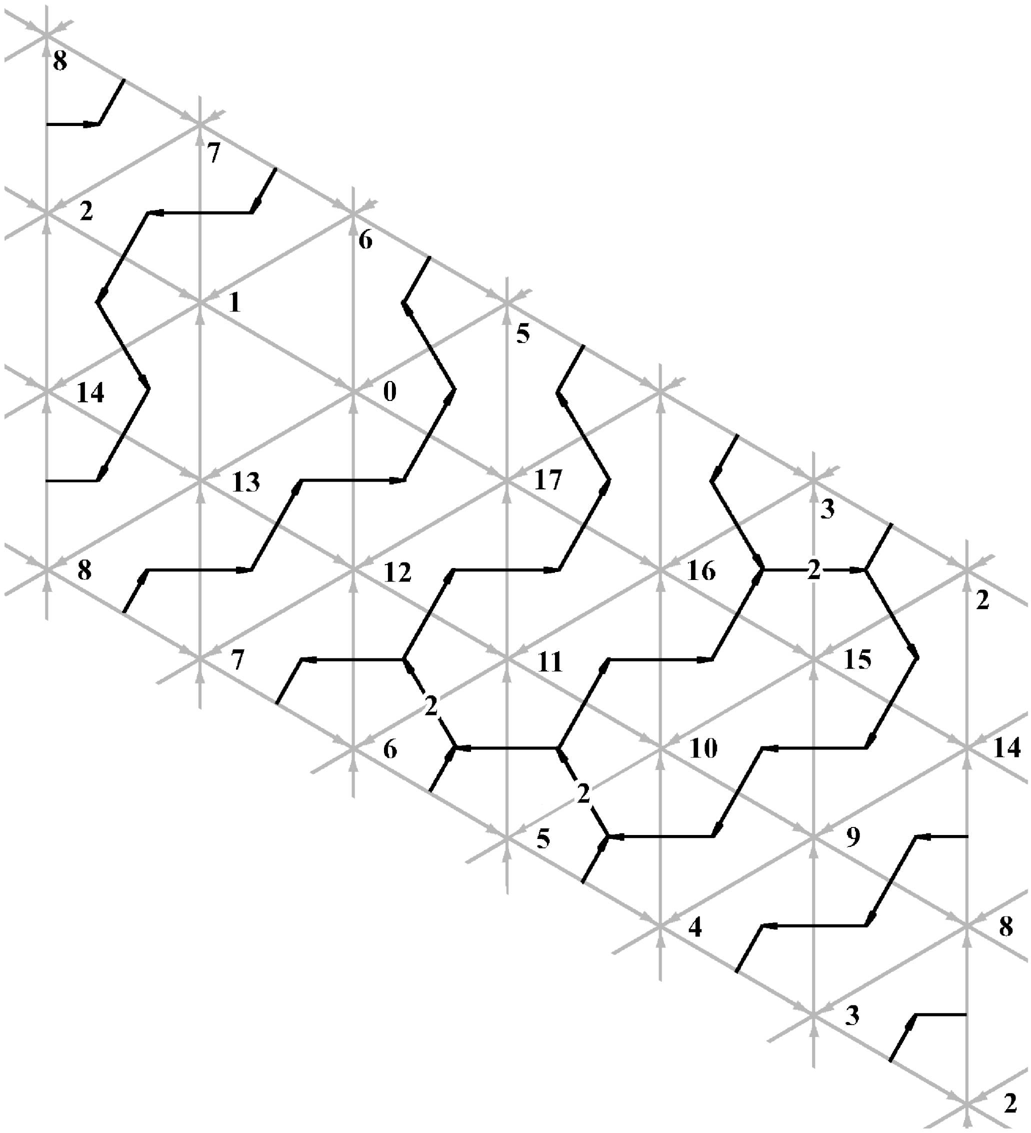


Observe that taking a union of strands for any pair of faces of σ yields subcomplex $\mathcal{N}$ of $T^\vee$, defined above, minus an isolated edge $(\chi_1, 1)^\vee$, an edge for which the corresponding diamond $D(\chi_1, 1)$ is simultaneously an e_1, e_4 and e_{10}-diamond. And removal of such an isolated edge, similar to what we've seen in the proof of Proposition 6.55, doesn't affect connectedness of the complement of $\mathcal{N}$.

We see that the problem lies in the strands of the two-dimensional faces of σ intersecting each other and together disconnecting a region of $\mathcal{T}^\vee$. Indeed, take the region R_2 above and consider $\delta(\chi_{10}^\vee + \chi_{15}^\vee)$: a 1-cocycle, whose support in $T^\vee$ encloses R_2.

We see that neither this cocycle, nor its inverse, are a subchain of either $S_{\mathcal{F},\langle e_{10},e_4\rangle,e_{10}}$, $S_{\mathcal{F},\langle e_1,e_4\rangle,e_1}$ or $S_{\mathcal{F},\langle e_1,e_{10}\rangle,e_1}$. Therefore we can not apply Proposition 6.60 to get rid of it.

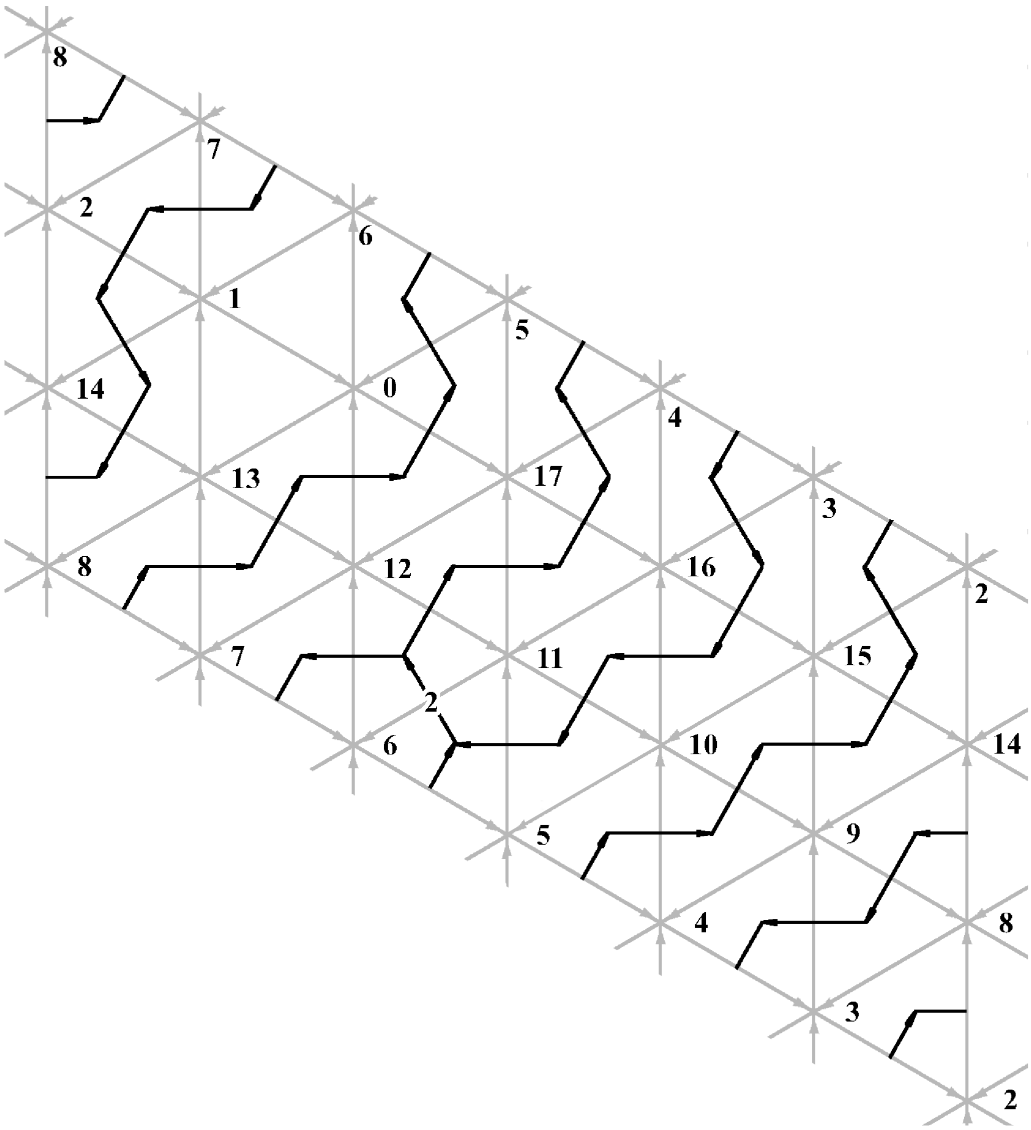
However, let us consider cocycle $S_{\mathcal{F},\langle e_{10},e_4\rangle,e_{10}} + S_{\mathcal{F},\langle e_{10},e_1\rangle,e_{10}}$, i.e. $2d_{\mathcal{F},e_{10}} - d_{\mathcal{F},e_1} - d_{\mathcal{F},e_4}$:



Recall that we mark by '2' those 1-cells $q^\vee$, which have multiplicity 2 in the cocycle depicted.

Observe that support of $S_{\mathcal{F},\langle e_{10},e_4\rangle,e_{10}} + S_{\mathcal{F},\langle e_{10},e_1\rangle,e_{10}}$ is precisely the union of the three strands of the two-dimensional faces of σ.

Furthermore, observe that $\delta(-\chi_{10}^\vee - \chi_{15}^\vee)$ is a subcycle of $2d_{\mathcal{F},e_{10}} - d_{\mathcal{F},e_1} - d_{\mathcal{F},e_4}$. Analogy with Proposition 6.60 suggests that there might be a way to modify our family in a way, which would remove this subcycle. And indeed, observe that in a family $\mathcal{F}'$, obtained from $\mathcal{F}$ by subtracting E_{10} from $D_{\chi_{15}}$ and $D_{\chi_{10}}$, $2d_{\mathcal{F},e_{10}} - d_{\mathcal{F},e_1} - d_{\mathcal{F}e_4}$ looks like:



Observe that its support no longer disconnects $T^\vee$, and one can verify that the new family $\mathcal{F}'$ is simple at the toric fixed point S_σ.

We therefore make the following definition:

Definition 6.66. Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j, e_k \rangle$ be a three-dimensional cone in the fan of Y.

We define cochain $N_{\mathcal{F},\sigma,e_i}$ in $\mathrm{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Z})$ by:

$$N_{\mathcal{F},\sigma,e_i} = 2d_{\mathcal{F},e_i} - d_{\mathcal{F},e_j} - d_{\mathcal{F},e_k} \tag{6.49}$$

We define σ-Necklace to be the support $\tilde{N}_{\mathcal{F},\sigma,e_i}$ of $N_{\mathcal{F},\sigma,e_i}$.

