

PROOF-THEORETIC SEMANTICS FOR SECOND-ORDER LOGIC

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ABSTRACT. We develop a proof-theoretic semantics (P-tS) for second-order logic (S-oL), providing an inferentialist alternative to both full and Henkin model-theoretic interpretations. Our approach is grounded in base-extension semantics (B-eS), a framework in which meaning is determined by inferential roles relative to atomic systems — collections of rules that encode an agent’s pre-logical inferential commitments. We show how both classical and intuitionistic versions of S-oL emerge from this set-up by varying the class of atomic systems. These systems yield modular soundness and completeness results for corresponding Hilbert-style calculi, which we prove equivalent to Henkin’s account of S-oL. In doing so, we reframe second-order quantification as systematic substitution rather than set-theoretic commitment, thereby offering a philosophically lightweight yet expressive semantics for higher-order logic. This work contributes to the broader programme of grounding logical meaning in use rather than reference and offers a new lens on the foundations of logic and mathematics.

1. INTRODUCTION

Second-order logic (S-oL) occupies a nuanced position in the foundations of logic and mathematics. Syntactically, it extends first-order logic (F-oL) by permitting quantification not only over individuals but also over properties, relations, and sets. Semantically and philosophically, however, it is often regarded as sitting between F-oL and full set theory: more expressive than the former, yet more constrained than the latter. As Väänänen [31] observes, this duality — being stronger than F-oL yet seemingly weaker than set theory — has long animated debates in logic and the philosophy of mathematics. He further notes that realizing the full strength of S-oL, especially in giving content to expressions like ‘for all properties’, requires invoking a set-theoretic background. This dependency raises foundational concerns: it seems to compromise S-oL’s logical autonomy by binding it to a certain mathematical metaphysics that one might have sought logic to justify.

Two primary approaches to the semantics of S-oL reflect this tension: standard semantics (also known as full semantics) and Henkin semantics. These approaches differ substantially in scope and foundational implications. Standard semantics supports the use of S-oL as a tool for categorically axiomatizing mathematical structures, while Henkin semantics enables its treatment as a syntactically well-behaved extension of F-oL.

Under standard semantics, second-order quantifiers range over the entire power set (or relation set) of the domain. Properties and relations are thus treated as

Key words and phrases. second-order logic, proof-theoretic semantics, base-extension semantics, inferentialism, soundness, completeness.

2010 Mathematics Subject Classification (MSC) codes: 03F05, 03B15, 03F03.

genuinely higher-order entities, modelled as sets or functions in a set-theoretic universe. This makes S-oL categorically powerful — capable of uniquely characterizing structures like the natural numbers and the real number continuum — but at a cost: the logic is incomplete, non-compact, and lacks any sound, recursively enumerable axiomatization. It thereby inherits the complexities of set-theoretic reasoning.

Henkin semantics, by contrast, restricts second-order quantifiers to range over a designated collection of subsets or relations, not necessarily the full power set. In this framework, S-oL is effectively a many-sorted or higher-order extension of F-oL. It admits a sound and complete deductive system and satisfies the compactness and Löwenheim–Skolem theorems. However, it loses much of the expressive and categorical strength that distinguishes S-oL in the standard setting. Notably, categoricity results for second-order Peano arithmetic or analysis no longer hold.

This paper proposes a new perspective on S-oL. Both standard and Henkin semantics are versions of *model-theoretic semantics* (M-tS) for S-oL. They are based in the philosophical position of *denotationalism*, where meaning and validity are grounded in reference and representation. This is what irrevocably connects them to set theory. In this paper, we provide an alternate account altogether based in terms of *proof-theoretic semantics* (P-tS).

The philosophical background to P-tS is *inferentialism*. Broadly speaking, the meaning of an expression is given entirely by the role that expression plays in inference (rather than, for example, its truth conditions). It has a long intellectual history and can be seen as a particular interpretation of Wittgenstein’s ‘meaning-as-use’ principle [33] in which ‘use’ is given by inference. Its formulation begun with Gentzen’s belief that introduction rules are definitional [5] and Dummett’s [4] anti-realist account of logic. It was later developed into a theory of meaning by Brandom [2].

