# How to calculate $A$-Hilb $\mathbb{C}^4$ for $\frac{1}{r}(1,1,a,b))$

Sarah Davis Miles Reid

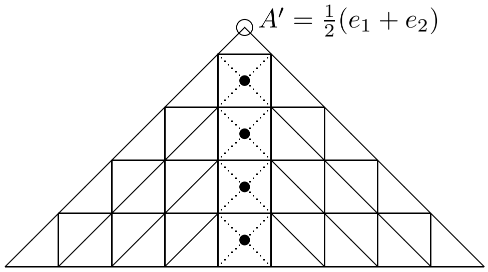
**Abstract**

A Craw–Reid algorithm computes $A$-Hilb $\mathbb{C}^4$ and crepant resolutions for the special groups $\frac{1}{r}(1,1,a,b)$. At the same time, we give a criterion in terms of HJ continued fractions for when a crepant resolution exists.

## First draft

### 0.1 New feature: the Trap

The key new feature of $A$-Hilb $\mathbb{C}^4$ in our cases is what we call the *trap* (isosceles trapezium) formed by two regularly tessellated triangles of size $r$ back to back, separated by an alley of parallelograms of width 1.




This is $A$-Hilb $\mathbb{C}^4$ of the group $\mathbb{Z}/s \times \mathbb{Z}/s$ with $s = 2r+1$

$$\frac{1}{s}(0,0,1,2r) + \frac{1}{s}(1,1,2r-1,0). \tag{0.1}$$

$A$-Hilb $\mathbb{C}^4$ subdivides the trap by tessellating the triangles into basic triangles (as we want), but subdividing the parallelograms by their centre, which provides a row of embossed studs standing out of the crepant junior plane; to obtain a crepant resolution, we need to break the symmetry and divide each parallelogram diagonally into two triangles.

### 0.2 Junior simplex and its median triangle

What makes $\frac{1}{r}(1,1,a,b)$ special is that all the junior lattice points other than the vertices $e_1, e_2$ themselves are contained in the median plane $x = y$. This plane intersects the junior simplex in the triangle with vertexes $e_4, e_3$ and the midpoint $A' = \frac{1}{2}(e_1 + e_2)$. We draw this median triangle with its junior lattice point, but with $A'$ marked as an open ring to emphasise that it really represents the axis $A = e_1 e_2$ out of the plane. A crepant resolution of the quotient $X = \frac{1}{r}(1, 1, a, b)$ is given by a subdivision of this triangle into "internal" basic lattice triangles $P_i P_j P_k$ in the median plane (with vertexes junior points), together with "external triangles" $A' P_i P_j$ involving the midpoint $A'$. We join up each internal triangle with $e_1$ and $e_2$ to give two basic tetrahedra $e_1 P_i P_j P_k$ and $e_2 P_i P_j P_k$, whereas an external triangle $A' P_i P_j$ gives a single basic tetrahedron $e_1 e_2 P_i P_j$.

If $A' = \frac{1}{2}(e_1 + e_2)$ is a lattice point (which happens if and only if $a, b$ are both even, say $a = 2a'$, $b = 2b'$, $r = 2r'$) then there are no external triangles. In this case the original Craw–Reid algorithm for the group $\frac{1}{r'}(1, a', b')$ gives a basic triangulation of the median triangle, and $A$-Hilb $\mathbb{C}^4$ is the crepant resolution corresponding to this. Assume from now on that one of $a, b$ is odd.

Our modified Craw–Reid algorithm uses "strong lines" from the two vertices $e_4$ and $e_3$ together with "strong planes" from the axis $A$ to subdivide the median triangle into regular triangles of side $\rho$ and regular traps of side $\rho$ adjacent to $A$. These are then tessellated into basis triangles as in Craw–Reid, and we prove that this defines $A$-Hilb $\mathbb{C}^4$ (the proof is not written, but it must work). The ingredients are the HJ continued fractions for $r/a$ and $r/b$ providing the strong lines from $e_4$ and $e_3$, plus the strong planes from axis $A$; the latter are slightly less obvious: they are given by the HJ continued fraction expansion of $r/(r - 2c - h)$, where

$$h = \operatorname{hcf}(r, a) \quad \text{and} \quad h = ac + xr, \tag{0.2}$$

so that $P_c =$ one of $(c, c, h, r - 2c - h)$ or $(c, c, h, 2r - 2c - h)$ is the lattice point closest to the face $e_4 A$ of the simplex.

## 0.3 Two main claims:

Quite remarkably, the construction just described gives either a sequence of junior lattice points that calculates $A$-Hilb $\mathbb{C}^4$ and provides a crepant resolution; or it generates an age 2 point that is not the sum of two juniors, so proves the nonexistence of a crepant resolution.

(1) In the cases where the crepant resolution exists, strong lines out of $e_4, e_3$ and strong planes out of $A$ give a big subdivision into regular triangles and traps. These can be subdivided into regular tesselations, giving $A$-Hilb $\mathbb{C}^4$. The result is completely parallel to Craw–Reid.

(2) We set $h = \operatorname{hcf}(a, r)$ and $P_c$ for the closest point to the face $e_4 A$, as in (0.2). This subdivision (and hence a crepant resolution) exists if and only if $P_c$ is junior (that is, $2c + h < r$), and the points given by running the continued fraction algorithm around $A$ are also junior. The latter holds if and only if the HJ continued fraction expansion of $r/(r - 2c - h)$ has only even entries.