Lemma 6.67. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j, e_k\rangle$ be a three-dimensional cone in the fan of Y. Let q be an arrow in the McKay quiver. Let μ be the coefficient of $e_q^\vee$ in $N_{\mathcal{F},\sigma,e_i}$.*

Then μ is depends on the restriction of the divisor of zeroes B_q to A_σ in the following fashion:

$B_q\vert_{A_\sigma}$	μ
E_i	2
$E_i + E_j$ *or* $E_i + E_k$	1
0 *or* $E_i + E_j + E_k$	0
E_j *or* E_k	-1
$E_j + E_k$	-2

Proof. The coefficients of E_i, E_j and E_k in B_q are all either 0 or 1, as demonstrated by (6.28). This gives eight possibilities for $B_q|_{A_\sigma}$. By Lemma 6.9, the coefficient of $e_q^\vee$ in $d_{\mathcal{F},e_i}$ is precisely the coefficient of E_i in B_q, and so for each of the eight possibilities we can use (6.49) to compute μ, giving the table above. □

Similar to the case of codimension 2 orbit, the family being simple at S_σ is equivalent to σ-Necklace not disconnecting $\mathcal{T}^\vee$:

Proposition 6.68. *Let $\mathcal{F}$ be a gnat-family across Y and $\sigma = \langle e_i, e_j, e_k\rangle$ be a three-dimensional cone in the fan of Y. Then Γ_σ is connected if and only if $\mathcal{T}^\vee \setminus \sigma$-Necklace is connected.*

Proof. We repeat the steps of the proof of 6.55. As before $\mathcal{T}^\vee \setminus \sigma$-Necklace is connected if and only if $\mathcal{T} \setminus \sigma\text{-Necklace}^\vee$ is connected. And $\mathcal{T} \setminus \sigma\text{-Necklace}^\vee$ and Γ_σ are, each, connected if and only if all their 0-cells are in the same connectivity class.

We again see that Γ_σ lies within 1-skeleton of $\mathcal{T} \setminus \sigma\text{-Necklace}^\vee$, and its complement consists of 1-cells $q^\vee$ for which $D(q)$ is an e_i, e_j and e_k-diamond.

To show that removal of these 1-cells from $\mathcal{T} \setminus \sigma$-Necklace$^\vee$ doesn't affect connectivity relation on the 0-cells, we, again, observe that given cell q, such that $B_q|_{A_\sigma} = E_i + E_j + E_k$, all the 1-cells in the boundary of $D(q)$ necessarily, by Proposition 6.34, lie in Γ_σ, connecting hq to tq. $\square$

For a codimension 2 orbit S_σ, given a subcycle s of $S_{\mathcal{F},\sigma,e_i}$ whose cohomology class is 0, we were able to modify family $\mathcal{F}$ in such a way as to reduce the σ-Strand by the support of s. We now show that the same is also true for codimension 3 orbit S_σ and σ-Necklace, albeit with certain restrictions:

Proposition 6.69. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j, e_k\rangle$ be a three-dimensional cone in the fan of Y. Let c be any subcycle of $N_{\mathcal{F},\sigma,e_i}$, such that its cohomology class $[c]$ is zero, and such that multiplicity of any 1-cell in c is no greater than 1.*

Then there exists a family $\mathcal{F}' = \oplus\mathcal{L}(-D'_\chi)$ such that

$$N_{\mathcal{F}',\sigma,e_i} = N_{\mathcal{F},\sigma,e_i} - 2c \tag{6.50}$$

$$S_{\mathcal{F}',\langle e_i,e_j\rangle,e_i} = S_{\mathcal{F},\langle e_i,e_j\rangle,e_i} - c \tag{6.51}$$

$$S_{\mathcal{F}',\langle e_i,e_k\rangle,e_i} = S_{\mathcal{F},\langle e_i,e_k\rangle,e_i} - c \tag{6.52}$$

$$S_{\mathcal{F}',\langle e_j,e_k\rangle,e_i} = S_{\mathcal{F},\langle e_j,e_k\rangle,e_i} \tag{6.53}$$

If $\mathcal{F}$ was normalised family, then we can have $\mathcal{F}'$ also to be normalised.

Proof. The proof follows along the lines of the proof of Proposition 6.60. As before, $[c] = 0$ implies existence of a cochain $s \in \mathrm{Hom}(\mathcal{C}_0^{\mathcal{T}}, \mathbb{Z})$ such that $\delta(s) = c$. Let $s = \sum \lambda_\chi e_\chi^\vee$.

We claim that the set of G-Weil divisors defined by

$$D'_{\chi^{-1}} = D_{\chi^{-1}} + \lambda_\chi E_i \tag{6.54}$$

satisfies reductor condition (5.5), and therefore $\mathcal{F}' = \oplus\mathcal{L}(-D'_\chi)$ is a gnat-family across Y.

We again have

$$B_{\mathcal{F}',q} = B_{\mathcal{F},q} - (\lambda_{hq} - \lambda_{tq})E_i \tag{6.55}$$

As before, if $(\lambda_{hq} - \lambda_{tq})$ is non-positive, then effectiveness of $B_{\mathcal{F}',q}$ is immediate, as $B_{\mathcal{F},q}$ is effective. Since $c = \sigma(s)$, $(\lambda_{hq} - \lambda_{tq})$ is the coefficient with which $e_q^\vee$ appears in c. This coefficient is, by assumption, 0 or ± 1.

Assume q is an arrow such that $(\lambda_{hq} - \lambda_{tq})$ equals to 1. Assumption that c is a subcycle of $N_{\mathcal{F},\sigma,e_i}$ yields that $N_{\mathcal{F},\sigma,e_i}(e_q) \geq 1$. Therefore, by Lemma 6.67, coefficient of E_i in $B_{\mathcal{F},q}$ is positive, therefore, as $B_{\mathcal{F}',q} = B_{\mathcal{F},q} - E_i$, we see that $B_{\mathcal{F}',q}$ is effective, as required.

Showing that (6.50) hold and that if $\mathcal{F}$ was normalised, we can ensure $\mathcal{F}'$ would also be normalised, is identical to the way it is done in the proof of Proposition 6.60. $\square$

The result appears to not be as satisfactory as that of Proposition 6.60, since instead of removing a contractible subcycle of $N_{\mathcal{F},\sigma,e_i}$ we appear to be replacing it by its inverse. Indeed, if a 1-cell $q^\vee$ has multiplicity 1 both in c and $N_{\mathcal{F},\sigma,e_i}$, then in $N_{\mathcal{F}',\sigma,e_i}$ it will still have multiplicity 1, but its orientation will change to the opposite. But observe that if $q^\vee$ has multiplicity 1 in c, but multiplicity 2 in $N_{\mathcal{F},\sigma,e_i}$, then the contraction procedure in Proposition 6.69 shall remove it from $N_{\mathcal{F},\sigma,e_i}$ entirely. Since we do not add any new 1-cells to $N_{\mathcal{F},\sigma,e_i}$, we see that the number of 1-cells in σ-Necklace will decrease. Recall the Example 6.65: there we saw this procedure removing entirely the edges $(\chi_{15}, 3)^\vee$ and $(\chi_{10}, 2)^\vee$ from $N_{\mathcal{F}_{\max},\sigma,e_{10}}$, connecting up the region R_2 with both the other connected components of $\mathcal{T}^\vee$.

We, therefore, set out to study the structure of N_{σ,e_i} and specifically its subchains of multiplicity 2.

Definition 6.70. We define the multiplicity 2 component of $N_{\mathcal{F},\sigma,e_i}$, denoted $N^2_{\mathcal{F},\sigma,e_i}$, to be the maximal co-chain $n \in \operatorname{Hom}(\mathcal{C}_1^{\mathcal{T}}, \mathbb{Z})$ such that $2n$ is a subchain of $N_{\mathcal{F},\sigma,e_i}$.

Lemma 6.71. *Let $q^\vee$ be a 1-cell of $\mathcal{T}^\vee$. Let μ denote the coefficient of $e_q^\vee$ in $N^2_{\mathcal{F},\sigma,e_i}$, then it depends on $B_q|_{A_\sigma}$ in the following fashion:*

$B_q\|_{A_\sigma}$	μ
E_i	1
$E_j + E_k$	-1
Any other	0

Proof. We consider eight possible values of $B_q|_{A_\sigma}$, and then apply Lemma 6.67 to establish coefficient μ' of $e_q^\vee$ in $N_{\mathcal{F},\sigma,e_i}$. Then, by Definition 6.70, μ is a half of μ', rounded down. □

Lemma 6.72. *Let $T^\vee$ be a 0-cell of $\mathcal{T}^\vee$. Then it belongs to the support of $N^2_{\mathcal{F},\sigma,e_i}$ if and only if restrictions to A_σ of divisors of zeroes of sides of T are either E_i, E_j and E_k, or E_i, $E_j + E_k$ and 0.*

Proof. $T^\vee$ lies in $N^2_{\mathcal{F},\sigma,e_i}$, if and only if, for one of the sides q of T, $q^\vee$ lies in $N^2_{\mathcal{F},\sigma,e_i}$. Let q be such a side, then by, Lemma 6.71, $B_q|_{A_\sigma} = E_i$ or $B_q|_{A_\sigma} = E_j + E_k$. By Proposition 6.34, if $B_q|_{A_\sigma} = E_i$, then the restrictions to A_σ of divisors of zeroes of the other two sides of T must be E_j and E_k or $E_j + E_k$ and 0. Similarly, if $B_q|A_\sigma = E_j + E_k$, then these restrictions must be E_i and 0. □

Lemma 6.73. *Let co-chain $n \in \mathrm{Hom}(\mathbb{C}_1^{\mathcal{T}}, \mathbb{Z})$ be a connected component of $N^2_{\mathcal{F},\sigma,e_i}$. Then it is an non self-intersecting edge-path. Let $T^\vee$ be a 0-cell of $\mathcal{T}^\vee$, which lies in the support of n. Then:*

1. *$T^\vee$ is a startpoint of n if and only if T is a 'minus' triangle and divisors of zeroes of the sides of T restrict to A_σ as E_i, E_j and E_k.*
2. *$T^\vee$ is an endpoint of n if and only if T is a 'plus' triangle and divisors of zeroes of the sides of T restrict to A_σ as E_i, E_j and E_k.*
3. *$T^\vee$ is an internal point of n if and only if divisors of zeroes of the sides of T restrict to A_σ as E_i, $E_j + E_k$ and 0.*

Proof. By Lemma 6.71, the multiplicity of any 1-cell in n is no greater than 1.

Take any 0-cell $T(\chi, \sigma)^\vee$, which lies in the support of n. Denote by q_1, q_2 and q_3 the sides of triangle $T(\chi, \sigma)$. By Lemma 6.72, the restrictions of B_{q_1}, B_{q_2} and B_{q_3} to A_σ are either E_i, E_j and E_k, or E_i, $E_j + E_k$ and 0.

First, assume, without loss of generality, that $B_{q_1}|_{A_\sigma} = E_i$, $B_{q_2}|_{A_\sigma} = E_j$ and $B_{q_3}|_{A_\sigma} = E_k$. Then by Lemma 6.71, we see that $n(e_{q_1}) = 1$, $n(e_{q_2}) = 0$, and $n(e_{q_3}) = 0$. Therefore, there is exactly one 1-cell of n attached to $T(\chi, \sigma)^\vee$. Furthermore, by equation (6.20) in the definition of boundary map δ for $\mathcal{C}^{\mathcal{T}}$, we have:

$$\delta n(e_{T(\chi,\sigma)}) = n(\delta e_{T(\chi,\sigma)}) = \epsilon(\sigma) n(e_{q_1} + e_{q_2} + e_{q_3}) = \epsilon(\sigma) \tag{6.56}$$

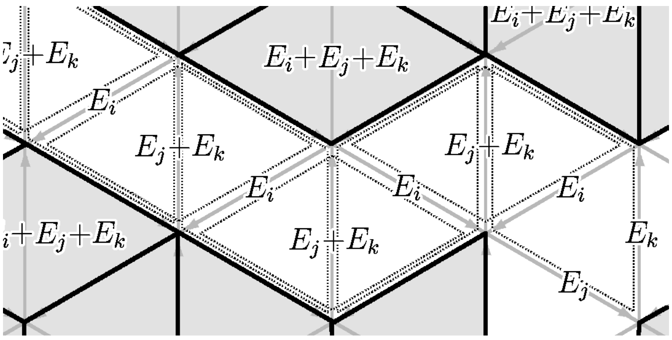
Now, assume that $B_{q_1}|_{A_\sigma} = E_i$, $B_{q_2}|_{A_\sigma} = E_j + E_k$ and $B_{q_3}|_{A_\sigma} = 0$. Then Lemma 6.71 yields $n(e_{q_1}) = 1$, $n(e_{q_2}) = -1$, and $n(e_{q_3}) = 0$. Therefore, there are exactly two 1-cells of n attached to $T(\chi, \sigma)^\vee$, and

$$\delta n(e_{T(\chi,\sigma)}) = n(\delta e_{T(\chi,\sigma)}) = \epsilon(\sigma) n(e_{q_1} + e_{q_2} + e_{q_3}) = 0 \tag{6.57}$$

Since choice of 0-cell $T(\chi, \sigma)^\vee$ in the support of n was arbitrary, this demonstrated that n satisfies all the conditions in Lemma 6.32 and thus n is a non self-intersecting edgepath.