The inferentialist approach to meaning is quite natural. What does the proposition ‘Tammy is a vixen’ mean? Intuitively, it means ‘Tammy is female’ and ‘Tammy is a fox’. In particular, from the fact that Tammy is a vixen, we can infer both that she is a fox and that she is female. In the standard format of proof rules, we have:

$$\frac{\text{Tammy is a vixen}}{\text{Tammy is a fox}} \quad \text{and} \quad \frac{\text{Tammy is a vixen}}{\text{Tammy is female}}$$

Similarly, if we have that Tammy is a fox and that Tammy is female, then we may infer that she is a vixen:

$$\frac{\text{Tammy is female} \quad \text{Tammy is a fox}}{\text{Tammy is a vixen}}$$

The inferentialist claim is that in providing these rules, we have provided the meaning of the expression. Observe the paradigmatic shift from denotationalism to inferentialism: rather than interpreting ‘Tammy’ and ‘vixen’ in a model, we express their meaning through inferential behaviours.

That explains the inferentialism as a theory of meaning in natural language. Within logic this conception of meaning required grounding validity in ‘proof’ rather than ‘truth’. This requires some unpacking as ‘proof’ here cannot mean proof in a given deduction system since it must be pre-logical in order to define the logical signs. Therefore we require a pre-logical notion of proof. Such an account has been given by Prawitz [20].

To begin, let us consider how we might express the example concerning Tammy in a symbolic language? This requires fixing an idea of rules over atomic propositions (i.e., propositions without any logical signs). Using natural deduction in the form of Gentzen [5], we take rules of the form,

$$\frac{}{C} \quad \frac{P_1 \quad \dots \quad P_n}{C} \quad \frac{\frac{[\mathbb{P}_1]}{P_1} \quad \dots \quad \frac{[\mathbb{P}_n]}{C}}{C}$$

Here C, P_1, \dots, P_n are all atomic propositions and $\mathbb{P}_1, \dots, \mathbb{P}_n$ are sets thereof. The latter rule allows each of the p_i s to be proved from a set of discharged hypotheses $[\mathbb{P}_i]$. Given such rule format, it is clear that the example above concerning Tammy can be encoded with the rules

$$\frac{V(t)}{Fe(t)} \quad \frac{V(t)}{Fo(t)} \quad \frac{Fe(t) \quad Fo(t)}{V(t)}$$

in which Fe is the predicate ‘is female’, Fo is the predicate ‘is a fox’, V is the predicate ‘is a vixen’ and t stands for ‘Tammy’.

Piecha and Schroeder-Heister [18, 29] and Sandqvist [26] have considered the philosophical commitments made in the choice of format for atomic rule. A collection of such atomic rules is called an ‘atomic system’ or a ‘base’. We think of such bases \mathcal{B} as the set of inferential commitments that an agent poses and that these commitments represents their semantic universe.

Gentzen [5] observed that rules may be thought to explicate the meaning of logical constants in this way. For example, the rules

$$\frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi} \quad \frac{\varphi \quad \psi}{\varphi \wedge \psi}$$

intuitively define conjunction in the same way that the rules above define ‘Tammy is a vixen’. Indeed, Gentzen [5] specifically thought that the introduction rules of his natural deduction systems define the logical signs and that the eliminations rules followed by some principle. Prawitz [20] used his normalization results to deliver on this idea. We provide a terse account here and defer to Schroeder-Heister [27] for a more complete picture.

Rather than dealing with the semantics of logical constants themselves, Prawitz [20] first ask what makes a ‘proof’ valid? This is because a corollary of his normalization theory is that proofs in Gentzen’s NJ, without loss of generality, end with the introduction rules. Let \mathbf{L} be a system of proof rules and let \mathbf{J} be a procedure on proof-structures that yields proof-structures (e.g., Prawitz’s reductions [21] for normalizing proofs). Define validity relative to a base \mathcal{B} and the procedures \mathbf{J} as follows:

- There is an *a priori* collection $\mathbf{C}(\mathbf{L})$ of $\langle \mathcal{B}, \mathbf{J} \rangle$ -valid proofs of \mathbf{L} .
- A completed proof-structure Φ is $\langle \mathcal{B}, \mathbf{J} \rangle$ -valid if \mathbf{J} can be applied to Φ to yield an element of $\mathbf{C}(\mathbf{L})$.
- An incomplete proof-structure Φ is $\langle \mathcal{B}, \mathbf{J} \rangle$ -valid if any completion of it is $\langle \mathcal{C}, \mathbf{J} \rangle$ -valid for any $\mathcal{C} \supseteq \mathcal{B}$.

Prawitz [20] conjectured that in his set-up (i.e., \mathbf{L} as NJ and \mathbf{J} as his reduction) a proof-structure Φ would be $\langle \mathcal{B}, \mathbf{J} \rangle$ -valid for arbitrary \mathcal{B} iff the conclusion of Φ is valid in intuitionistic logic. This turns out to be false.

