The Heisenberg category of a category, part II

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Heisenberg algebras (review)

A lattice is a free $\mathbb{Z}$-module M of finite rank equipped with a nondegenerate bilinear form

$$\chi\colon M \times M \to \mathbb{Z}, \quad v, w \mapsto \langle v, w \rangle_\chi.$$

We do not require the form χ to be symmetric nor antisymmetric.

Let (M, χ) be a lattice. The Heisenberg algebra $\underline{H}_M := \underline{H}_{(M,\chi)}$ is the unital $\Bbbk$-algebra with generators $p_a^{(n)}$, $q_a^{(n)}$ for $a \in M$ and integers $n \geq 0$ modulo the following relations for all $a, b \in M$ and $n, m \geq 0$:

$$p_a^{(0)} = 1 = q_a^{(0)}, \tag{1}$$

$$p_{a+b}^{(n)} = \sum_{k=0}^{n} p_a^{(k)} p_b^{(n-k)}, \; q_{a+b}^{(n)} = \sum_{k=0}^{n} q_a^{(k)} q_b^{(n-k)}, \tag{2}$$

$$p_a^{(n)} p_b^{(m)} = p_b^{(m)} p_a^{(n)}, \; q_a^{(n)} q_b^{(m)} = q_b^{(m)} q_a^{(n)}, \tag{3}$$

$$q_a^{(n)} p_b^{(m)} = \sum_{k=0}^{\max(m,n)} s^k \langle a, b \rangle_\chi \, p_b^{(m-k)} q_a^{(n-k)}. \tag{4}$$

Heisenberg algebra of a category

Let $\mathcal{V}$ be a $\Bbbk$-linear category with finite dimensional Hom-spaces. The Grothendieck group $K_0(\mathcal{V})$ of $\mathcal{V}$ comes equipped with the Euler (or Mukai) pairing

$$[a],\ [b] \mapsto \langle [a], [b] \rangle_\chi := \chi(\mathrm{Hom}_\mathcal{V}(a,b)) = \sum_{n\in\mathbb{Z}} (-1)^n H^n \,\mathrm{Hom}_\mathcal{V}(a,b).$$

A Serre functor on $\mathcal{A}$ is an autoequivalence S of $\mathcal{A}$ equipped with a isomorphisms

$$\eta_{a,b}\colon \mathrm{Hom}_\mathcal{A}(a,b) \simeq \mathrm{Hom}_\mathcal{A}(b, Sa)^*,$$

natural in $a, b \in \mathcal{A}$.

Heisenberg algebra of a category

If there is a Serre functor on $\mathcal{V}$, the left and right kernels of χ agree.
Proof:

$$\langle [a], [b] \rangle_\chi = \langle [b], [Sa] \rangle_\chi = \langle [S^{-1} b], [a] \rangle_\chi$$

Numerical Grothendieck group: $K_0^{\mathrm{num}}(\mathcal{V}) \coloneqq K_0(\mathcal{V}) / \ker(\chi)$.
Under certain conditions, this is a finite rank free abelian group.

Example

If X is a smooth and proper variety over $\Bbbk$, then $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ ($=K^{\mathrm{b}}(\mathrm{Coh}(X))$ modulo quasi-isomorphisms) admits a Serre functor $S = (-) \otimes_X \omega_X[\dim X]$, where ω_X is the canonical line bundle of X. Via Hirzebruch-Riemann-Roch:

$$\chi(\mathrm{Hom}(a, b)) = \chi(a^\vee \otimes b) = \int_X \mathrm{ch}(a^\vee \otimes b) \cdot \mathrm{td}(T_X).$$

$K_0^{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) = K_0^{\mathrm{num}}(X) := K_0(X) / \ker(\mathrm{ch})$
$\underline{H}_X := \underline{H}_{(K_0^{\mathrm{num}}(X), \chi)}$

Categorification

When categorifying a $\Bbbk$-algebra A, we are looking for a $\Bbbk$-linear monoidal category $\mathcal{A}$ with the following (compatible) data:

- for $a, b \in \mathcal{A}$, there is $a \circ_1 b \in \mathcal{A}$ (1-composition),
- for $\alpha \in \mathrm{Hom}_{\mathcal{A}}(a, b), \beta \in \mathrm{Hom}_{\mathcal{A}}(b, c)$ there is $\alpha \circ_2 \beta \in \mathrm{Hom}_{\mathcal{A}}(a, c)$ (2-composition)

such that $K_0^{\mathrm{num}}(\mathcal{A}) \simeq A$ as a $\Bbbk$-algebra.