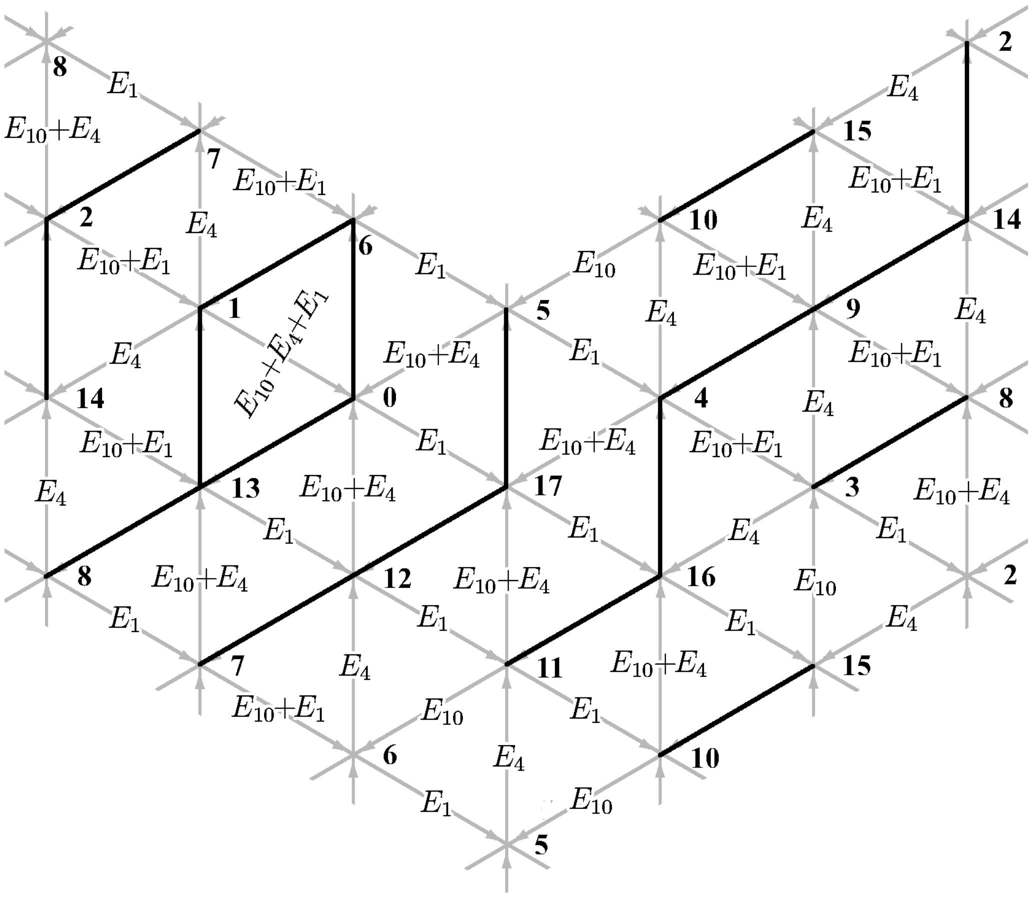
For the second part of the claim, let $T^\vee$ be any 0-cell in the support of n. If the restrictions to A_σ of divisors of zeroes of sides of T are E_i, $E_j + E_k$ and 0, then (6.57) demonstrates (see Definition 6.31) that $T^\vee$ is an internal point of n. And if the restrictions are E_i, E_j and E_k, then (6.56) demonstrates that if T is a 'minus' triangle (resp. a 'plus' triangle), then $T^\vee$ is the startpoint (resp. the endpoint) of n. This proves the claim. □

Example 6.74. Similar to the case of σ-Strand in the Example 6.50, we can visualise a connected component n of N^2_{σ,e_i} as a sequence interlocked e_i-diamonds and (e_j, e_k)-diamonds in $\mathcal{T}$, through the centres of which runs the edgepath n:



Observe that the sequence consists of triangles, divisors of zeroes of whose sides restrict to A_σ as E_i, $E_j + E_k$ and 0 and terminates with a triangle, divisors of zeroes of whose sides restrict to A_σ as E_i, E_j and E_k, illustrating Lemma 6.73. Note that, as on the figure this triangle is a 'minus' triangle, Lemma 6.73 tells us that it is a startpoint of the edgepath n.

For a concrete example, let the setup be as in the Example 6.65, with σ again denoting $\langle e_{10}, e_1, e_4 \rangle$, and recall the diagram of $N_{\mathcal{F}_{\max},\sigma,e_{10}}$. On the diagram below, we mark each arrow q of McKay quiver by restriction of $B_{\mathcal{F}_{\max},q}$ to A_σ, marking in bold the arrows where this restriction is zero:



Observe that there are 6 triangles, divisors of zeroes of whose sides restrict to A_σ as E_{10}, E_4 and E_1:

$$\text{'minus' triangles:} \qquad T(\chi_6, 132),\ T(\chi_5, 132) \text{ and } T(\chi_{16}, 132) \tag{6.58}$$

$$\text{'plus' triangles:} \qquad T(\chi_{11}, 123),\ T(\chi_3, 123) \text{ and } T(\chi_{12}, 123) \tag{6.59}$$

Observe that $N^2_{\mathcal{F}_{\max},\sigma,e_{10}}$ splits, as in Lemma 6.73, into three edgepaths, which join up the triangles of (6.58) to triangles of (6.59). Writing them down as formal edgepaths, and denoting each oriented cell (q, o_q) by o_q for brevity, we obtain:

$$\begin{aligned}
T(\chi_6, 132) \to T(\chi_{12}, 123): &\quad \{+e^\vee_{\chi_{11},2}\}\\
T(\chi_{16}, 132) \to T(\chi_3, 123): &\quad \{+e^\vee_{\chi_{15},3}\}\\
T(\chi_5, 132) \to T(\chi_{11}, 123): &\quad \{+e^\vee_{\chi_{10},2}\}
\end{aligned}$$

Similarly, the three edgepaths composing $N^2_{\mathcal{F}_{\max},\sigma,e_1}$ are:

$$\begin{aligned}
&T(\chi_6, 132) \to T(\chi_3, 123):\\
&\quad \{+e^\vee_{\chi_6,1}, -e^\vee_{\chi_5,2}, +e^\vee_{\chi_0,1}, -e^\vee_{\chi_{12},3}, +e^\vee_{\chi_{13},1}, -e^\vee_{\chi_7,3}, +e^\vee_{\chi_8,1}, -e^\vee_{\chi_2,3}, +e^\vee_{\chi_3,1}\}\\
&T(\chi_{16}, 132) \to T(\chi_{11}, 123): \quad \{+e^\vee_{\chi_{16},1}, -e^\vee_{\chi_{10},3}, +e^\vee_{\chi_{11},2}\}\\
&T(\chi_5, 132) \to T(\chi_{12}, 123): \quad \{+e^\vee_{\chi_5,1}, -e^\vee_{\chi_4,2}, +e^\vee_{\chi_{17},1}, -e^\vee_{\chi_{11},3}, +e^\vee_{\chi_{12},1}\}
\end{aligned}$$

Finally, the three edge-paths composing $N^2_{\mathcal{F}_{\max},\sigma,e_4}$ are:

$$\begin{aligned}
&T(\chi_6, 132) \to T(\chi_{11}, 123): \quad \{+e^\vee_{\chi_5,3}\}\\
&T(\chi_{16}, 132) \to T(\chi_{12}, 123):\\
&\quad \{+e^\vee_{\chi_6,3}, -e^\vee_{\chi_7,1}, +e^\vee_{\chi_1,3}, -e^\vee_{\chi_2,1}, +e^\vee_{\chi_1,2}, -e^\vee_{\chi_{14},1}, +e^\vee_{\chi_8,3}, -e^\vee_{\chi_9,1}, +e^\vee_{\chi_3,3}, -e^\vee_{\chi_4,1}, +e^\vee_{\chi_3,2}\}\\
&T(\chi_5, 132) \to T(\chi_3, 123): \quad \{+e^\vee_{\chi_4,3}, -e^\vee_{\chi_{10},1}, +e^\vee_{\chi_9,3}, -e^\vee_{\chi_{15},1}, +e^\vee_{\chi_2,2}\}
\end{aligned}$$

Observe that σ-Necklace is a union of supports of $N^2_{\sigma,e_{10}}$, N^2_{σ,e_1} and N^2_{σ,e_4}. Thus we can think of σ-Necklace as a graph with 6 vertices, which are the triangles listed in (6.58) and (6.59), and 9 edges, which are the 3 edge-paths of $N^2_{\sigma,e_{10}}$, the 3 edge-paths of N^2_{σ,e_1} and the 3 edge-paths of N^2_{σ,e_4}.

Finally, observe that there is another way to divide the six triangles, listed in (6.58) and (6.59), into two groups. For each of them, if we take first the side q such that $B_q|_{A_\sigma} = E_{10}$, then such that $B_q|_{A_\sigma} = E_4$ and then such that $B_q|_{A_\sigma} = E_1$, we shall follow around the boundary of the triangle in one of the two possible directions. On the diagram above, we see that in cases of $T(\chi_{12}, 123)$, $T(\chi_{16}, 132)$, $T(\chi_{11}, 123)$ and $T(\chi_5, 132)$ this direction is 'clockwise', while in cases of $T(\chi_3, 123)$ and $T(\chi_{16}, 132)$ it is 'anti-clockwise'.

Definition 6.75. Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and σ be a three-dimensional cone in the fan of Y. Let n be a 1-cochain in $\mathrm{Hom}(\mathcal{C}^{\mathcal{T}}_1, \mathbb{Z})$.

Given a generator $e_i \in \sigma$, we say that n is a (σ, e_i)-`link` if it is a connected component of $N^2_{\mathcal{F},\sigma,e_i}$. We say that n is an σ-link if it is a (σ, e_i)-link for one of the generators $e_i \in \sigma$.

Definition 6.76. Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j, e_k\rangle$ be a three-dimensional cone in the fan of Y. Let $T^\vee$ be a 0-cell of $\mathcal{T}^\vee$. Let T be a dual 2-cell in $\mathcal{T}$ and let q_1, q_2 and q_3 be its x_1, x_2 and x_3-oriented sides.

We say that $T^\vee$ is a σ-Clasp if there exists a bijection $\theta : \{1, 2, 3\} \to \{i, j, k\}$ such that

$$B_{q_a}|_{A_\sigma} = E_{\theta(a)} \quad \forall\, a \in \{1, 2, 3\} \tag{6.60}$$

We say that $T^\vee$ is a `regular` σ-Clasp if the ordered triple of points $e_{\theta(1)}$, $e_{\theta(2)}$, $e_{\theta(3)}$ have the same orientation as the ordered triple e_1, e_2 and e_3. In other words, if $(e_{\theta(1)} - e_{\theta(2)}) \wedge (e_{\theta(1)} - e_{\theta(3)})$ is a positive multiple of $(e_1 - e_2) \wedge (e_1 - e_3)$. Otherwise, we say that $T^\vee$ is a `reversed` σ-Clasp.