Before continuing, we would like to briefly remark on the treatment of ‘incomplete’ proof-structures. For Prawitz [20], these were open natural deduction derivations — that is, derivations with undischarged hypotheses. Schroeder-Heister [28] has explained that Prawitz’s semantics is closely related to the BHK-interpretation of intuitionism. Intuitively, a ‘construction’ of an implication $\varphi \rightarrow \psi$ is one that given a construction of φ yields a construction of ψ . However, such a condition on a construction of $\varphi \rightarrow \psi$ would be satisfied vacuously if there be no construction of φ (relative to \mathcal{B} and \mathbf{J}). It follows (cf. Kripke’s semantics [15]) that we should consider arbitrary extensions $\mathcal{C} \subseteq \mathcal{B}$ that enrich our ‘understanding’ so that we have constructions for φ .

We thus have a semantics of proofs. From this we get a semantics of the logical constants *in terms of* proofs. A formula φ is \mathcal{B} -valid iff, for any base \mathcal{B} , there is a $\langle \mathcal{B}, \mathbf{J} \rangle$ -valid proof-structure Φ concluding φ . Let’s write $\Vdash_{\mathcal{B}} \varphi$ (read ‘ \mathcal{B} supports φ ’) to denote this situation. Depending on the specifics of the set-ups (e.g., the choice of \mathbf{J}), we may unfold this into a set of clauses. For example, Prawitz’s [20] set-up of disjunction may given by the clause:

$$\Vdash_{\mathcal{B}} \varphi \vee \psi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \varphi \text{ or } \Vdash_{\mathcal{B}} \psi$$

That is, there is a $\langle \mathcal{B}, \mathbf{J} \rangle$ -valid proof Θ concluding $\varphi \vee \psi$ iff there is a $\langle \mathcal{B}, \mathbf{J} \rangle$ -valid proof Φ concluding either φ and ψ . This follows from the fact that normalized NJ-proofs concluding a disjunction end by \vee -introduction and so contain a sub-proof of one of the disjuncts.

Piecha et al. [3, 17, 19] studied the judgment $\Vdash_{\mathcal{B}}$ that arises from systematically writing its clauses like this. The implication is handled using base-extension following the remark above. That is, we have

$$\Vdash_{\mathcal{B}} \varphi \rightarrow \psi \quad \text{iff} \quad \varphi \Vdash_{\mathcal{B}} \psi$$

where, for $\Delta \neq \emptyset$,

$$\Delta \Vdash_{\mathcal{B}} \psi \quad \text{iff} \quad \text{for all } \mathcal{C} \supseteq \mathcal{B}, \text{ if } \Vdash_{\mathcal{C}} \psi \text{ for } \psi \in \Gamma, \text{ then } \Vdash_{\mathcal{C}} \psi$$

At first glance, this semantics appear to recall Kripke’s semantics for intuitionistic logic [15] using bases \mathcal{B} for worlds. This impression is misleading. Piecha et al. [3, 17, 19] discovered that the resulting semantics validates the Kreisel-Putnam (KP) axiom. That is, for any \mathcal{B} ,

$$\Vdash_{\mathcal{B}} (a \rightarrow (b \vee c)) \rightarrow ((a \rightarrow b) \vee (a \rightarrow c))$$

However, KP is not intuitionistically valid and, therefore, this is not a semantics for intuitionistic logic and Prawitz’s conjecture fails. Stafford [30] later showed that this actually provides a semantics for an intermediate logic he called *generalized inquisitive logic*.

In parallel to this approach to P-tS, Sandqvist [25] developed a similar ‘support’ (\Vdash) relation. The biggest difference is the treatment of disjunction for which he uses the following clause instead:

$$\Vdash_{\mathcal{B}} \varphi \vee \psi \quad \text{iff} \quad \forall \mathcal{C} \supseteq \mathcal{B}, \forall P, \text{ if } \varphi \Vdash_{\mathcal{C}} P \text{ and } \psi \Vdash_{\mathcal{C}} P, \text{ then } \Vdash_{\mathcal{C}} P$$

The semantics is summarized in Figure 1 where $\vdash_{\mathcal{B}}$ denotes derivability in the \mathcal{B} and Δ is a non-empty set of formulae. He established the soundness and completeness of NJ [5] (\vdash) with respect to this semantics. For finite Γ ,