Khovanov has found a categorification of the Heisenberg algebra $\underline{H}_{\mathbb{Z}}$ of the rank 1 lattice (free boson).

Theorem (Gy-K-L)

Let $\mathcal{V}$ be a $\Bbbk$-linear (triangulated) category with a Serre functor. There is a $\Bbbk$-linear (triangulated) monoidal category $\mathcal{H}_{\mathcal{V}}$ such that $K_0^{\mathrm{num}}(\mathcal{H}_{\mathcal{V}}) \simeq \underline{H}_{K_0^{\mathrm{num}}(\mathcal{V})}$.

Some examples where the Theorem can be applied

1. $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ where X is a smooth and proper variety over $\Bbbk$,
   $S = (-) \otimes_X \omega_X[\dim X]$, $K_0^{\mathrm{num}}(\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)) = K_0^{\mathrm{num}}(X)$
2. $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(\Bbbk)) = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{pt})$, $S = \mathrm{Id}$, $K_0^{\mathrm{num}}(\mathrm{pt}) = \mathbb{Z}$
   $\rightsquigarrow$ our $\mathcal{H}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{pt})}$ reproduces Khovanov's Heisenberg category
3. $\Gamma < \mathrm{SL}(2, \mathbb{C})$ a finite subgroup, $Y = \mathbb{C}^2/\Gamma$, X its minimal resolution. Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ be the bounded derived category of coherent sheaves on X, and $\mathcal{V}$ be full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ consisting of sheaves supported on the exceptional divisor E. Then
   - the Serre functor is given by the shift $[2]$.
   - $K_0^{\mathrm{num}}(\mathcal{V}) = \mathbb{Z}\Delta$, the root lattice corresponding to Γ
   - χ is given by the Cartan matrix = the intersection form on E.

Categorification of the HA

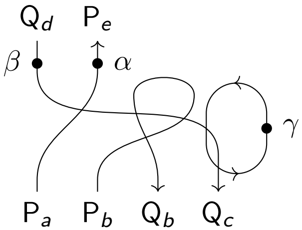
Let $\mathcal{V}$ be a $\Bbbk$-linear category with a Serre functor.

Provisional Heisenberg category: $\mathcal{H}'_{\mathcal{V}}$

- objects: finite words on 2 sets of symbols: P_a & Q_a where $a \in \mathrm{Ob}(\mathcal{V})$
- morphisms: $\{$ planar string diagrams $\}$ / $\sim_{\text{a set of relations}}$

Example

A planar string diagram:



Represents a morphism in $\mathrm{Hom}_{\mathcal{H}_{\mathcal{V}}}(\mathsf{P}_a\mathsf{P}_b\mathsf{Q}_b\mathsf{Q}_b, \mathsf{Q}_d\mathsf{P}_e)$

Diagrams are read from bottom to top!

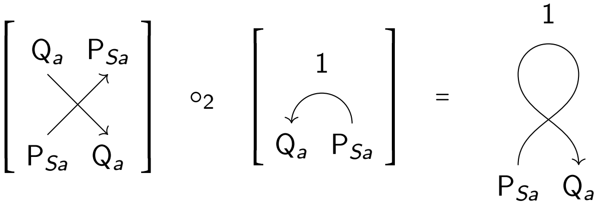
Categorification of the HA

Provisional Heisenberg category: $\mathcal{H}'_{\mathcal{V}}$ (continued)

- 1-composition: concatenations of words

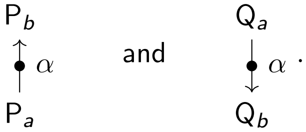
$$(\mathsf{P}_a\mathsf{P}_b\mathsf{Q}_c) \circ_1 (\mathsf{Q}_d\mathsf{P}_e) = \mathsf{P}_a P_b \mathsf{Q}_c\mathsf{Q}_d\mathsf{P}_e$$

- 2-compositon: composition of string diagrams

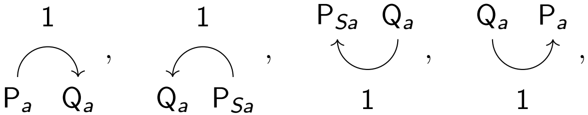


String diagrams - Generators

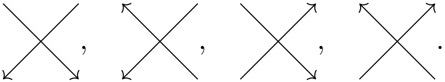
- dots for $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, b)$:



- cups and caps (1 = empty word):



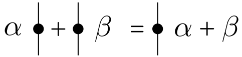
- crossings in all possible orientations

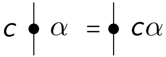


String diagrams - Relations

The diagrams are considered up to boundary preserving isotopies and a number of relations:

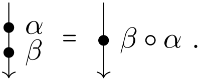
1. Linearity relations:



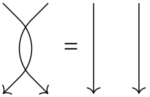


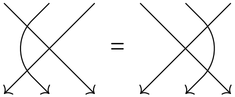
   for any scalar $c \in \Bbbk$ and any compatible orientation of the strings.