Example 6.77. Let us return to the observation made in the end of Example 6.74. Let $T^\vee$ be one of the σ-Clasps listed in (6.58) and (6.59), let q_1, q_2 and q_3 be its x_1, x_2 and x_3-oriented, sides and let θ be the bijection satisfying (6.60). We observed that if we follow the sides of T in order $q_{\theta^{-1}(10)}$, $q_{\theta^{-1}(4)}$, $q_{\theta^{-1}(1)}$, then, if T is one of $T(\chi_{12}, 123)$, $T(\chi_{16}, 132)$, $T(\chi_{11}, 123)$ or $T(\chi_5, 132)$, we shall go around the boundary of T 'clockwise', while, if T is one of $T(\chi_3, 123)$ or $T(\chi_{16}, 132)$, we shall go around it 'anti-clockwise'.

Observe that ordering sides as q_1, q_2, q_3 goes around the boundary of every T 'clockwise'. Therefore the ordering $q_{\theta^{-1}(10)}$, $q_{\theta^{-1}(4)}$, $q_{\theta^{-1}(1)}$ goes around the boundary of T 'clockwise', if and only if the ordered triple of points e_1, e_2, e_3 defines the same orientation of plane L' as the ordered triple $e_{\theta^{-1}(10)}$, $e_{\theta^{-1}(4)}$, $e_{\theta^{-1}(1)}$. And the latter happens if and only if the ordered triple $e_{\theta(1)}$, $e_{\theta(2)}$, $e_{\theta(3)}$ defines the same orientation of L' as the ordered triple e_{10}, e_4, e_1.

On the other hand, by looking at the diagram in the Example 6.17, we see that e_{10}, e_4 and e_1 define the same, 'anti-clockwise', orientation of L' as e_1, e_2, e_3.

Thus we see that $T(\chi_{12}, 123)$, $T(\chi_{16}, 132)$, $T(\chi_{11}, 123)$ and $T(\chi_5, 132)$ are regular σ-Clasps, while $T(\chi_3, 123)$ and $T(\chi_{16}, 132)$ are the reversed ones.

Recall the local approximation map ψ, introduced in Definition 6.44. Let $e_i \in \mathfrak{E}$, and recall that e_i-diamond cochain d_{e_i}, introduced in Definition 6.48, maps e_q to 1, if $E_i \in B_q$, and to 0, otherwise.

Now take any triangle T in $\mathcal{T}$ and let q_1, q_2 and q_3 be its x_1, x_2 and x_3-oriented sides. Then, for any $a \in \{1,2,3\}$, the element $\psi(d_{e_i})(T)$ of L' maps $[x_a]$ to 1, if $E_i \in B_{q_a}$, and to 0, otherwise. Proposition 6.34 implies that the divisor of zeroes of precisely one of the sides of T contains E_i. Let this side be q_b, for some $b \in \{1,2,3\}$. Then $\psi(d_{e_i})(T)$ is the vertex e_b of the junior simplex Δ.

We conclude that $\psi(d_{e_\bullet})(T)$ defines a map from $\mathfrak{E}$ to $\{1,2,3\}$ such that for any $e_i \in \mathfrak{E}$ and $a \in \{1,2,3\}$

$$\psi(d_{e_i})(T) = e_a \text{ if and only if } E_i \in B_{q_a} \tag{6.61}$$

This allows for a following algebraic criterion, which determines whether a 1-cell $T^\vee$ is an σ-Clasp:

Proposition 6.78. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j, e_k\rangle$ be a three-dimensional cone in the fan of Y. Let $T^\vee$ be a 0-cell of $\mathcal{T}^\vee$. Then one of the following always holds:*

1. *$T^\vee$ is not an σ-*Clasp *and*

$$\psi(d_{e_i} - d_{e_j})(T) \wedge \psi(d_{e_i} - d_{e_k})(T) = 0 \tag{6.62}$$

2. *$T^\vee$ is a regular σ-*Clasp *and*

$$\psi(d_{e_i} - d_{e_j})(T) \wedge \psi(d_{e_i} - d_{e_k})(T) = +|G|(e_i - e_j) \wedge (e_i - e_k) \tag{6.63}$$

3. *$T^\vee$ is a reversed σ-*Clasp *and*

$$\psi(d_{e_i} - d_{e_j})(T) \wedge \psi(d_{e_i} - d_{e_k})(T) = -|G|(e_i - e_j) \wedge (e_i - e_k) \tag{6.64}$$

Proof. Observe the map $\psi(d_{e_\bullet})(T)$ from $\{i,j,k\}$ to $\{1,2,3\}$ is a bijection if and only if $T^\vee$ is an σ-Clasp. This is because if the requisite bijection θ from $\{1,2,3\}$ to $\{i,j,k\}$ defined by (6.60) exists, then, by (6.61), it has to be the inverse of $\psi(d_{e_\bullet})(T)$.

On the other hand, clearly $\psi(d_{e_\bullet})(T)$ is a bijection if and only if

$$\psi(d_{e_i} - d_{e_j})(T) \wedge \psi(d_{e_i} - d_{e_k})(T) \neq 0$$

Suppose $T^\vee$ is an σ-Clasp. Then observe that

$$(\psi(d_{e_i})(T) - \psi(d_{e_j})(T)) \wedge (\psi(d_{e_i})(T) - \psi(d_{e_j})(T))$$

is a basis for $\Lambda^2((\mathbb{Z}^3)^\vee \cap (1,1,1)^\perp)$.

Since $(e_i - e_j) \wedge (e_i - e_k)$ is a basis of $\Lambda^2(L \cap (1,1,1)^\perp)$, we have:

$$(\psi(d_{e_i})(T) - \psi(d_{e_j})(T)) \wedge (\psi(d_{e_i})(T) - \psi(d_{e_j}))(T) = \pm |L : (\mathbb{Z}^3)^\vee| \, (e_i - e_j) \wedge (e_i - e_k)$$

And since, as seen in Section 4.1, $L/(\mathbb{Z}^3)^\vee$ is isomorphic to G, we see that $|L : (\mathbb{Z}^3)^\vee|$, the index of $(\mathbb{Z}^3)^\vee$ in L, must be equal to $|G|$. The result follows. $\square$

One particular consequence of Proposition 6.78 is that, when using Lemma 6.46 to calculate the cup product of cohomology classes of $d_{e_i} - d_{e_j}$ and $d_{e_i} - d_{e_k}$, only σ-Clasps contribute to the answer.

Proposition 6.79. *Let $\mathcal{F} = \oplus \mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j, e_k \rangle$ be a three-dimensional cone in the fan of Y.*

Then

$$(\#\ \textit{of regular}\ \sigma\text{-Clasps}) = (\#\ \textit{of reversed}\ \sigma\text{-Clasps}) + 2$$

Proof. Proposition 6.57 tells us that the cohomology classes of $d_{e_i} - d_{e_j}$ and $d_{e_i} - d_{e_k}$ are $e_i - e_j$ and $e_i - e_k$, respectively. Then Proposition 6.46 tells us that their cup product is an average of wedge product of their local approximations across $\mathcal{T}_2$, that is:

$$\sum_{T \in \mathcal{T}_2} \psi(d_{e_i} - d_{e_j})(T) \wedge \psi(d_{e_i} - d_{e_k})(T) = 2|G|(e_i - e_j) \wedge (e_i - e_k)$$

Proposition 6.78 tells us that only σ-Clasps contribute a non-zero term to the sum in LHS. It further says that each regular σ-Clasp contributes $+|G|(e_i - e_j) \wedge (e_i - e_k)$, while each reversed σ-Clasp contributes $-|G|(e_i - e_j) \wedge (e_i - e_k)$. Thus, we see that there must be two more regular σ-Clasps than reversed σ-Clasps. $\square$

Recall how in Example 6.74 we saw that σ-Necklace was union of σ-Clasps and supports of σ-links, with the only cells, belonging to a support of more then one σ-link, being σ-Clasps.

We now prove that this is always the case:

Lemma 6.80. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and σ be a three-dimensional cone in the fan of Y. Then*

1. σ-Necklace *is a disjoint union of all* σ-Clasp*s and interiors of supports of all* σ-link*s.*

2. *For any two-dimensional face σ' of σ, σ'-Strand is a disjoint union of all* σ-Clasp*s and interiors of supports of all (σ, e_i)-links, for $e_i \in \sigma'$.*

Proof. Let $q^\vee$ be a 1-cell, which lies in σ-Necklace. Then Lemma 6.67 lists all six possible values of $B_q|_{A_\sigma}$, and applying Lemma 6.71 for each of them, shows that $q^\vee$ lies in $N^2_{\mathcal{F},\sigma,e_i}$ for precisely one $e_i \in \sigma$. E.g., if $B_q|_{A_\sigma} = E_i$, then $q^\vee \in N^2_{\mathcal{F},\sigma,e_i}$, and if $B_q|_{A_\sigma} = E_i + E_j$, then $q^\vee \in N^2_{\mathcal{F},\sigma,e_k}$. Thus every 1-cell of σ-Necklace belongs to an unique σ-link.

Let $T^\vee$ be a 0-cell, which belongs to σ-Necklace. We must show that it is either an σ-Clasp or an internal point of an unique σ-link. Since every 1-cell of σ-Necklace belongs to some σ-link n, so must $T^\vee$. Then, by Lemma 6.73, $T^\vee$ is either an σ-Clasp or an interior point of n.

If $T^\vee$ is an interior point of n, assume, without loss of generality, that n is an (σ, e_i)-link. Since all (σ, e_i) are, by definition, disjoint, it remains to show that $T^\vee$ doesn't belong to any (σ, e_j) or (σ, e_k)-link.

Denote by q_1, q_2 and q_3 the sides of triangle T. Since T is an interior point of a (σ, e_i)-link, Lemma 6.73 says that the restrictions of B_{q_1}, B_{q_2} and B_{q_2} to A_σ are, in some order, E_i, $E_j + E_k$ and 0. Then, by Lemma 6.71, none of the three 1-cells $q_1^\vee$, $q_2^\vee$ or $q_3^\vee$, attached to $T^\vee$, lie in a (σ, e_j)-link or in a (σ, e_j)-link. Therefore $T^\vee$ doesn't either, as required.

This shows 1. The proof of 2 is similar. We first use Lemmas 6.52 and 6.71 to establish that any 1-cell in $\langle e_i, e_j\rangle$-Strand belongs to an unique (σ, e_i) or (σ, e_j)-link, and then proceed as before. $\square$

Proposition 6.81. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j, e_k\rangle$ be a three-dimensional cone in the fan of Y. Assume, further, that* σ-Necklace *is connected.*

The number of connected components in $\mathcal{T}^\vee \setminus \sigma$-Necklace *is half the number of* σ-Clasp*s.*

Proof. We argue by passing to T_G, a geometrical realisation of $\mathcal{T}^\vee$. Let $\dot{N}$ be an open neighbourhood which retracts onto the image of σ-Necklace in T_G. Consider the long exact sequence for relative homology:

$$\begin{array}{ccccccc} 0 & \rightarrow & H_2(T_G) & \rightarrow & H_2(T_G, \dot{N}) & \rightarrow & \\ H_1(\dot{N}) & \xrightarrow{\delta_1} & H_1(T_G) & \rightarrow & H_1(T_G, \dot{N}) & \rightarrow & \\ H_0(\dot{N}) & \xrightarrow{\delta_2} & H_0(T_G) & \rightarrow & H_0(T_G, \dot{N}) & \rightarrow & 0 \end{array}$$

As both $\dot{N}$ and T_G are connected, δ_2 is an isomorphism. Since $\dot{N}$ contains the images of σ'-Strands for the three two-dimensional faces σ' of σ, it supports a basis for $H_1(T_G)$ and hence δ_1 is a surjection. Consequently, $H_1(T_G, \dot{N}) = H_0(T_G, \dot{N}) = 0$.