- Soundness: $\Gamma \vdash \varphi$ implies $\Gamma \Vdash \varphi$

$\Vdash_{\mathcal{B}} P$	iff	$\vdash_{\mathcal{B}} P$	(At)
$\Vdash_{\mathcal{B}} \varphi \wedge \psi$	iff	$\Vdash_{\mathcal{B}} \varphi$ and $\Vdash_{\mathcal{B}} \psi$	(\wedge)
$\Vdash_{\mathcal{B}} \varphi \vee \psi$	iff	$\forall \mathcal{C} \supseteq \mathcal{B}, \forall P$, if $\varphi \Vdash_{\mathcal{C}} P$ and $\psi \Vdash_{\mathcal{C}} P$, then $\Vdash_{\mathcal{C}} P$	(\vee)
$\Vdash_{\mathcal{B}} \perp$	iff	$\Vdash_{\mathcal{B}} P$ for any atom P	(\perp)
$\Vdash_{\mathcal{B}} \varphi \rightarrow \psi$	iff	$\varphi \Vdash_{\mathcal{B}} \psi$	(\rightarrow)
$\Delta \Vdash_{\mathcal{B}} \varphi$	iff	$\forall \mathcal{C} \supseteq_{\mathfrak{B}} \mathcal{B}$, if $\Vdash_{\mathcal{C}} \psi$ for all $\psi \in \Delta$, then $\Vdash_{\mathcal{C}} \varphi$	(Inf)

FIGURE 1. Sandqvist's Base-extension Semantics

- Completeness: $\Gamma \Vdash \varphi$ implies $\Gamma \vdash \varphi$.

Pym et al. [22] have shown that this semantics is fully natural in the sense of categorical logic. Importantly, for these theorems to hold, the set of atoms (P s) is assumed to be denumerably infinite and we must admit all atomic rules of the forms given above including discharge. Sandqvist [23, 24] (see also Makinson [16]) also showed that when restricting to atomic systems that do not include discharge, the result yields classical logic.

Having sketched the background to, and key aspects of, P-tS, we can ask why does having a P-tS for S-oL matter? As mentioned, the standard interpretations of S-oL inherit not just technical features but also the philosophical burdens of set theory because of its grounding in denotationalism. For example, the expression ‘for all properties’ presupposes a background ontology of sets or relations — an assumption that ties the logic to precisely the kind of mathematical metaphysics it might otherwise have been expected to illuminate. Even Henkin semantics, while more proof-theoretically tractable, remains embedded in a model-theoretic framework, treating rules of inference as subordinate to pre-given semantic structures. From an inferentialist standpoint, this order of explanation is inverted: what gives logical expressions their meaning is not their denotation, but their use in inference.

A P-tS for S-oL thus serves a dual purpose. Philosophically, it offers an anti-realist account of higher-order logic that avoids the ontological commitments of standard semantics. Rather than appealing to a universe of sets, it aims to ground the meaning of second-order quantification in the roles such quantifiers play in deduction. Technically, it reframes S-oL not as a logic whose expressiveness must be justified by set-theoretic resources, but as a system whose coherence and content can be understood internally, through the rules governing introduction, elimination, and transformation.

Moreover, such a perspective opens the possibility of treating S-oL as a kind of inferentialist analogue to set theory. The expressive power that allows S-oL to formulate comprehension, categoricity, and induction principles need not be seen as a reflection of ontological depth; rather, it can be reconstructed as a consequence of the inferential structure of the logic itself. This resonates with long-standing intuitions that much of mathematics — particularly arithmetic and analysis — can be grounded in logical reasoning, without presupposing a realist ontology of sets.

Finally, S-oL plays a crucial role as a metalanguage: it is often the formal system within which we specify semantics, express reflection principles, or formulate general

logical theories. But if the semantics of this metalanguage is itself given only model-theoretically, we risk circularity or infinite regress. P-tS offers a way of internalizing the metalanguage, grounding its expressions in rules rather than in external models. In this way, S-oL becomes not just an object of inferentialist analysis, but a vehicle for advancing the broader project of P-tS as a foundational framework.

In Section 2, we introduce the syntax of second-order logic and present a proof-theoretic semantics using base-extension semantics. This framework interprets logical expressions through their inferential role relative to a class of atomic systems. In Section 3, we develop Hilbert-style calculi for classical and intuitionistic S-oL and prove soundness and completeness theorems with respect to our semantics. Section 4 demonstrates the equivalence of these systems with Henkin-style natural deduction calculi, thereby showing that our proof-theoretic semantics recovers standard S-oL in its Henkin form. We conclude in Section 5 by reflecting on the philosophical significance of our approach and outlining directions for further research, including the challenge of recovering full second-order semantics within the inferentialist paradigm.

