Rainbow k-connection in Dense Graphs (Extended Abstract)

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Abstract

An edge-colouring of a graph G is *rainbow k-connected* if, for any two vertices of G, there are k internally vertex-disjoint paths joining them, each of which is *rainbow* (i.e., all edges of each path have distinct colours). The minimum number of colours for which there exists a rainbow k-connected colouring for G is the *rainbow k-connection number* of G, and is denoted by $\mathrm{rc}_k(G)$. The function $\mathrm{rc}_k(G)$ was introduced by Chartrand et al. in 2008, and has since attracted considerable interest. In this note, we shall consider the function $\mathrm{rc}_k(G)$ for complete bipartite and multipartite graphs, highly connected graphs, and random graphs.

Keywords: rainbow colouring, k-connected, random graph

1 Introduction

Connectivity is one of the most important and fundamental graph-theoretic concepts, in both the combinatorial setting and the algorithmic setting. The connectivity concept and its variants, such as k-connectivity, requirement of Hamiltonicity, requirement of bounded diameter, and so on, have been intensely studied. In 2008, Chartrand, Johns, McKeon and Zhang [4,5] introduced the concept of *rainbow k-connectivity*, as follows. An edge-colouring of a graph G is *rainbow k-connected* if, for any two vertices of G, there are k internally vertex-disjoint *rainbow paths* joining them (i.e., the edges of each path have distinct colours). The minimum number of colours for which there exists a rainbow k-connected colouring for G is the *rainbow k-connection number* of G, and is denoted by $\mathrm{rc}_k(G)$. We write $\mathrm{rc}(G) = \mathrm{rc}_1(G)$. Note that, by Menger's theorem, any two vertices of a graph G have k internally vertex-disjoint paths joining them if and only if G is k-connected. Hence, the function $\mathrm{rc}_k(G)$ will only be defined for k-connected graphs G.

In their original papers, Chartrand et al. [4,5] studied $\mathrm{rc}_k(G)$ for many basic families of graphs, notably in the cases when G is complete, and complete bipartite and multipartite. They also introduced the *strong rainbow connection number* $\mathrm{src}(G)$, and considered some relationships between $\mathrm{rc}(G)$ and $\mathrm{src}(G)$. Since then, the function $\mathrm{rc}_k(G)$ has attracted considerable interest. Among the concepts considered are minimum degree conditions, higher connectivity, random graphs, and the time complexity of determining $\mathrm{rc}_k(G)$, and further analogous functions were also introduced, including the *rainbow vertex connection number* $\mathrm{rvc}(G)$ and the *k-rainbow index* $\mathrm{rx}_k(G)$ (Caro et al. [2]; Krivelevich and Yuster [9]; Chartrand et al. [6], among others). Very

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recently, Li and Sun [12] published an informative survey which summarised the current status of rainbow connection.

In this note, we report new results on rainbow connection for complete bipartite and multipartite graphs, highly connected graphs, and random graphs.

2 Complete Bipartite and Multipartite Graphs

For $1 \leq n_1 \leq \cdots \leq n_t$ with $t \geq 2$, let $K_{n_1,\ldots,n_t}$ denote the complete multipartite graph with classes $n_1, \ldots, n_t$. Chartrand et al. [4] determined $\mathrm{rc}(K_{n_1,\ldots,n_t})$.

Theorem 2.1 **([4])** *Let $1 \leq n_1 \leq \cdots \leq n_t$ with $t \geq 2$, let $\sum_{i=1}^{t-1} n_i = m$ and $n_t = n$. Then*

$$\mathrm{rc}(K_{n_1,\ldots,n_t}) = \begin{cases} n & \text{if } t = 2 \text{ and } n_1 = 1, \\ \min(\lceil \sqrt[m]{n} \rceil, 4) & \text{if } t = 2 \text{ and } 2 \leq n_1 \leq n_2, \\ 1 & \text{if } t \geq 3 \text{ and } n_t = 1, \\ 2 & \text{if } t \geq 3, n_t \geq 2 \text{ and } m > n, \\ \min(\lceil \sqrt[m]{n} \rceil, 3) & \text{if } t \geq 3 \text{ and } m \leq n. \end{cases}$$

For $k \geq 2$ and the complete bipartite graph $K_{n,n}$, they proved the following for $\mathrm{rc}_k(K_{n,n})$.

Theorem 2.2 **([5])** *For $k \geq 2$ and $n = 2k\lceil \frac{k}{2} \rceil$, we have $\mathrm{rc}_k(K_{n,n}) = 3$.*

They also asked whether we have $\mathrm{rc}_k(K_{n,n}) = 3$ for all sufficiently large n. Li and Sun [11] proved that this is the case, when $n \geq 2k\lceil \frac{k}{2} \rceil$. This result and Theorem 2.2 both considered explicit colourings. With a random construction, we are able to improve the result to $n \geq 2k + o(k)$, as follows.

Theorem 2.3 **([7])** *For $\varepsilon > 0$, there exists a function $f(\varepsilon)$ such that for $k \geq f(\varepsilon)$ and $n \geq (2 + \varepsilon)k$, we have $\mathrm{rc}_k(K_{n,n}) = 3$.*

Using a similar random method, we can in fact extend Theorem 2.3 to complete multipartite graphs with equipartitions. For $t \geq 3$, let $K_{t\times n}$ denote the complete t-partite graph, with n vertices in each class.

Theorem 2.4 **([7])** *For $\varepsilon > 0$ and $t \geq 3$, there exists a function $f(\varepsilon, t)$ such that for $k \geq f(\varepsilon, t)$ and $n \geq \frac{(2+\varepsilon)k}{t-2}$, we have $\mathrm{rc}_k(K_{t\times n}) = 2$.*

There remains the open problem of determining similar results for complete multipartite graphs, whose partitions are not equal. Chartrand et al. [5] asked this question in the bipartite case.