Let n_R denote the number of the connected components of $T^\vee \setminus \sigma$-Necklace. By a duality theorem ([Hat01], Theorem 3.46) $H_2(T_G, \dot{N})$ is isomorphic to $H^0(T_G - \dot{N})$. Therefore $\operatorname{rk} H_2(T_G, \dot{N}) = n_R$.

Therefore the long exact sequence yields:

$$\chi(N) - \chi(T_G) + n_R = 0 \tag{6.65}$$

The Euler characteristic $\chi(T_G)$ of the torus T_G is 0. By ([Hat01], p. 146) the Euler characteristic $\chi(N)$ of a CW-complex N equals to $n_0 - n_1$, where n_0 and n_1 are total numbers of 0-cells and 1-cells in N, respectively. Every 1-cell in N is attached to two 0-cells. A 0-cell in N has three 1-cells attached to it, if it is an σ-Clasp, and two 1-cells attached to it otherwise. Therefore $n_1 = n_0 + \frac{1}{2}n_C$, where n_C is a total number of σ-Clasps. Thus $\chi(N) = -\frac{1}{2}n_C$.

Substituting all this into (6.65) yields

$$n_R = \frac{1}{2}n_C$$

as required. □

Thus the reversed σ-Clasps represent an obstruction to the connectedness of $\mathcal{T}^\vee \setminus \sigma$-Necklace: the latter is connected if and only if there exist exactly two

σ-Clasps, which, by Proposition 6.79, is equivalent to there not existing any reversed σ-Clasps.

We now, given a reversed σ-Clasp $T^\vee$, explicitly construct a contractible component of σ-Necklace, whose boundary $T^\vee$ lies on. Provided that σ-Strands corresponding to faces of σ-Necklace are all connected, i.e. the disconnectedness of $\mathcal{T}^\vee \setminus \sigma$-Necklace doesn't come from any single σ-Strand, but from the way they intersect each other. But, most importantly, we construct this contractible component in such a way that we can apply Proposition 6.69 to it:

Proposition 6.82. *Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y and $\sigma = \langle e_i, e_j, e_k\rangle$ be a three-dimensional cone in the fan of Y. Assume further that, for each of the three two-dimensional faces σ' of σ, σ'-Strand is connected.*

If there exists a reversed σ-Clasp, then one of $N_{\mathcal{F},\sigma,e_i}$, $N_{\mathcal{F},\sigma,e_j}$ or $N_{\mathcal{F},\sigma,e_k}$ contains subcycle c, which is a closed edge-path and which has cohomology class 0.

Proof. Proposition 6.79 implies that, if there exists a reversed σ-Clasp, then there exists a regular one. Let σ' be any two-dimensional face of σ. By Lemma 6.80, σ'-Strand is a disjoint union of all σ-Clasps and interiors of such (σ, e_a)-links, that $e_a \in \sigma'$. Since, by assumption, σ'-Strand is also connected, we see that there must exist an open σ-link n, connecting a regular σ-Clasp and a reversed one. Without a loss of generality, let n be a (σ, e_i)-link. Let $T_0^\vee$ denote the startpoint of n and $T_1^\vee$ denote the endpoint, i.e. $\delta n = e^\vee_{T_1} - e^\vee_{T_0}$.

Consider $S_{\langle e_j,e_k\rangle,e_j} = d_{e_j} - d_{e_k}$. By assumption it is connected, therefore by Corollary 6.54 it is a closed non self-intersecting edgepath. Lemma 6.80 implies that every σ-Clasp belongs to it. In particular, $T_0^\vee$ and $T_1^\vee$ do, splitting $S_{\langle e_j,e_k\rangle,e_j}$ in two open edge-paths: one that goes from $T_0^\vee$ to $T_1^\vee$ and one that goes from $T_1^\vee$ to $T_0^\vee$. Let m_1 and m_2, respectively, denote the corresponding subchains of $S_{\langle e_j,e_k\rangle,e_j}$, so that we have $S_{\langle e_j,e_k\rangle,e_j} = m_1 + m_2$, $\delta m_1 = \delta n$ and $\delta m_2 = -\delta n$. Then $m_1 - n$ and $-m_2 - n$ are closed edge-paths.

By applying Lemmas 6.71 and 6.67 for each cell $q^\vee$ in n, we see that $-N^2_{\mathcal{F},\sigma,e_i}$ is subchain of both $N_{\mathcal{F},\sigma,e_j}$ and $N_{\mathcal{F},\sigma,e_k}$. Similarly, by applying Lemmas 6.52 and 6.67, we see that $S_{\langle e_j,e_k\rangle,e_j}$ is a subchain of $N_{\mathcal{F},\sigma,e_j}$ and $-S_{\langle e_j,e_k\rangle,e_j}$ is a subchain of $N_{\mathcal{F},\sigma,e_k}$. We conclude that $m_1 - n$ is a subcycle of $N_{\mathcal{F},\sigma,e_j}$ and $-m_2 - n$ is a subcycle of $N_{\mathcal{F},\sigma,e_k}$.

Therefore, if we can show that one of the cohomology classes $[m_1 - n]$ or $[-m_2 - n]$ is zero, we are done.

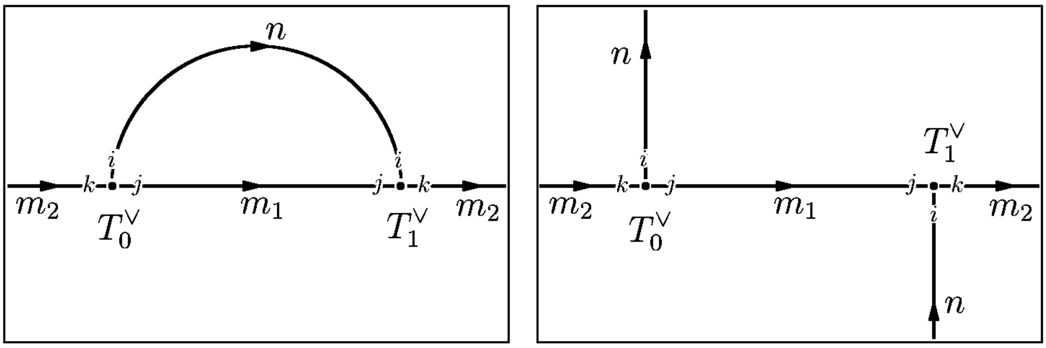
A key step is to establish that

$$[m_1 - n] \wedge [m_1 + m_2] = 0 \tag{6.66}$$

Suppose we have done that. Then $[m_1 - n] = k[m_1 + m_2]$ for some integer k. Consequently, $[-m_2 - n] = [m_1 - n] - [m_1 + m_2] = (k-1)[m_1 + m_2]$. On the other hand, both $n - m_1$ and $-n - m_2$ are closed, non self-intersecting edge-paths, and therefore, by Lemma 6.33, both k and $(1-k)$ have to be 0, $+1$ or -1. This necessitates one of them to be zero, and the proof is finished.

It remains to demonstrate (6.66). We give first a geometric proof, using intersection theory on torus T_G, then we follow the sketch with a purist algebraic proof working entirely in abstract complex $T^\vee$.

The geometric proof is as follows: we claim that since T_G is an orientable manifold, there are only two principally different configurations of n, m_1 and m_2:



The Case 1 is the situation, which we desire: $m_1 - n$ is the contractible cycle we seek. Observe, that both $m_1 - n$ and $-m_2 - n$ can clearly be homotopied away from $m_1 + m_2$, therefore the cup products $[m_1 - n] \wedge [m_1 + m_2]$ and $[-m_2 - n] \wedge [m_1 + m_2]$ are zero. On the other hand, in Case 2, neither $m_1 - n$, nor $-m_2 - n$ are contractible and $m_1 - n$ and $-m_2 - n$ are both homotopic to a curve, which intersect $m_1 + m_2$ transversally in one point, and therefore neither of the respective cup products is zero. We claim that the Case 1 configuration happens precisely when one of the two clasps $T_0^\vee$ and $T_1^\vee$ is regular and the other reversed, while Case 2 configuration happens whenever both clasps are of the same type. Indeed, observe that each of n, m_1 and m_2 is attached to each clasp by precisely one 1-cell of $\mathcal{T}_1^\vee$. On the figure we have, in each case, marked by i, j, or k whether this cell is a part of (σ, e_i), (σ, e_j) or (σ, e_k)-link, or equivalently, whether the restriction to A_σ of the divisor of zeroes of its dual in $\mathcal{T}_1$ is E_i, E_j or E_k. Recalling the discussion in the beginning of the Example 6.77, observe that, in Case 1, markings i, j and k go clockwise around $\mathcal{T}_0^\vee$ and anti-clockwise around $\mathcal{T}_1^\vee$, while in Case 2, they go clockwise around both of the clasps.

We now proceed to an algebraic proof. To calculate $[m_1 - n] \wedge [m_1 + m_2]$, we use Proposition 6.46, which tells us that it is an average of $\psi(m_1-n)(T) \wedge \psi(m_1+m_2)(T)$ across all the triangles T in $\mathcal{T}_2$. By construction, the only 0-cells in $\mathcal{T}^\vee$, which belong to more than one of n, m_1 and m_2, are $T_0^\vee$ and $T_1^\vee$. Therefore, for all $T \neq T_0$ or T_1, $\psi(m_1 - n)(T) \wedge \psi(m_1 + m_2)(T) = 0$ and thus:

$$2|G|[m_1 - n] \wedge [m_1 + m_2] = \sum_{T=T_0,T_1} \psi(m_1 - n)(T) \wedge \psi(m_1 + m_2)(T) \tag{6.67}$$

Consider a triangle T_0. Let q_i, q_j and q_k denote the sides of T_0, restrictions of whose divisors of zeroes to A_σ are E_i, E_j and E_k, respectively. By Lemma 6.71, n maps e_{q_i} to 1 and maps e_{q_j} and e_{q_k} to 0. So does d_{e_i}, and therefore $\psi(n)(T_0) = \psi(d_{e_i})(T_0)$.

On the other hand, $m_1 + m_2 = d_{e_j} - d_{e_k}$, and therefore $m_1 + m_2(e_{q_i}) = 0$, $m_1 + m_2(e_{q_j}) = +1$ and $m_1 + m_2(e_{q_k}) = -1$. Therefore, the edgepath $m_1 + m_2$ contains two oriented 1-cells attached to $T_0^\vee$: $(q_j^\vee, +e_{q_j}^\vee)$ and $(q_k^\vee, -e_{q_k}^\vee)$. By Lemma 6.73, T_0 is a 'minus' triangle, as it is a startpoint of an σ-link. Using (6.21), we have:

$$\begin{aligned}
\delta(+e_{q_j}^\vee)(e_{T_0}) &= +e_{q_j}^\vee(\delta(e_{T_0})) = +e_{q_j}^\vee(-e_{q_i} - e_{q_j} - e_{q_j}) = -1 \\
\delta(-e_{q_k}^\vee)(e_{T_0}) &= -e_{q_k}^\vee(\delta(e_{T_0})) = -e_{q_k}^\vee(-e_{q_i} - e_{q_j} - e_{q_j}) = +1
\end{aligned}$$

Therefore T_0 is an origin of $(q_j^\vee, +e_{q_j}^\vee)$ and an end of $(q_k^\vee, -e_{q_k}^\vee)$. Since m_1 is a subchain of $m_1 + m_2$, which is an open edgepath starting at $T_0^\vee$, m_1 must contain $(q_j^\vee, +e_{q_j}^\vee)$, and not $(q_k^\vee, -e_{q_k}^\vee)$. We conclude, that m_1 maps e_{q_j} to $+1$ and maps e_{q_i} and e_{q_k} to 0, and therefore $\psi(m_1)(T_0) = \psi(d_{e_j})(T_0)$. By the same reasoning, $\psi(m_2)(T_0) = \psi(-d_{e_k})(T_0)$.