2. SECOND-ORDER LOGIC

The P-tS of S-oL in this paper builds on that of F-oL provided by Gheorghiu [8]. In the first-order setting, the key steps are the following:

- Defining an appropriate notion of base to ground the semantics. Atomic rules are defined over closed atomic formulas only. Using open formulas does not increase the expressive power of the semantics.
- Providing semantics clauses for the quantifiers \forall and \exists . The signs are read literally over the syntax — for example, $\forall x\varphi$ is understood to mean any instance of φ with x replaced by a closed term t .
- Proving sound and completeness with respect to deduction systems. It adapts a technique by Sandqvist [25] requiring careful handling of discharge and using eigenvariables (fresh constants) to handle the quantifiers.

The treatment of S-oL adopts and simplifies this set-up. An important difference is that we do not only consider predicates $P(-, \dots, -)$ with positive arity, but also propositional atoms P . This enables us to give a cleaner and more contained treatment of the semantics and the proof of completeness.

One feature that merits a remark is that the ontology is given by the syntax which is fixed before the logic is defined. By contrast, in the M-tS of quantifier logics the quantifiers are handled by a ‘model update’ in which we name things in a given universe. Let \mathfrak{M} be a first-order structure over a domain D and let s be an assignment of variables. Then:

$$\mathfrak{M}, s \models \forall x \varphi(x) \quad \text{iff} \quad \text{for all } d \in D, \mathfrak{M}, s[x \mapsto d] \models \varphi(x)$$

That we have soundness and completeness results with a fixed ontology may be surprising to those holding on to a denotational or realist account of meaning. It is, however, as expected in the inferentialist, anti-realist character of P-tS.

2.1. Syntax of Second-order Logic. To begin, we fix disjoint denumerable sets of symbols for every $n \geq 0$:

- ICON — individual-constants a, b, c, \dots
- IVAR — individual-variables x, y, z, \dots

- PCONS_n — predicate-constant of arity n , P, Q, R, \dots
- PVAR_n — predicate-variables of arity n , X, Y, Z, \dots

We define the following sets as the collections of all predicate-constants and predicate-variables, respectively:

$$\text{PCONS} := \bigcup_n \text{PCONS}_n \quad \text{PVAR} := \bigcup_n \text{PVAR}_n$$

We may write $P^{(n)}$ (resp. $X^{(n)}$) to denote that P is a predicate-constant (resp. predicate-variable) of arity n . Given a predicate-constant P or predicate-variable X , we may write $\alpha(P)$ and $\alpha(X)$ to denote their arities.

The elements $\text{ICON} \cup \text{IVAR}$ are *terms*. The atomic formulae are defined by the following:

- If t_1, \dots, t_n are terms and $P \in \text{ICON}_n$, then $R(t_1, \dots, t_n)$ is an atom
- If t_1, \dots, t_n are terms and $X \in \text{PVAR}_n$, then $X(t_1, \dots, t_n)$ is an atom

The set of all atoms is denoted ATOM . We define formulae by the following grammar as x ranges over IVAR and X ranges over IVAR :

$$\varphi ::= A \in \text{ATOM} \mid \varphi \rightarrow \varphi \mid \forall x \varphi \mid \forall X \varphi$$

The set of all formulae is denoted FORM . We may use the following abbreviations:

- $\perp := \forall X^0(X^0)$
- $\varphi \wedge \psi := \forall X^0((\varphi \rightarrow (\psi \rightarrow X^0)) \rightarrow X^0)$
- $\varphi \vee \psi := \forall X^0((\varphi \rightarrow X^0 \wedge \psi \rightarrow X^0) \rightarrow X^0)$
- $\exists x \varphi := \forall X^0((\forall x \varphi \rightarrow X^0) \rightarrow X^0)$
- $\exists X \varphi := \forall X^0((\forall X \varphi \rightarrow X^0) \rightarrow X^0)$

To handle quantifiers we will require the usual notion of *free variable*. This concept is doubtless familiar (cf. van Dalen [32]) so we simply fix some notation:

- we may write $\text{FIV}(\varphi)$ to denote the set of free individual-variables in φ
- we may write $\text{FPV}(\varphi)$ to denote the set of free predicate-variables in φ

The notations may be extended to sets through point-wise union; that is,

$$\text{FIV}(\Gamma) := \bigcup_{\varphi \in \Gamma} \text{FIV}(\varphi) \quad \text{and} \quad \text{FPV}(\Gamma) := \bigcup_{\varphi \in \Gamma} \text{FPV}(\varphi)$$

The point of individual- and predicate-variables is that they may be replaced by terms (of the appropriate type and arity). To this end, we require substitutions. We write $[x \mapsto t]$, where x is an individual-variables and t is a term, to denote the substitution of x by t :

$$\varphi[x \mapsto t] := \begin{cases} P(t_1[x \mapsto t], \dots, t_n[x \mapsto t]) & \text{if } \varphi = P(t_1, \dots, t_n) \\ \forall y(\psi[x \mapsto t]) & \text{if } \varphi = \forall y \psi \text{ and } y \neq x \\ \forall y \psi & \text{if } \varphi = \forall y \psi \text{ and } y = x \\ \forall X^n(\psi[x \mapsto t]) & \text{if } \varphi = \forall X^n \psi \end{cases}$$