3 Highly Connected Graphs

A natural question one can ask is, what can we say about $\mathrm{rc}(G)$ if G is highly connected? In this direction, Caro et al. were the first to prove a result.

Theorem 3.1 ([2]) *If G is a 2-connected graph on n vertices, then $\mathrm{rc}(G) \le \frac{2n}{3}$, and $\mathrm{rc}(G) \le \frac{n}{2} + O(\sqrt{n})$.*

For 3-connected graphs, Li and Shi proved the following.

Theorem 3.2 ([10]) *If G is a 3-connected graph on n vertices, then $\mathrm{rc}(G) \le \frac{3(n+1)}{5}$.*

For higher connectivity, the following result of Chandran et al. is known.

Theorem 3.3 ([3]) *If G is a connected graph on n vertices with minimum degree δ, then $\mathrm{rc}(G) \le \frac{3n}{\delta+1} + 3$. Hence, if G is ℓ-connected, then $\mathrm{rc}(G) \le \frac{3n}{\ell+1} + 3$.*

We can ask the more general question with $\mathrm{rc}_k(G)$ instead. For 2-connected graphs, we have the following.

Theorem 3.4 ([7]) *If G is a 2-connected graph on n vertices, then $\mathrm{rc}_2(G) \le \frac{3n}{2}$.*

Sketch of proof. We may assume that G is minimally 2-connected. Take an ear decomposition of G. We first colour the initial cycle with distinct colours. Then, for each new ear that we attach, we colour it with distinct new colours, except when the ear has length 2, in which case we colour both edges with one new colour. Then one can check that for G, at most $\frac{3n}{2}$ colours are used, and that the resulting colouring is rainbow 2-connected. $\square$

A well-known sub-family of 2-connected graphs are those which are *series-parallel graphs*. A *2-connected series-parallel graph* is a (simple) graph which can be obtained from a K_3, and repeatedly applying a sequence of operations, each of which is a subdivision or replacement of an edge by a double edge. For these graphs, we can prove the following result.

Theorem 3.5 ([7]) *If G is a 2-connected series-parallel graph on n vertices, then $\mathrm{rc}_2(G) \le n$.*

Sketch of proof. This is similar to the previous proof. Again, assume that G is minimally 2-connected, and take an ear decomposition of G. Colour the initial cycle with distinct colours. But now, for each new ear that we attach, the fact that G is series-parallel allows us to colour it in such a way that exactly one edge uses a colour that has previously been used, the other edges are given distinct and new colours, while the rainbow 2-connected property is preserved. Then for the final graph G, n colours are used in total, and it is rainbow 2-connected. $\square$

We can ask the following, more general question. For $1 \leq k \leq \ell$, derive a sharp upper bound for $\mathrm{rc}_k(G)$, if G is an ℓ-connected graph on n vertices. Is there a constant $c = c(k, \ell)$ such that $\mathrm{rc}_k(G) \leq cn$?

4 Random Graphs

Another interesting question is to consider $\mathrm{rc}_k(G)$ in the *random graphs* setting. As usual, for $0 < p < 1$, let $G_{n,p}$ denote the random graph on n vertices, with edge probability p. For a graph property Q, we say that the function $f(n)$ is a *sharp threshold function* for Q if there are constants $c, C > 0$ such that $G_{n,cf(n)}$ almost surely does not satisfy Q, and $G_{n,p}$ almost surely satisfies Q for all $p \geq Cf(n)$. A result of Bollobás and Thomason [1] implies that any monotone (increasing) graph property has a sharp threshold function. Since the property $\mathrm{rc}(G_{n,p}) \leq 2$ is monotone, it has a sharp threshold function. For this property, Caro et al. proved the following result.

Theorem 4.1 **([2])** *$p = \sqrt{\log n/n}$ is a sharp threshold function for the property $\mathrm{rc}(G_{n,p}) \leq 2$.*

This result was generalised substantially by He and Liang, as follows.

Theorem 4.2 **([8])** *Let $d \geq 2$ be a fixed integer and $k = k(n) \leq O(\log n)$. Then $p = \frac{(\log n)^{1/d}}{n^{(d-1)/d}}$ is a sharp threshold function for the property $\mathrm{rc}_k(G_{n,p}) \leq d$.*

Here, our generalisation of Theorem 4.1 is as follows.

Theorem 4.3 **([7])** *$p = \sqrt{\log n/n}$ is a sharp threshold function for the property $\mathrm{rc}_k(G_{n,p}) \leq 2$ for all $k \geq 1$.*

We can also consider other models of random graphs. Recall that $G_{n,m,p}$ denotes the random bipartite graph with classes n and m, and edge probability p. Also, $G_{n,M}$ denotes the random graph on n vertices and M edges, with the space of all such graphs having the uniform probability distribution. We can likewise consider sharp threshold functions for these models, and again by the result of Bollobás and Thomason [1], all monotone properties in these models have a sharp threshold function. We have the following results.

Theorem 4.4 ([7]) *$p = \sqrt{\log n/n}$ is a sharp threshold function for the property $\mathrm{rc}_k(G_{n,n,p}) \leq 3$ for all $k \geq 1$.*

Theorem 4.5 ([7]) *$M = \sqrt{n^3 \log n}$ is a sharp threshold function for the property $\mathrm{rc}_k(G_{n,M}) \leq 2$ for all $k \geq 1$.*

It would be interesting to derive similar results for other models of random graphs. For example, random regular graphs.

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