Thus we have

$$\psi(m_1 - n)(T_0) \wedge \psi(m_1 + m_2)(T_0) = \psi(d_{e_j} - d_{e_i})(T_0) \wedge \psi(d_{e_j} - d_{e_k})(T_0) \tag{6.68}$$

Repeating the same calculation for T_1, we obtain that also

$$\psi(m_1 - n)(T_1) \wedge \psi(m_1 + m_2)(T_1) = \psi(d_{e_j} - d_{e_i})(T_0) \wedge \psi(d_{e_j} - d_{e_k})(T_1) \quad (6.69)$$

By assumption one of T_0 and T_1 is a regular clasp and the other is reversed. Therefore the sum of RHSs in (6.68) and (6.69) is zero, and substituting (6.68) and (6.69) into (6.67), we conclude that $[n - m_1] \wedge [m_1 + m_2] = 0$, as required. $\square$

It might be tempting to use Proposition 6.82 to locate contractible components of σ-Necklace and Proposition 6.69 to modify $\mathcal{F}$, over and over, until $\mathcal{T} \setminus \sigma$-Necklace becomes connected. However, we can only apply Proposition 6.82 whenever, for each two-dimensional face σ' of σ, σ'-Strand is connected. And there is no guarantee that applying Proposition 6.69 wouldn't disconnect one of them. We can reconnect any given σ'-Strand by using Proposition 6.63, but doing so might disconnect one of the other two strands.

However it turns out that it is possible to make modifications to $\mathcal{F}$ in such a way that the number of 1-cells in σ-Necklace strictly decreases with each modification.

The key observation is that applying Proposition 6.69 to contractible component c of $N_{\mathcal{F},\sigma,e_i}$, we add no new 1-cells to σ-Necklace, but remove from it every 1-cell $q^\vee$, whose multiplicity in c is 1 and whose multiplicity in $N_{\mathcal{F},\sigma,e_i}$ is 2.

Indeed, recalling Proof of Proposition 6.69, new family $\mathcal{F}'$ was constructed in such a way that

$$N_{\mathcal{F}',\sigma,e_i} = N_{\mathcal{F},\sigma,e_i} - 2c \quad (6.70)$$

Let $q^\vee$ be a 1-cell in $\mathcal{T}^\vee$, which doesn't lie in σ-Necklace of $\mathcal{F}$. Then $N_{\mathcal{F},\sigma,e_i}(e_q) = 0$. Since c is a subcycle of $N_{\mathcal{F},\sigma,e_i}(e_q)$, we also have $c(e_q) = 0$. Therefore $N_{\mathcal{F}',\sigma,e_i}(e_q) = 0$, and therefore $q^\vee$ doesn't lie in the σ-Necklace of $\mathcal{F}'$.

Suppose $q^\vee$ is a 1-cell, whose multiplicity in c is 1 and whose multiplicity in $N_{\mathcal{F},\sigma,e_i}$ is 2. Therefore $N_{\mathcal{F},\sigma,e_i}(e_q) = \pm 2\ c(e_q)$. Since c is a subchain of $N_{\mathcal{F},\sigma,e_i}$, orientation of $q^\vee$ must be the same in both of them (see equation (6.19)), and therefore $N_{\mathcal{F},\sigma,e_i}(e_q) = +2\ c(e_q)$. Substituting this into 6.70 yields $N_{\mathcal{F}',\sigma,e_i}(e_q) = 0$ and therefore $q^\vee$ doesn't lie in σ-Necklace of $\mathcal{F}'$.

We can now prove the main theorem of this chapter.

Theorem 6.83. *Let G be a finite, abelian subgroup of $\mathrm{SL}_3(\mathbb{C})$. Let Y be any crepant toric resolution of $X = \mathbb{C}^3/G$. Let $\mathcal{F} = \oplus\mathcal{L}(-D_\chi)$ be a gnat-family on Y. Let $\sigma = \langle e_i, e_j, e_k\rangle$ be any three-dimensional cone in the fan of Y.*

Then there exists an algorithm, which modifies $\mathcal{F}$ until it produces a new gnat-family $\mathcal{F}'$, which is simple restricted to A_σ.

Proof. Each step of the algorithm consists of applying one of the following two modifications to $\mathcal{F}$. The algorithm terminates if, at some step, we obtain a family $\mathcal{F}'$, to which neither modification can be applied.

Modification A: This can be applied if σ has a two-dimensional subface σ', such that σ'-Strand is disconnected. If this is the case, let $\sigma' = \langle e_i, e_j\rangle$, without loss of generality. Then we use Proposition 6.59 to decompose $S_{\mathcal{F},\langle e_i,e_j\rangle,e_i}$ into two disjoint subcycles a and c, such a is connected and $[c] = 0$. By Lemma 6.52, no 1-cell of $\mathcal{T}^\vee$ belongs to n with multiplicity greater than 1. Also, by applying Lemmas 6.52 and 6.67 , we see that c is a subcycle of $N_{\mathcal{F},\sigma,e_i}$, while $-c$ is a subcycle of $N_{\mathcal{F},\sigma,e_j}$.

By assumption, c is non-empty. Let $q^\vee$ be a 1-cell of c. By Lemma 6.80, $q^\vee$ has multiplicity 2 in either $N_{\mathcal{F},\sigma,e_i}$ or $N_{\mathcal{F},\sigma,e_j}$.

If $q^\vee$ has multiplicity 2 in $N_{\mathcal{F},\sigma,e_i}$, apply Proposition 6.69 to c, as a subcycle of $N_{\mathcal{F},\sigma,e_i}$, to obtain a new family $\mathcal{F}'$. The number of 1-cells in σ-Necklace of $\mathcal{F}'$ will be strictly less that the number of 1-cells in σ-Necklace of $\mathcal{F}$.

If $q^\vee$ has multiplicity 2 in $N_{\mathcal{F},\sigma,e_i}$, apply Proposition 6.69 to $-c$, as a subcycle of $N_{\mathcal{F},\sigma,e_j}$, to obtain a new family $\mathcal{F}'$. Again, the number of 1-cells in σ-Necklace of $\mathcal{F}'$ is strictly less that the number of 1-cells in σ-Necklace of $\mathcal{F}$.

Modification B: This can be applied whenever σ'-Strand is connected for all three two-dimensional faces σ' of σ and there exists reversed σ-Clasps. If this is the case, we can apply Proposition 6.82, to identify a subcycle c of, without loss of generality, $N_{\mathcal{F},\sigma,e_i}$, such that c is a closed edge-path and $[c] = 0$. Then we claim that c must contain a 1-cell $q^\vee$ of $N^2_{\mathcal{F},\sigma,e_i}$. Assume otherwise. then by 6.80 its support is a subset of $\langle e_j, e_k\rangle$-Strand. By assumption, $\langle e_j, e_k\rangle$-Strand is connected, so by Corollary 6.54, it is a support of a closed, non self-intersecting edge-path. Since c is a closed edge-path itself, the support of c must be the whole of $\langle e_j, e_k\rangle$-Strand. Then we can apply Lemma 6.33, to show that $T^\vee \setminus \langle e_j, e_k\rangle$-Strand has two connected components. Which is impossible, as then by Proposition 6.61, $\langle e_j, e_k\rangle$-Strand would also have two connected components.

Since at each step of the algorithm we decrease the number of 1-cells in σ-Necklace of $\mathcal{F}$, the process must terminate. This can happen only when at some step we produced family $\mathcal{F}'$, such that σ'-Strand is connected for all three two-dimensional faces σ' of σ and no reversed σ-Clasps exist. Then by Proposition 6.79, the total number of σ-Clasps is 2, hence by Proposition 6.81, $T^\vee \backslash \sigma$-Necklace has a single connected component. Therefore, by Proposition 6.68, $\mathcal{F}|_{A_\sigma}$ is simple $\square$

6.11 Conclusion

The goal of this chapter was to prove that for any abelian $G \subset \mathrm{SL}_3(\mathbb{C})$ and any crepant toric resolution Y of $\mathbb{C}^3/G$, there exists a (globally) simple gnat-family $\mathcal{F}$. Since every gnat-family is simple on codimension 0 and 1 orbits (Corollary 6.13 and Proposition 6.35), the problem reduces to considering codimension 2 and 3 orbits. Indeed, by Proposition 6.12, if $\mathcal{F}$ is simple on an orbit S_σ, it is simple on the affine open piece A_σ. Therefore if $\mathcal{F}$ is simple at all toric fixed points S_σ, where σ ranges across all three-dimensional cones in the fan of Y, then $\mathcal{F}$ is simple on the whole of Y.

For any codimension 2-orbit S_σ, we have found the following criterion: $\mathcal{F}$ is simple along S_σ if and only if σ-Strand is connected (Proposition 6.55). We know how to modify any gnat-family $\mathcal{F}$ to make it satisfy this criterion (Proposition 6.60). Furthermore, by making these modifications in a controlled way, we can ensure that eventually we reach a gnat-family which is simple along all the codimension 2 orbits (Proposition 6.63). In fact, we can also see that, for any G and Y, there is a particular gnat-family, the maximal shift family, which is simple everywhere in codimension 2 (Corollary 6.64).

Now, for any toric fixed point S_σ, we also have a criterion: $\mathcal{F}$ is simple at S_σ if and only if σ-Necklace is connected and there are no reversed σ-Clasps (Propositions 6.81 and 6.79). We have an algorithm which repeatedly modifies any gnat-family $\mathcal{F}$ until this criterion is satisfied at a given fixed point (Theorem 6.83). However, using this algorithm may affect the simplicity of $\mathcal{F}$ at other fixed points and we have no way to control the repeated use of this algorithm to ensure that we can make $\mathcal{F}$ simple at all fixed points simultaneously.

Thus, the existence of a (globally) simple gnat-family for an arbitrary choice of a finite abelian $G \subset \mathrm{SL}_3(\mathbb{C})$ and of a toric crepant resolution Y of $\mathbb{C}^3/G$ still remains unproven.

An alternative approach to trying to control the repeated use of the algorithm in Theorem 6.83 would be, similar to the case of the maximal shift family and codimension 2 orbits, to try to directly construct a gnat-family $\mathcal{F}$ in a way which would ensure that no reversed σ-Clasps could exist.

6.12 Non-projective example

Although we do not prove that a simple gnat-family always exists, the methods we developed provide the tools, with which, for any given G and Y, one could attempt to construct such a family or to verify that a given family is simple.

For any crepant projective resolution Y, in case finite of abelian $G \subset \mathrm{SL}_3(\mathbb{C})$, Theorem 1.1 [CI02] proves there exists θ, such that Y is a moduli space of θ-stable constellations. Then the pushdown to Y of the universal θ-stable G-constellation on $Y \times \mathbb{C}^3$ can be shown to be an orthonormal gnat-family. In particular, this shows that for each crepant projective Y, there is at least one family $\mathcal{F}$, which is (globally) simple on Y.