Similarly, we may write $[X^n \mapsto P^n]$, where X^n is predicate-variable of arity n and P is a predicate-constant of arity n , to denote substitution of X^n by P^n :

$$\varphi[X^n \mapsto P^n] := \begin{cases} P(t_1, \dots, t_n) & \text{if } \varphi = X^n(t_1, \dots, t_n) \\ \forall y(\psi[X^n \mapsto P^n]) & \text{if } \varphi = \forall y \psi \\ \forall Y^n(\psi[X^n \mapsto P^n]) & \text{if } \varphi = \forall Y^n \psi \text{ and } Y^n \neq X^n \\ \forall Y^n \psi & \text{if } \varphi = \forall Y^n \psi \text{ and } Y^n = X^n \end{cases}$$

As these substitutions always respect arity, henceforth we may elide it so that in writing $[X \mapsto P]$ it is assumed that X and P have the same arity.

2.2. Base-extension Semantics. The semantics is grounded by derivability in an *atomic system*. An atomic system may be thought to represent an agent's inferential commitments about some basic sentences. By ‘inference’ here, we mean that if an agent is committed to the premises and the inference, then they are likewise committed to the conclusion. For example, Aristotle might accept the basic sentence

Socrates is human

as well as the inference

$$\frac{\text{Socrates is human}}{\text{Socrates is mortal}}$$

As a consequence of these commitments, Aristotle accepts the assertion ‘Socrates is mortal’, as it follows from them.

Working within predicate logic we are faced with a choice: should open atomic formulas count as atomic sentences or only closed ones? If so, one could naturally express general commitments such as ‘whoever is human is also mortal’ using the inference:

$$\frac{x \text{ is human}}{x \text{ is mortal}}$$

However, we aim to model positions that capture inferential relationships between complete thoughts. For this reason, we restrict our attention to atomic *sentences*. Accordingly, we represent generalizations such as the above by including, for each name t , the inference:

$$\frac{t \text{ is human}}{t \text{ is mortal}}$$

Both approaches are mathematically viable, but we believe the use of closed atoms more faithfully reflects our intended interpretation. The equivalence in expressive power between the two approaches follows from Gheorghiu [8].

Definition 2.1 (Atomic Rule). *An atomic rule is an expression of the form*

$$\{\mathbb{P}_1 \Rightarrow P_1, \dots, \mathbb{P}_n \Rightarrow P_n\} \Rightarrow P$$

where each $\mathbb{P}_i \subseteq \overline{ATOM}$ and each $P_i \in \overline{ATOM}$. The rule is:

- zero-level if $n = 0$,
- first-level if $n > 0$ and $\mathbb{P}_1 = \dots = \mathbb{P}_n = \emptyset$,
- second-level otherwise.

Definition 2.2 (Atomic System). *An atomic system \mathcal{S} is a set of atomic rules.*

Definition 2.3 (Derivability in an Atomic System). *Let \mathcal{S} be an atomic system. The derivability relation $\mathbb{P} \vdash_{\mathcal{S}} P$ is defined inductively as follows:*

- (REF) If $P \in \mathbb{P}$, then $\mathbb{P} \vdash_{\mathcal{S}} P$.
- (APP) If $\{\mathbb{P}_1 \Rightarrow P_1, \dots, \mathbb{P}_n \Rightarrow P_n\} \Rightarrow P \in \mathcal{S}$, and for each i , $\mathbb{P} \cup \mathbb{P}_i \vdash_{\mathcal{S}} P_i$, then $\mathbb{P} \vdash_{\mathcal{S}} P$.

Let's return to the example of Aristotle. His position can now be formalized as an atomic system \mathcal{A} containing the rules

$$H(s) \quad \text{and} \quad H(s) \Rightarrow M(s)$$

From this system, we can derive that $\vdash_{\mathcal{A}} M(s)$, reflecting Aristotle’s inferential commitment to the claim that Socrates is mortal.

The structure of this formalism suggests an analogy to natural deduction, in the style of Gentzen [5]:

$$\frac{\begin{array}{ccc} [\mathbb{P}_1] & & [\mathbb{P}_n] \\ P_1 & \cdots & P_n \end{array}}{P}$$

This resemblance is confirmed by Definition 2.3 which treats \Rightarrow for atomic systems just as the horizontal bar is treated in natural deduction. However, since rule application in this setting does not involve substitution, the framework is not exactly the same. A closer structural resemblance to hereditary Harrop formulas (cf. Gheorghiu and Pym [12]) or intuitionistic resolution (cf. Gheorghiu [6]).