2. Neighboring dots can merge (with sign when upwards):

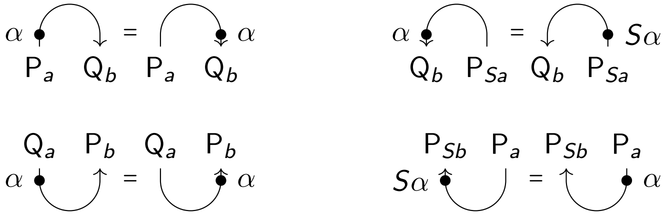


3. Symmetric group downwards (also holds upwards)

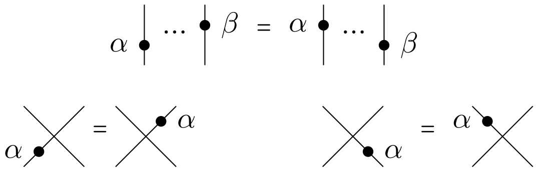




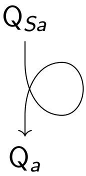
4. Dots may slide through caps and cups as follows:



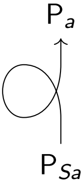
5. Dots may freely slide along strands as well as through all types of crossings.



6. Left curls vanish



$= 0$

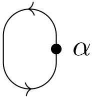


$= 0$

7. The Serre functor S induces a Serre trace map

$$\mathrm{Tr}\colon \mathrm{Hom}_{\mathcal{A}}(a, Sa) \to \Bbbk, \qquad \alpha \mapsto \eta_{a,a}(\mathrm{id}_a)(\alpha).$$

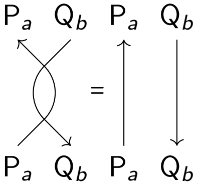
We impose



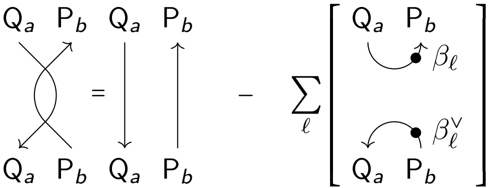
$= \mathrm{Tr}(\alpha),$

where $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, Sa)$.

8.



9. Fix a basis $\{\beta_\ell\}$ of $\mathrm{Hom}(a, b) \rightsquigarrow$ dual basis $\{\beta_\ell^\vee\}$ of $\mathrm{Hom}(b, Sa) \cong \mathrm{Hom}(a, b)^\vee$. We impose



Cups & caps ≈ adjunction units & counits

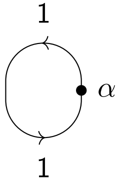
An adjoint triple of functors:

$$\mathsf{P}_a \dashv \mathsf{Q}_a \dashv \mathsf{P}_{Sa}.$$

By relation 7, for any $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, Sa)$ the composition

$$1 \xrightarrow{\text{unit}} \mathsf{Q}_a\mathsf{P}_a \xrightarrow{(\mathrm{id}_{\mathsf{Q}_a})\alpha} \mathsf{Q}_a\mathsf{P}_{Sa} \xrightarrow{\text{counit}} 1,$$

pictorially



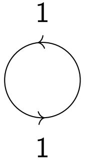
is the multiplication by $\mathrm{Tr}(\alpha) \in \mathbb{k}$.

Example

In $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{pt})$, Q and P are biadjoint:

$$1 \xrightarrow{\text{unit}} \mathsf{QP} \xrightarrow{\text{counit}} 1,$$

pictorially

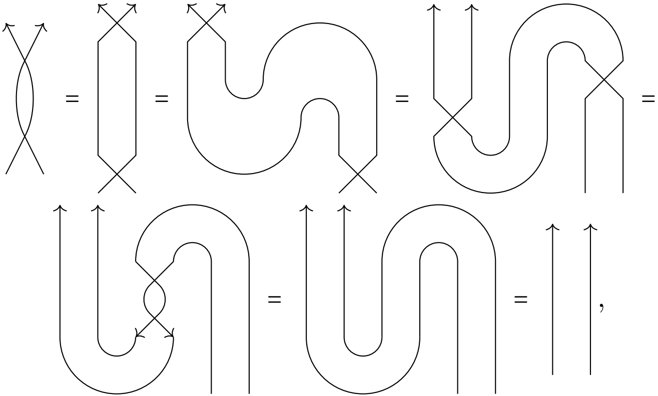


, is the identity.