In this section, we turn our attention to a case when the resolution Y of $\mathbb{C}^3/G$ is non-projective. Such Y can not be a moduli space M_θ of θ-stable G-constellations for some parameter θ, since these are constructed (cf. [CI02], Section 2.1) as GIT quotient spaces and therefore are projective. Thus, the existence of a gnat-family which is (globally) simple on Y is not guaranteed.

The following example establishes existence of a simple gnat-family for one particular non-projective resolution:

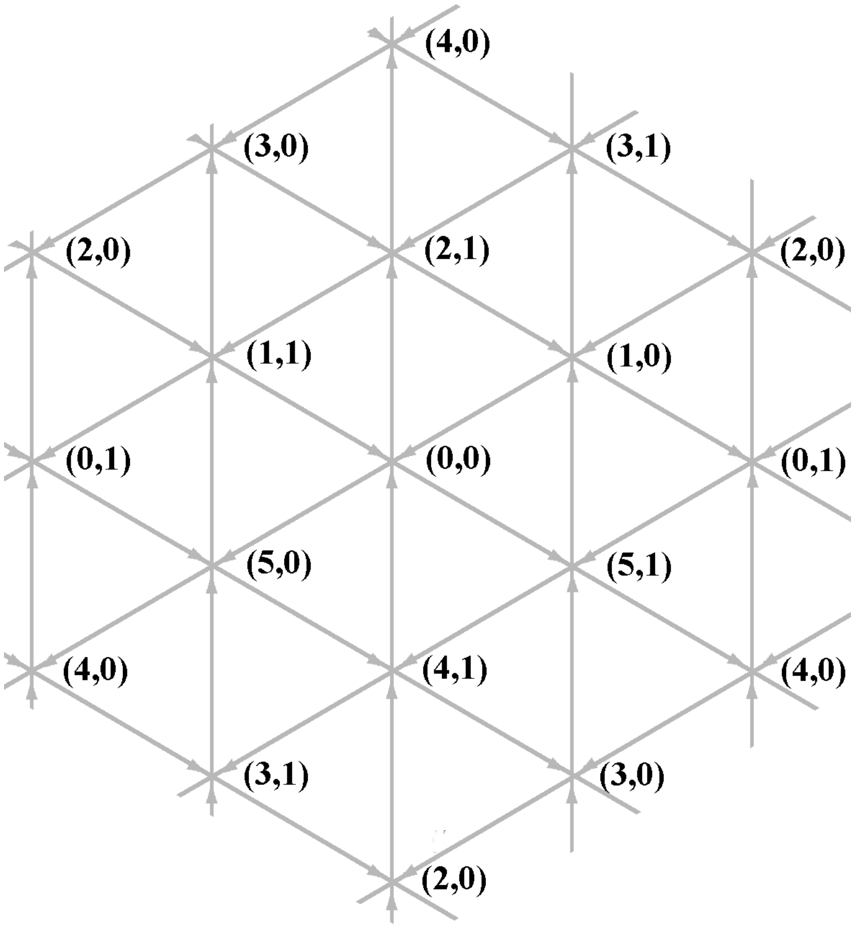
Example 6.84. We set the group G to be $\frac{1}{6}(1,1,4) \oplus \frac{1}{2}(1,0,1)$. That is, the image in $\mathrm{SL}_3(\mathbb{C}^n)$ of the product $\mu_6 \times \mu_2$ of the groups of 6th and 2nd roots of unity, respectively, under the embedding:

$$(\xi_1, \xi_2) \mapsto \begin{pmatrix} \xi_1\xi_2 & & \\ & \xi_1 & \\ & & \xi_1^4\xi_2 \end{pmatrix} \tag{6.71}$$

By $\chi_{i,j}$ we shall denote the character on G induced from the character $(\xi_1, \xi_2) \mapsto$

$\xi_1^i \xi_2^j$ on $\mu_6 \times \mu_2$. Thus, as in Definition 3.1, the monomial x_1 is of weight $\chi_{1,1}$ and G acts on it by the character $\chi_{1,1}^{-1} = \chi_{5,0}$. Hence in the McKay quiver quiver of G, there is an arrow from each $\chi \in G^\vee$ to $\chi\chi_{5,0}$.

Calculating the whole of the McKay quiver of G, we obtain the following:

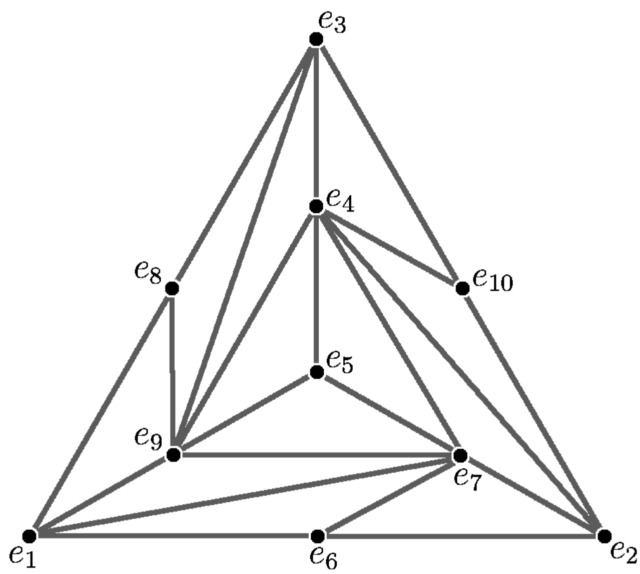


The above is a diagram of the fundamental domain of the McKay quiver of G in the universal cover quiver $\mathcal{U}$, the latter being embedded into $\mathbb{R}^2$ as described in Section 6.4.

The lattice L, defined in Section 4.1, is generated in $(\mathbb{Z}^3)^\vee \otimes \mathbb{Q}$ by $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $\frac{1}{6}(1,1,4)$ and $\frac{1}{2}(1,0,1)$. Calculating set $\mathfrak{E}$ of elements of L, which lie in the junior simplex Δ we obtain:

$$\begin{array}{lll} e_1 = (1,0,0) & e_2 = (0,1,0) & e_3 = (0,0,1) \\ e_4 = \frac{1}{6}(1,1,4) & e_5 = \frac{1}{3}(1,1,1) & e_6 = \frac{1}{2}(1,1,0) \\ e_7 = \frac{1}{6}(1,4,1) & e_8 = \frac{1}{2}(1,0,1) & e_9 = \frac{1}{6}(4,1,1) \\ e_{10} = \frac{1}{2}(0,1,1) & & \end{array} \tag{6.72}$$

We choose Y to be the resolution of $X = \mathbb{C}^3/G$, whose fan Σ triangulates the junior simplex as follows:



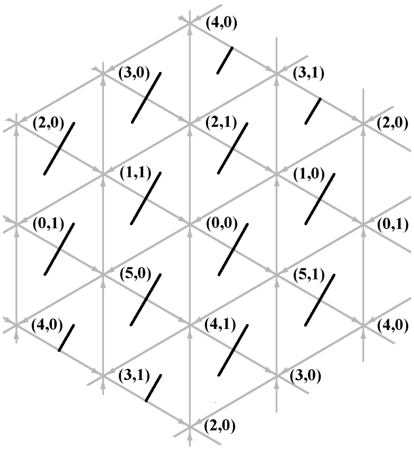
We choose gnat-family $\mathcal{F}$ on Y to be the maximal shift family $\oplus\mathcal{L}(-M_\chi)$, where χ-Weil divisors M_χ are the maximal shift divisors introduced in Definition 5.18. We calculate them, as shown in Example 5.21, for the above choices of G and Y, and obtain that the coefficients of E_i in M_χ are given by the following table:

	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
$M_{\chi_{0,0}}$	0	0	0	0	0	0	0
$M_{\chi_{2,0}}$	$\frac{2}{6}$	$\frac{4}{6}$	0	$\frac{2}{6}$	0	$\frac{2}{6}$	0
$M_{\chi_{4,0}}$	$\frac{4}{6}$	$1\frac{2}{6}$	0	$\frac{4}{6}$	0	$\frac{4}{6}$	0
$M_{\chi_{1,1}}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	0
$M_{\chi_{1,0}}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	0	$\frac{1}{6}$	$\frac{3}{6}$
$M_{\chi_{4,1}}$	$\frac{4}{6}$	$\frac{2}{6}$	0	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{3}{6}$
$M_{\chi_{3,1}}$	$\frac{3}{6}$	1	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$	1	0
$M_{\chi_{3,0}}$	$\frac{3}{6}$	1	$\frac{3}{6}$	1	0	$\frac{3}{6}$	$\frac{3}{6}$
$M_{\chi_{0,1}}$	1	1	0	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{3}{6}$
$M_{\chi_{5,1}}$	$\frac{5}{6}$	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	0
$M_{\chi_{5,0}}$	$\frac{5}{6}$	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{2}{6}$	0	$\frac{5}{6}$	$\frac{3}{6}$
$M_{\chi_{2,1}}$	$\frac{2}{6}$	$\frac{4}{6}$	0	$\frac{5}{6}$	$\frac{3}{6}$	$\frac{5}{6}$	$\frac{3}{6}$

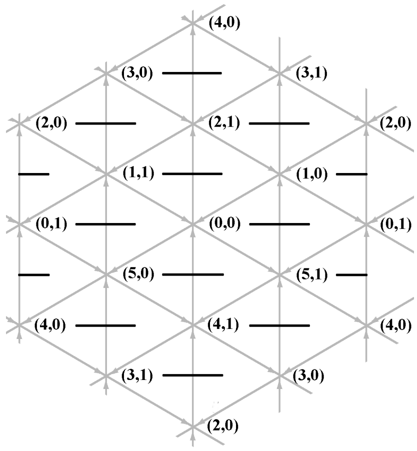
We omit E_1, E_2 and E_3 as they have coefficient 0 in all M_χ. Recall also that, by Corollary 3.14, E_i are the only prime Weil divisors on Y, which can have non-zero coefficients in any of M_χ. Next, using equation 6.12, we calculate B_q for each arrow q in the McKay quiver of G. We obtain:

$$
\begin{array}{ll}
B_{\chi_{0,0},1} = E_1 & B_{\chi_{1,1},1} = E_1 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9 \\
B_{\chi_{0,0},2} = E_2 & B_{\chi_{1,1},2} = E_2 + E_6 + E_7 \\
B_{\chi_{0,0},3} = E_3 & B_{\chi_{1,1},3} = E_3 + E_4 + E_8 \\
B_{\chi_{4,0},1} = E_1 & B_{\chi_{1,0},1} = E_1 + E_6 + E_9 \\
B_{\chi_{4,0},2} = E_2 & B_{\chi_{1,0},2} = E_2 + E_4 + E_5 + E_6 + E_7 + E_9 + E_{10} \\
B_{\chi_{4,0},3} = E_3 & B_{\chi_{1,0},3} = E_3 + E_4 + E_{10} \\
B_{\chi_{2,0},1} = E_1 + E_5 + E_9 & B_{\chi_{4,1},1} = E_1 + E_8 + E_9 \\
B_{\chi_{2,0},2} = E_2 + E_5 + E_7 & B_{\chi_{4,1},2} = E_2 + E_7 + E_{10} \\
B_{\chi_{2,0},3} = E_3 + E_4 + E_5 & B_{\chi_{4,1},3} = E_3 + E_4 + E_5 + E_7 + E_8 + E_9 + E_{10} \\
B_{\chi_{5,1},1} = E_1 + E_6 + E_8 + E_9 & B_{\chi_{3,1},1} = E_1 + E_6 + E_8 + E_9 \\
B_{\chi_{5,1},2} = E_2 + E_6 & B_{\chi_{3,1},2} = E_2 + E_5 + E_6 + E_7 + E_9 \\
B_{\chi_{5,1},3} = E_3 + E_8 & B_{\chi_{3,1},3} = E_3 + E_4 + E_5 + E_8 + E_9 \\
B_{\chi_{5,0},1} = E_1 + E_6 & B_{\chi_{3,0},1} = E_1 + E_5 + E_6 + E_7 + E_9 \\
B_{\chi_{5,0},2} = E_2 + E_6 + E_7 + E_{10} & B_{\chi_{3,0},2} = E_2 + E_6 + E_7 + E_{10} \\
B_{\chi_{5,0},3} = E_3 + E_{10} & B_{\chi_{3,0},3} = E_3 + E_4 + E_5 + E_7 + E_{10} \\
B_{\chi_{2,1},1} = E_1 + E_8 & B_{\chi_{0,1},1} = E_1 + E_4 + E_5 + E_8 + E_9 \\
B_{\chi_{2,1},2} = E_2 + E_{10} & B_{\chi_{0,1},2} = E_2 + E_4 + E_5 + E_7 + E_{10} \\
B_{\chi_{2,1},3} = E_3 + E_4 + E_8 + E_{10} & B_{\chi_{0,1},3} = E_3 + E_4 + E_8 + E_{10}
\end{array}
$$

