Example of bad A-Hilb

Miles Reid

A convex lattice polytope in $\mathbb{A}^2_{\mathbb{Z}}$ has a basic triangulation. $A\text{-Hilb}\,\mathbb{C}^3$ behaves well for diagonal subgroup $A \subset \mathrm{SL}(3,\mathbb{C})$.

Triangulations that are essentially 3-dimensional are frequently much worse. For example, it is known that $A\text{-Hilb}\,\mathbb{C}^3$ for the terminal cyclic quotient orbifold point $A = \frac{1}{r}(1,a,r-a)$ is singular and much more discrepant than necessary.

For diagonal subgroups $A \subset \mathrm{SL}(4,\mathbb{C})$ it often happens that $A\text{-Hilb}\,\mathbb{C}^4$ is very bad. The first reducible case seems to be $A = \frac{1}{30}(1,6,10,13)$; its A-Hilb has 158 monomial ideals, with as many as 19 equations, some giving rise to reducible deformation spaces. One of the champions is

$$I := \left\langle \begin{matrix} x^6, x^3y, x^3t, x^2z, x^2t^2, xy^2, xyt, xzt, xt^3, \\ y^5, y^4z, y^3t, y^2zt, yz^2, yt^2, z^3, z^2t, zt^2, t^4 \end{matrix} \right\rangle \tag{1}$$

The monomial basis of $\mathbb{C}[x,y,z,t]/I$ consists of all monomials not in I. Deforming I involves replacing the 19 monomials by equations; the affine piece of A-Hilb with $\mathbb{C}[Z]$ based by this A-set is locally disconnected. Indeed, the ideal I_Z needs 19 generators, so his neighbours need 19 equations such as

$$xyt = az^2, \quad yz^2 = bt^2, \quad xzt = cy^4, \quad y^2zt = dx^5 \tag{2}$$

The four ratios here

$$a = xyt/z^2, \quad b = yz^2/t^2, \quad c = xzt/y^4, \quad d = y^2zt/x^5 \tag{3}$$

base the lattice of invariant monomials, and are parameters on A-Hilb. From these a standard syzygy manipulation proves that

$$\begin{aligned}
xy^2 &= abt \quad \text{because } t^2 \text{ is basic} && (4)\\
x^2z &= abcy^2 \quad \text{because } y^4 && (5)\\
x^3t &= a^2bcyz \quad \text{because } y^2z\text{:} \qquad x^3yt = ax^2z^2 = a^2bcy^2z && (6)\\
x^6 &= a^4b^3c^2y \quad \text{because } z \text{ (or } yt \text{ or almost anything)} && (7)\\
y^4z &= abdx^4 \quad \text{because } x^5 && (8)\\
z^2t &= abcdx^3 \quad \text{because } x^4 && (9)\\
yt^2 &= a^2bcdx^2 \quad \text{because } x^3 && (10)\\
y^3t &= a^3b^2cdx \quad \text{because } xt && (11)\\
y^5 &= a^4b^3cd \quad \text{because } t && (12)\\
zt^2 &= a^3b^2c^2dy \quad \text{because } y^4 && (13)\\
xt^3 &= a^4b^2c^2dz \quad \text{because } z^2 && (14)\\
z^3 &= a^3b^3c^2d \quad \text{because } y && (15)\\
xyzt &= a^4b^3c^2d \quad \text{because } 1 && (16)
\end{aligned}$$

These relations can be proved assuming only that (**) is a monomial basis. For example, there must exist some relation

$$x^3t = \lambda yz \tag{17}$$

because yz bases the ε^{16} eigenspace. Multiply that by y, use the relation $xyt = az^2$, then the relation $x^2z = abcy^2$, then cancel y^2z, which is valid because y^2z is basic.

However, there are also relations

$$x^3y = et^3, \quad x^2t^2 = fy^3z, \quad t^4 = a^2bcdfy^2z, \tag{18}$$

involving new parameters e, f, about which one can only prove that

$$b(def - 1) = 0 \quad \text{and} \quad b(f - ac) = 0 \tag{19}$$

The mechanism here is that the monomials one would need to cancel to prove $def = 1$ or $f = ac$ are in the socle, so do not give rise to any syzygy deduction as used in (4)–(16).

Therefore A-Hilb is contained in the reducible subvariety of $\mathbb{A}^6_{\langle a,b,c,d,e,f\rangle}$ defined by (19). This has two components:

(I) either $b = 0$

(II) or $f = ac$ and $def = 1$ (so a, c, d, e, f cannot approach 0).

Both components work to give clusters. (I) gives a 5-dimensional component of A-Hilb, with every cluster supported at the origin, a distinct component not in the closure of the birational component.