We may refine the theory of atomic systems by fixing collections of them relative to which we work. Such collections result in different logics. We defer to Piecha and Schroeder-Heister [29, 18] for further discussion on the philosophical difference between them. Accordingly, we introduce the idea of a *basis*, which specifies a set of atomic systems.

Definition 2.4 (Basis). *A basis \mathfrak{B} is a set of atomic systems.*

Given a basis \mathfrak{B} , its elements are called *bases* \mathcal{B} . Once a basis is fixed, we always work relative to its bases. We therefore introduce the notion of *base-extension*, a restricted version of superset (or ‘extension’) of an atomic system that respects the given basis.

Definition 2.5 (Base-extension). *Given a basis \mathfrak{B} , base-extension is the least relation satisfying the following:*

$$\mathcal{Y} \supseteq_{\mathfrak{B}} \mathcal{X} \quad \text{iff} \quad \mathcal{X}, \mathcal{Y} \in \mathfrak{B} \text{ and } \mathcal{Y} \supseteq \mathcal{X}$$

Having fixed a basis, the meaning of the logical signs is given by clauses that collectively define a semantic judgment called *support* (\Vdash). We can justify the clauses in turn by the intended reading of the logical sign they define. Since a base \mathcal{B} represents an agent’s beliefs expressed inferentially, an atom A is said to be supported in \mathcal{B} just in case A can be derived from those beliefs. This is captured formally as:

$$\Vdash_{\mathcal{B}} A \quad \text{iff} \quad \vdash_{\mathcal{B}} A$$

An intuitive reading of implication $\varphi \rightarrow \psi$ is this: An agent believes the implication iff, supposing they were also to believe φ , would consequently believe ψ . We can express this condition schematically as:

$$\Vdash_{\mathcal{B}} \varphi \rightarrow \psi \quad \text{iff} \quad \varphi \Vdash_{\mathcal{B}} \psi$$

But what exactly do we mean by this? In general, we must specify what it should mean for an agent with beliefs \mathcal{B} to conclude ψ were they to commit to a (non-empty) set of formulae Δ . The use of the subjunctive mood in articulating this condition suggests an appeal to hypothetical scenarios — specifically, *extensions* of the belief base that ensure support for every formula in Δ . This motivates the notion of *base extension*. Formally, we define:

$$\Delta \Vdash_{\mathcal{B}} \varphi \quad \text{iff} \quad \text{for all } \mathcal{C} \supseteq_{\mathfrak{B}} \mathcal{B}, \text{ if } \Vdash_{\mathcal{C}} \psi \text{ for each } \psi \in \Delta, \text{ then } \Vdash_{\mathcal{C}} \varphi$$

It remains to consider quantification. Consider the statement ‘for any person, if that person is human, then they are mortal’. What we typically mean is that for

$\Vdash_{\mathcal{B}} P$	iff $\vdash_{\mathcal{B}} P$	(At)
$\Vdash_{\mathcal{B}} \varphi \rightarrow \psi$	iff $\varphi \Vdash_{\mathcal{B}} \psi$	(\rightarrow)
$\Vdash_{\mathcal{B}} \forall x \varphi$	iff $\Vdash_{\mathcal{B}} \varphi[x \mapsto a]$ for any $a \in \text{ICON}$	(\forall)
$\Vdash_{\mathcal{B}} \forall X \varphi$	iff $\Vdash_{\mathcal{B}} \varphi[X \mapsto P]$ for any appropriate $P \in \text{PCONS}$	(\forall)
$\Delta \Vdash_{\mathcal{B}} \varphi$	iff $\forall \mathcal{C} \supseteq_{\mathfrak{B}} \mathcal{B}$, if $\Vdash_{\mathcal{C}} \psi$ for any $\psi \in \Delta$, then $\Vdash_{\mathcal{C}} \varphi$	(Inf)
$\Gamma \Vdash \varphi$	iff $\Gamma \Vdash_{\mathcal{B}} \varphi$ for any $\mathcal{B} \in \mathfrak{B}$	

FIGURE 2. Base-extension Semantics for Second-order Logic

any name we can introduce — say, *Socrates* — the corresponding instance (e.g., ‘if Socrates is human, then Socrates is mortal’) is one we accept. Here lies a key point of divergence between model-theoretic and proof-theoretic semantics: in model-theoretic accounts, quantifiers range over a realist ontology of actual individuals; in proof-theoretic semantics, the ontology is anti-realist, constructed from the names present in the language.