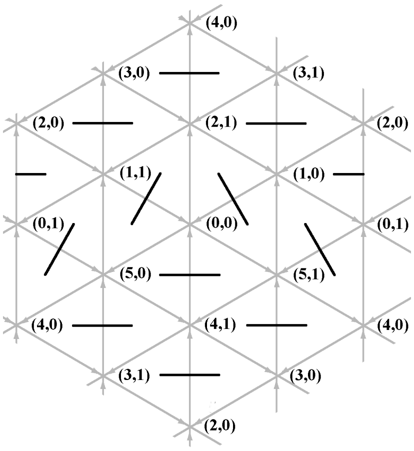
By Proposition 6.68, to demonstrate that $\mathcal{F}$ is a simple family it suffices to demonstrate that, for every three-dimensional cone σ in the fan of Y, the σ-Necklace is connected. By symmetry of the fan Σ and divisors M_χ, we only need to check that σ-Necklace is connected for the cones $\langle e_1, e_8, e_9\rangle$, $\langle e_9, e_8, e_3\rangle$, $\langle e_9, e_3, e_4\rangle$ and $\langle e_9, e_4, e_5\rangle$. First we calculate diamond cochains d_{e_i} (cf. Definition 6.48) for e_1, e_3, e_4, e_5, e_8 and e_9. We use the fact that $e_q^\vee$ has coefficient 1 in d_{e_i} if $E_i \in B_q$ and 0 otherwise. We obtain the following:



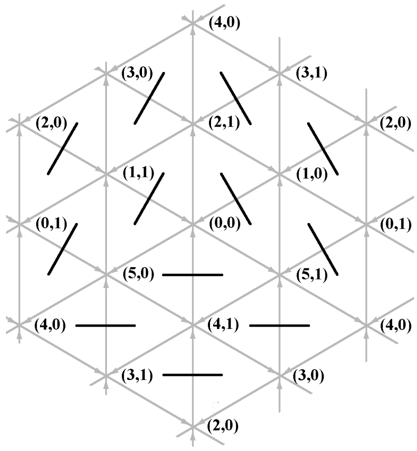
d_{e_1}



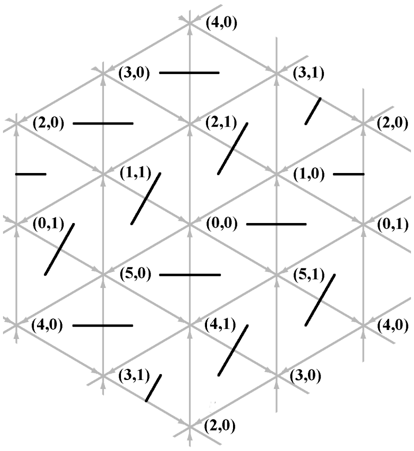
d_{e_3}



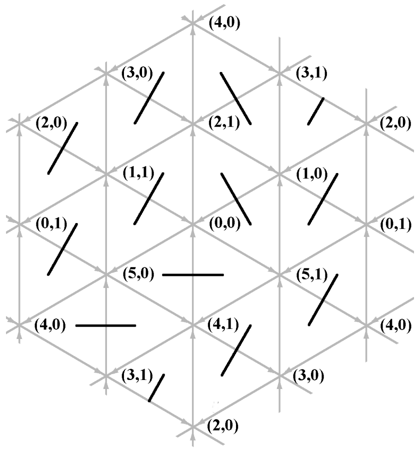
d_{e_4}



d_{e_5}

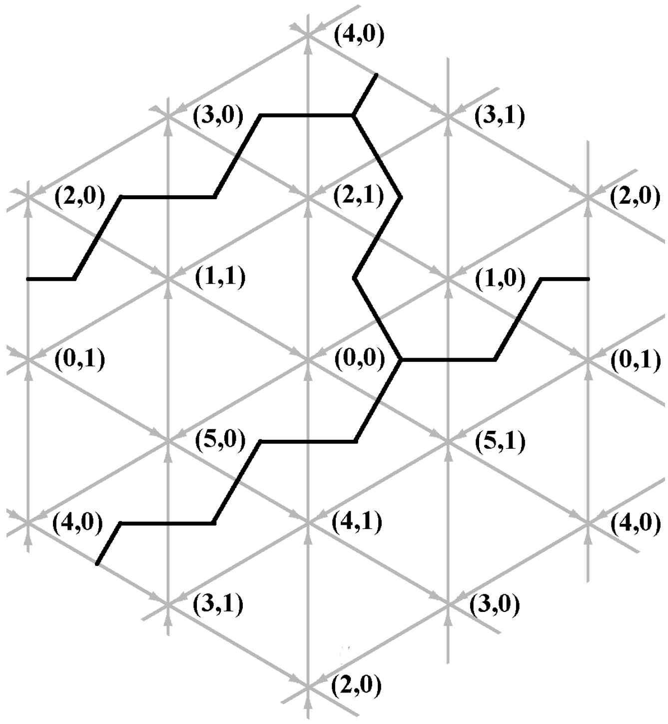


d_{e_8}

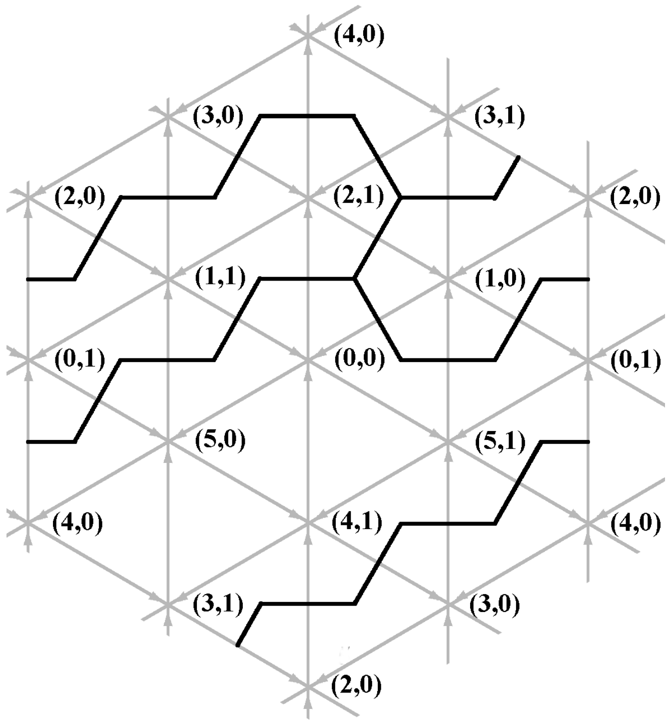


d_{e_9}

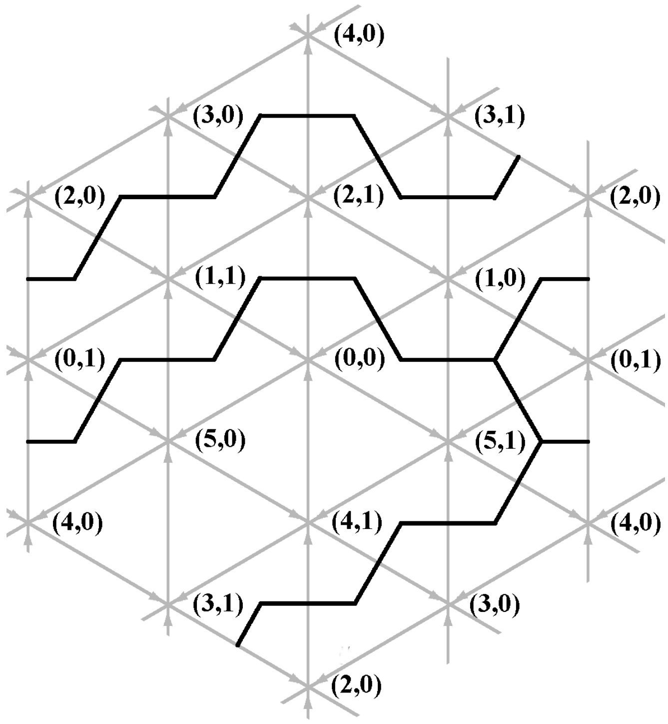
We then calculate σ-Necklaces. We obtain that $\langle e_1, e_8, e_9 \rangle$-Necklace is:



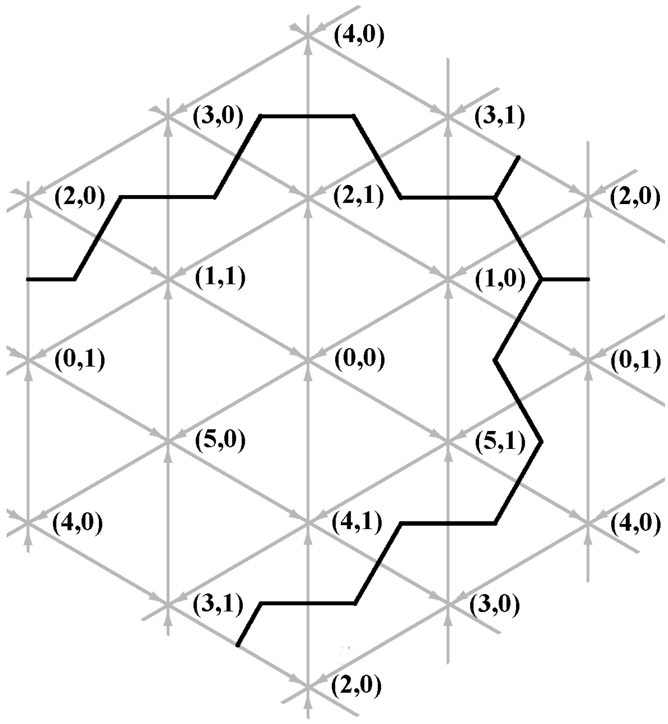
that $\langle e_9, e_8, e_3 \rangle$-Necklace is:



that $\langle e_9, e_3, e_4 \rangle$-Necklace is:



and that $\langle e_9, e_4, e_5 \rangle$-Necklace is:



The reader can now verify that each of the σ-Necklaces on the diagrams above is connected and has only two clasps. Therefore, by Proposition 6.81, the family $\mathcal{F}$ is simple on affine open pieces $A_{\langle e_1, e_8, e_9 \rangle}$, $A_{\langle e_9, e_8, e_3 \rangle}$, $A_{\langle e_9, e_3, e_4 \rangle}$ and $A_{\langle e_9, e_4, e_5 \rangle}$. And hence, by symmetry, on the whole of Y.

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