Our relations imply further ones

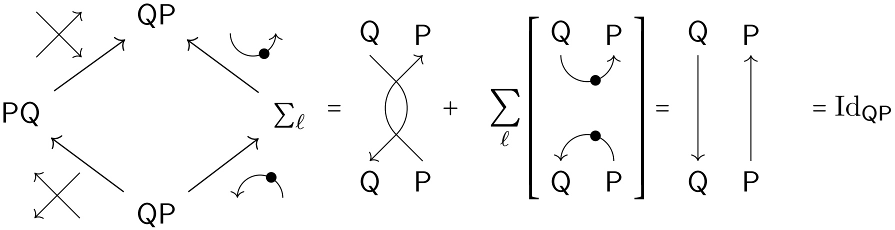
Example

Upward crossing:

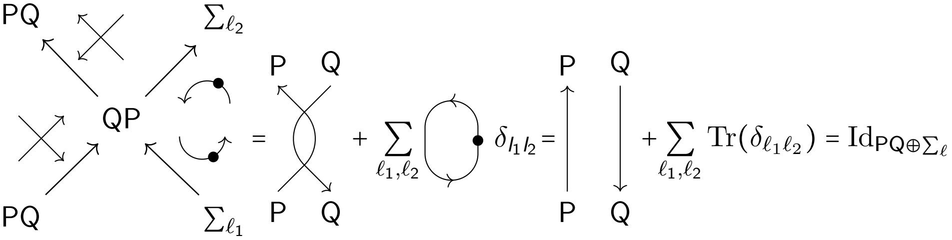


Example: Heisenberg relation

A.



B.



$$(A) + (B) \Longrightarrow QP \simeq PQ \oplus \textstyle\sum_\ell \beta_\ell \simeq PQ \oplus \mathrm{Hom}(a,b)$$

Idempotent completion

The Heisenberg category $\mathcal{H}_\mathcal{V}$ associated with $\mathcal{V}$ is the idempotent completion of $\mathcal{H}'_\mathcal{V}$:

- objects are pairs (R, e), where R is an object of $\mathcal{H}'_\mathcal{V}$ and $e\colon R \to R$ is a idempotent in $\mathrm{End}(R)$.
- morphisms $(R_1, e_1) \to (R_2, e_2)$ are morphisms $f\colon R_1 \to R_2$ from $\mathcal{H}'_\mathcal{V}$ which satisfy $f = e_2 \circ f \circ e_1$.

The symmetric group relations imply that we have an action of the symmetric group S_n on n parallel upward/downwards strands, i.e. morphism $\Bbbk[S_n] \to \mathrm{End}(\mathrm{P}_a^n)$ and $\Bbbk[S_n] \to \mathrm{End}(\mathrm{Q}_a^n)$.
Let

$$e_{\mathrm{triv}} = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \in \Bbbk[S_n]$$

be the symmetrizer idempotent of $\Bbbk[S_n]$.
Define $\mathrm{P}_a^{(n)} := (\mathrm{P}_a^n,\, e_{\mathrm{triv}})$ and $\mathrm{Q}_a^{(n)} := (\mathrm{Q}_a^n,\, e_{\mathrm{triv}})$.

Theorem (Gy-K-L)

$\mathrm{Q}_a^{(m)}\mathrm{P}_b^{(n)} \cong \bigoplus_{i=0}^{\min(m,n)} \mathrm{Sym}^i \,\mathrm{Hom}_\mathcal{V}(a, b) \otimes_\Bbbk \mathrm{P}_b^{(n-i)}\mathrm{Q}_a^{(m-i)}$.

Symmetric powers of a category

N-fold tensor power $\mathcal{V}^{\otimes N}$ has

- objects: finite direct sums of N-tuples $a_1 \otimes \cdots \otimes a_N$ of objects of $\mathcal{V}$
- morphism spaces:

$$\mathrm{Hom}_{\mathcal{V}^{\otimes N}}\left(a_1 \otimes \cdots \otimes a_N,\, b_1 \otimes \cdots \otimes b_N\right) \coloneqq \mathrm{Hom}_{\mathcal{V}}(a_1, b_1) \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{V}}(a_N, b_N).$$

The category $\mathcal{V}^{\otimes N}$ can be endowed with an S_N-action given on objects by

$$\sigma(a_1 \otimes \cdots \otimes a_N) \coloneqq a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma_{-1}(N)}.$$

We let

$$\mathrm{Sym}^N \mathcal{V} \coloneqq (\mathcal{V}^{\otimes N})^{S_N}$$

be the category of S_N-equivariant objects in $\mathcal{V}^{\otimes N}$, i.e. of tuples $\left(\underline{a}, (\epsilon_\sigma)_{\sigma \in S_N}\right)$ with $\underline{a} \in \mathcal{V}^{\otimes N}$ and $\epsilon_\sigma \colon \underline{a} \xrightarrow{\sim} \sigma(\underline{a})$ isomorphisms compatible with the S_N-action.