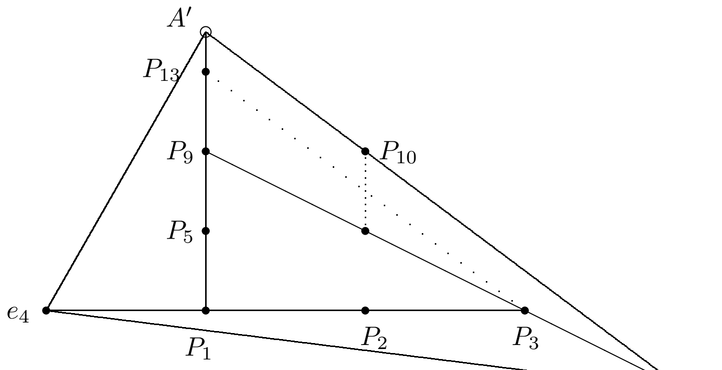
## 0.4 Example $\frac{1}{30}(1,1,7,21)$

**Lines out of $e_4$** First, $30/7 = [5,2,2,3]$; from this, the vectors out of $e_4$ are $e_4P_1$ tagged with 5, then by the continued fraction algorithm the arithmetic progression $e_4P_5$, $e_4P_9$ both tagged with 2, $e_4P_{13}$ tagged with 3 and $e_4A'$. You have to get used to the little paradoxical point that the tag 3 on $e_4P_{13}$ means $P_9, P_{13}, A'$ are coplanar.

$$3 \times P_{13} - P_9 = e_1 + e_2. \tag{0.3}$$

(If we draw $A'$ at the top of triangle, it is not a vertex, but the midpoint of axis $A = e_1e_2$.) Here

$$P_1 = (1,1,7,21), \quad P_5 = (5,5,5,15), \quad P_9 = (9,9,3,9), \quad P_{13} = (13,13,1,3)$$
$$\text{and} \quad 3 \times P_{13} - P_9 = (30,30,0,0) = e_1 + e_2.$$




**Lines out of $e_3$** We calculate $30/21 = [2,2,4]$ with hcf $= 3$; the continued fraction algorithm gives vectors $e_3P_1$, $e_3P_2$ tagged with 2, then $e_3P_3$ tagged with 4, followed by $e_3P_{10}$ with $P_{10} = (10,10,10,0)$, coplanar with $e_3$ and axis $A$.

**Planes out of axis $A$** This is the new and slightly tricky bit. The calculation starts from the midpoint

$$A' = \frac{1}{2}(e_1 + e_2) = (15,15,0,0) \tag{0.4}$$

and the vectors

$$A'e_4 = (-15,-15,0,30) \tag{0.5}$$
$$A'P_{13} = (-2,-2,1,3) \tag{0.6}$$

(Here $3 = r - 2c - 1 = 30 - 2 \times 13 - 1$ is the 4th coordinate of $P_{13}$, the nearest point to the $e_4A$ plane.) The continued fraction we need is $30/3 = [10]$, so the only other plane through $A$ is given by

$$10 \times (0.5) - (0.6) = (-5, -5, 10, 0) = A'P_{10}. \tag{0.7}$$

The fact that 10 is even makes 5 odd, so that

$$(15, 15, 0, 0) + (-5, -5, 10, 0) = (10, 10, 10, 0) \tag{0.8}$$

is a lattice point, namely $P_{10}$. Claim (2) says this guarantees that the crepant resolution exists.

These lines and planes subdivide the median triangle into

- a regular triangle $P_1P_3P_9$ of side 2 (a Meeting of Champions in the sense of Craw–Reid, 2.8.2, bounded by lines from all three corners)
- 3 regular triangles of side 1 spanned by $e_4$ and the line segments $P_1P_5$, $P_5P_9$, $P_9P_{13}$
- 3 regular triangles of side 1, spanned by $e_3$ and the line segments $e_1P_1$, $P_1P_2$, $P_2P_3$
- a trap $AP_9e_3$ of side 3;
- a trap $Ae_4P_{13}$ of side 1 (that is, a basic tetrahedron $e_1e_2e_4P_{13}$.

## 0.5 Example $\frac{1}{17}(1, 1, 5, 10)$ fails

The nearest point to the $Ae_4$ axis is $P_7 = (7, 7, 1, 2)$. This is junior, so we haven't lost yet. We look for the other strong planes out of axis $A$. For this, write vectors

$$Ae_4 = (-17/2, -17/2, 0, 17) \tag{0.9}$$
$$AP_7 = (-3/2, -3/2, 1, 2) \tag{0.10}$$

Now the continued fraction procedure is to do $9 \times (0.10) - (0.9)$, where $9 = \text{Ceiling}(17/2)$. This leads to vector $(-5, -5, 9, 1)$, so gives the point

$$(-5, -5, 9, 1) + (17/2, 17/2, 0, 0) = (7/2, 7/2, 9, 1) \tag{0.11}$$

of the median triangle in the junior simplex. Now this is not a lattice point. The nearest lattice point on the same plane is

$$(-5, -5, 9, 1) + 2 \times (17/2, 17/2, 0, 0) = (12, 12, 9, 1), \tag{0.12}$$

which is age 2. This is a point of age 2 that is not a sum of two juniors, which implies that no crepant resolution exists.

The point here is that the odd HJ entry 9 in $17/2 = [9, 2]$ leads directly to an age 2 lattice point that contradicts JunSuff.