

# INTRODUCTION TO SPHERICAL AND $\mathbb{P}$ -TWISTS

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**Exercise 1.1.** Let  $X$  be a K3 surface and  $C \subset X$  a smooth rational curve with  $C^2 = -2$ . Show that  $\mathcal{O}_C(k)$  is a spherical object. That is, show that  $\text{Ext}_X^*(\mathcal{O}_C(k), \mathcal{O}_C(k)) \simeq \mathbb{C} \oplus \mathbb{C}[-2]$ .

**Exercise 1.2.** Recall that the spherical twist  $T_{\mathcal{E}}$  associated to a spherical object  $\mathcal{E} \in \mathcal{D}(X)$  is defined by the following distinguished triangle

$$\text{Hom}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \rightarrow \mathcal{F} \rightarrow T_{\mathcal{E}}(\mathcal{F}).$$

In the setting of Exercise 1.1, compute  $T_{\mathcal{O}_C}(\mathcal{O}_x)$  where  $\mathcal{O}_x$  is the skyscraper sheaf of a point  $x \in X$  and observe the difference between the answers when  $x \notin C$  and  $x \in C$ .

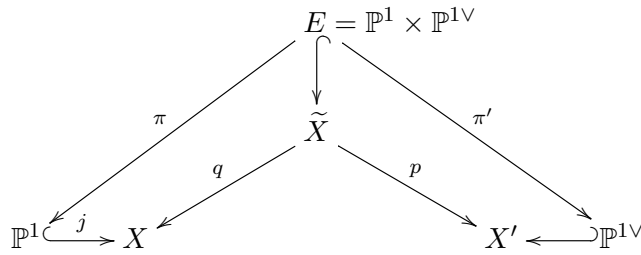
**Exercise 1.3.** Let  $\mathcal{E} \in \mathcal{D}(X)$  be any object. Show that  $\mathcal{E} \cup \mathcal{E}^\perp$  is a spanning class for  $\mathcal{D}(X)$ .

(Recall that a collection of objects  $\Omega$  in  $\mathcal{D}(X)$  is a spanning class if for all  $\mathcal{G} \in \mathcal{D}(X)$  the following two conditions hold: (i) if  $\text{Hom}(\mathcal{F}, \mathcal{G}[i]) = 0$  for all  $\mathcal{F} \in \Omega$  and all  $i \in \mathbb{Z}$  then  $\mathcal{G} \simeq 0$ , (ii) if  $\text{Hom}(\mathcal{G}[i], \mathcal{F}) = 0$  for all  $\mathcal{F} \in \Omega$  and all  $i \in \mathbb{Z}$  then  $\mathcal{G} \simeq 0$ .)

**Exercise 1.4.** Show that  $T_{\mathcal{E}}(\mathcal{E}) \simeq \mathcal{E}[1 - \dim X]$  and  $T_{\mathcal{E}}(\mathcal{F}) \simeq \mathcal{F}$  for all  $\mathcal{F} \in \mathcal{E}^\perp$  and deduce that  $\text{Hom}(\mathcal{G}, \mathcal{G}'[i]) \rightarrow \text{Hom}(T_{\mathcal{E}}(\mathcal{G}), T_{\mathcal{E}}(\mathcal{G}')[i])$  is an isomorphism for all  $\mathcal{G}, \mathcal{G}' \in \mathcal{E} \cup \mathcal{E}^\perp$  and all  $i \in \mathbb{Z}$ .

**Exercise 1.5.** Let  $X$  be a (strict) CY3-fold and  $C \subset X$  a smooth rational curve with normal bundle  $\mathcal{N}_{C/X} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ . Show that  $\mathcal{O}_C(k)$  is again a spherical object. That is, show that  $\text{Ext}_X^*(\mathcal{O}_C(k), \mathcal{O}_C(k)) \simeq \mathbb{C} \oplus \mathbb{C}[-3]$ . (Hint: Use the fact that  $i^*i_*\mathcal{O}_C \simeq \bigoplus \bigwedge^k \mathcal{N}_{C/X}[k]$ .)

**Exercise 1.6.** Consider the Atiyah flop of  $C \subset X$  in the setting of Exercise 1.3:



Show that there are natural isomorphisms  $q_*p^*p_*q^*\mathcal{O}_X(-k) \simeq T_{\mathcal{O}_{\mathbb{P}^1}(-1)}^{-1}(\mathcal{O}_X(-k))$  for  $k = 0, 1$  where  $T_{\mathcal{E}}^{-1}$  is defined by the following distinguished triangle

$$T_{\mathcal{E}}^{-1}(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \text{Hom}^*(\mathcal{F}, \mathcal{E})^\vee \otimes \mathcal{E}.$$

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