Representation

We identify S_{N-1} with the subgroup $1 \times S_{N-1}$ of S_N. Correspondingly, we have a forgetful functor

$$\mathrm{Res}_{S_N}^{1\times S_{N-1}} : \mathrm{Sym}^N \mathcal{V} \to \mathcal{V} \times \mathrm{Sym}^{N-1} \mathcal{V}.$$

It has a left and right adjoint

$$\mathrm{Ind}_{1\times S_{N-1}}^{S_N} : \mathcal{V} \times \mathrm{Sym}^{N-1} \mathcal{V} \to \mathrm{Sym}^N \mathcal{V}.$$

A. Krug introduced (in a more specialized context) the functors

$$P_{N,a} : \mathrm{Sym}^{N-1} \mathcal{V} \xrightarrow{a\otimes -} \mathcal{V} \otimes \mathrm{Sym}^{N-1} \mathcal{V} \xrightarrow{\mathrm{Ind}_{1\times S_{N-1}}^{S_N}} \mathrm{Sym}^N \mathcal{V},$$

and

$$Q_{N,a} : \mathrm{Sym}^N \mathcal{V} \xrightarrow{\mathrm{Res}_{S_N}^{1\times S_{N-1}}} \mathcal{V} \otimes \mathrm{Sym}^{N-1} \mathcal{V} \xrightarrow{\mathrm{Hom}_{\mathcal{V}}(a,-)\otimes \mathrm{id}} \mathrm{Sym}^{N-1} \mathcal{V}.$$

Representation

Let

$$\mathcal{F}_{\mathcal{V}} := \bigoplus_{N=0}^{\infty} \mathrm{Sym}^N \mathcal{V}$$

and for every $a \in \mathcal{V}$

$$P_a := \oplus_{N \geq 1} P_{N,a} : \mathcal{F}_{\mathcal{V}} \to \mathcal{F}_{\mathcal{V}} \qquad Q_a := \oplus_{N \geq 1} Q_{N,a} : \mathcal{F}_{\mathcal{V}} \to \mathcal{F}_{\mathcal{V}}$$

Theorem (Gy-K-L)

The correspondence $\mathsf{P}_a \mapsto P_a$ *and* $\mathsf{Q}_a \mapsto Q_a$ *extends to a monoidal functor* $\mathcal{H}_{\mathcal{V}} \to \mathrm{End}(\mathcal{F}_{\mathcal{V}})$.

Under certain technical conditions, this categorifies the Fock space representation
$\underline{H}_{K_0^{\mathrm{num}}(\mathcal{V})} \to \mathrm{End}(\underline{F}_{K_0^{\mathrm{num}}(\mathcal{V})}) = \mathrm{End}(\oplus_{N=0}^{\infty} \mathrm{Sym}^N K_0^{\mathrm{num}}(\mathcal{V}))$.

Example

 α for $\alpha \in \mathrm{Hom}_{\mathcal{V}}(a, b)$ is mapped to the natural transformation
$\mathrm{Ind}_{1 \times S_{N-1}}^{S_N} \circ (\alpha \otimes \mathrm{Id}) : P_a \Longrightarrow P_b$

The geometric example

For $\mathcal{V} = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$,

$$\mathcal{F}_{\mathcal{V}} = \bigoplus_{N=0}^{\infty} \mathsf{Sym}^N \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \bigoplus_{N=0}^{\infty} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}([X^N/S_N]).$$

Bridgeland-King-Reid: If X is a smooth projective surface, there is an equivalence

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}([X^N/S_N]) = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{[N]})$$

Corollary

For a smooth projective surface X the representation

$$\mathcal{H}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)} \to \mathrm{End}(\mathcal{F}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)})$$

strongly categorifies the representation

$$\underline{H}_{K_0^{\mathrm{num}}(X)} \to \mathrm{End}(\oplus_{N=0}^{\infty} K_0^{\mathrm{num}}(X^{[N]}))$$

constructed by Grojnowki-Nakajima.

Thank you for your attention!

Questions?