INTRODUCTION TO SPHERICAL AND $\mathbb{P}$-TWISTS

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Exercise 1.1. Let X be a K3 surface and $C \subset X$ a smooth rational curve with $C^2 = -2$. Show that $\mathcal{O}_C(k)$ is a spherical object. That is, show that $\mathrm{Ext}^*_X(\mathcal{O}_C(k), \mathcal{O}_C(k)) \simeq \mathbb{C} \oplus \mathbb{C}[-2]$.

Exercise 1.2. Recall that the spherical twist $T_{\mathcal{E}}$ associated to a spherical object $\mathcal{E} \in \mathcal{D}(X)$ is defined by the following distinguished triangle

$$\mathrm{Hom}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F} \to T_{\mathcal{E}}(\mathcal{F}).$$

In the setting of Exercise 1.1, compute $T_{\mathcal{O}_C}(\mathcal{O}_x)$ where $\mathcal{O}_x$ is the skyscraper sheaf of a point $x \in X$ and observe the difference between the answers when $x \notin C$ and $x \in C$.

Exercise 1.3. Let $\mathcal{E} \in \mathcal{D}(X)$ be any object. Show that $\mathcal{E} \cup \mathcal{E}^{\perp}$ is a spanning class for $\mathcal{D}(X)$.

(Recall that a collection of objects Ω in $\mathcal{D}(X)$ is a spanning class if for all $\mathcal{G} \in \mathcal{D}(X)$ the following two conditions hold: (i) if $\mathrm{Hom}(\mathcal{F}, \mathcal{G}[i]) = 0$ for all $\mathcal{F} \in \Omega$ and all $i \in \mathbb{Z}$ then $\mathcal{G} \simeq 0$, (ii) if $\mathrm{Hom}(\mathcal{G}[i], \mathcal{F}) = 0$ for all $\mathcal{F} \in \Omega$ and all $i \in \mathbb{Z}$ then $\mathcal{G} \simeq 0$.)

Exercise 1.4. Show that $T_{\mathcal{E}}(\mathcal{E}) \simeq \mathcal{E}[1 - \dim X]$ and $T_{\mathcal{E}}(\mathcal{F}) \simeq \mathcal{F}$ for all $\mathcal{F} \in \mathcal{E}^{\perp}$ and deduce that $\mathrm{Hom}(\mathcal{G}, \mathcal{G}'[i]) \to \mathrm{Hom}(T_{\mathcal{E}}(\mathcal{G}), T_{\mathcal{E}}(\mathcal{G}')[i])$ is an isomorphism for all $\mathcal{G}, \mathcal{G}' \in \mathcal{E} \cup \mathcal{E}^{\perp}$ and all $i \in \mathbb{Z}$.

Exercise 1.5. Let X be a (strict) CY3-fold and $C \subset X$ a smooth rational curve with normal bundle $\mathcal{N}_{C/X} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. Show that $\mathcal{O}_C(k)$ is again a spherical object. That is, show that $\mathrm{Ext}^*_X(\mathcal{O}_C(k), \mathcal{O}_C(k)) \simeq \mathbb{C} \oplus \mathbb{C}[-3]$. (Hint: Use the fact that $i^* i_* \mathcal{O}_C \simeq \bigoplus \bigwedge^k \mathcal{N}_{C/X}[k]$.)

Exercise 1.6. Consider the Atiyah flop of $C \subset X$ in the setting of Exercise 1.3:

$$\begin{tikzcd}
 & & E = \mathbb{P}^1 \times \mathbb{P}^{1\vee} \arrow[d, hook] \arrow[ddll, "\pi"'] \arrow[ddrr, "\pi'"] & & \\
 & & \widetilde{X} \arrow[dl, "q"'] \arrow[dr, "p"] & & \\
\mathbb{P}^1 \arrow[r, hook, "j"] & X & & X' & \mathbb{P}^{1\vee} \arrow[l, hook']
\end{tikzcd}$$

Show that there are natural isomorphisms $q_* p^* p_* q^* \mathcal{O}_X(-k) \simeq T^{-1}_{\mathcal{O}_{\mathbb{P}^1}(-1)}(\mathcal{O}_X(-k))$ for $k = 0, 1$ where $T^{-1}_{\mathcal{E}}$ is defined by the following distinguished triangle

$$T^{-1}_{\mathcal{E}}(\mathcal{F}) \to \mathcal{F} \to \mathrm{Hom}^*(\mathcal{F}, \mathcal{E})^{\vee} \otimes \mathcal{E}.$$

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