Accordingly, support for a universal formula is defined by substituting the quantified variable with any available individual constant:

$$\Vdash_{\mathcal{B}} \forall x \varphi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \varphi[x \mapsto a] \text{ for all } a \in \text{ICON}$$

We treat second-order quantification analogously:

$$\Vdash_{\mathcal{B}} \forall X \varphi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \varphi[X \mapsto P] \text{ for all } P \in \text{PCONS}$$

Strictly speaking, we should write $P \in \text{PCONS}_{\alpha(X)}$, where $\alpha(X)$ denotes the arity of X , but this should be clear from the context and the definition of substitution.

Definition 2.6 (Support). *Let \mathfrak{B} be a basis and $\mathcal{B} \in \mathfrak{B}$. Support is the smallest relation \Vdash defined by the clauses of Figure 2 in which all formulae are closed, Δ is a non-empty set of closed formulae, and Γ is a finite (possibly empty) set of formulae.*

Of course, using the abbreviations discussed above, we also have the following:

$$\Vdash_{\mathcal{B}} \perp \quad \text{iff} \quad \vdash_{\mathcal{B}} P \text{ for any } P \in \text{PCONS}_0 \quad (\perp)$$

$$\begin{aligned} \Vdash_{\mathcal{B}} \varphi \wedge \psi \quad \text{iff} \quad & \text{for any } \mathcal{C} \supseteq_{\mathfrak{B}} \mathcal{B} \text{ and } P \in \text{PCONS}_0, \\ & \text{if } \varphi, \psi \Vdash_{\mathcal{C}} P, \text{ then } \Vdash_{\mathcal{C}} P \end{aligned} \quad (\wedge)$$

$$\begin{aligned} \Vdash_{\mathcal{B}} \varphi \vee \psi \quad \text{iff} \quad & \text{for any } \mathcal{C} \supseteq_{\mathfrak{B}} \mathcal{B} \text{ and } P \in \text{PCONS}_0, \\ & \text{if } \varphi \Vdash_{\mathcal{C}} P \text{ and } \psi \Vdash_{\mathcal{C}} P, \text{ then } \Vdash_{\mathcal{C}} P \end{aligned} \quad (\vee)$$

$$\begin{aligned} \Vdash_{\mathcal{B}} \exists x \varphi \quad \text{iff} \quad & \text{for any } \mathcal{C} \supseteq_{\mathfrak{B}} \mathcal{B} \text{ and } t \in \text{ICON}, \\ & \text{if } \varphi[x \mapsto t] \Vdash_{\mathcal{C}} P, \text{ then } \Vdash_{\mathcal{C}} P \end{aligned} \quad (\exists)$$

$$\begin{aligned} \Vdash_{\mathcal{B}} \exists X \varphi \quad \text{iff} \quad & \text{for any } \mathcal{C} \supseteq_{\mathfrak{B}} \mathcal{B} \text{ and } Q \in \text{PCONS}, \\ & \text{if } \varphi[X \mapsto Q] \Vdash_{\mathcal{C}} P, \text{ then } \Vdash_{\mathcal{C}} P \end{aligned} \quad (\exists)$$

These clauses directly recover the B-eS of propositional and first-order intuitionistic and classical logic — see Sandqvist [23, 24, 25] and Gheorghiu [8].

We may also extend support to capture open formulae by treating them as quantified. If $x \in \text{FIV}(\varphi)$, then

$$\Vdash \varphi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \varphi[x \mapsto a] \text{ for any } a \in \text{ICON}.$$

$\varphi \rightarrow (\psi \rightarrow \varphi)$	(K)
$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$	(S)
$\forall x \varphi \rightarrow \varphi[t \mapsto x]$	(\forall E)
$\forall X \varphi \rightarrow \varphi[P \mapsto X]$	(\forall E)
$(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg \psi) \rightarrow \neg \varphi)$	(\neg I)
$(\varphi \rightarrow \perp) \rightarrow (\varphi \rightarrow \psi)$	(efq)
.....	
$\neg \neg \varphi \rightarrow \varphi$	(dne)

FIGURE 3. Axiomatization of Second-order Logic, HI and HC

If $X \in \text{FPV}(\varphi)$, then

$$\Vdash \varphi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \varphi[X \mapsto P] \text{ for any appropriate } P \in \text{PCONS}.$$

This completes the definition of the semantic clauses. Intuitively, logical consequences should not depend on any particular inferential commitments an agent may hold; they should follow solely from the logical form of the formulae involved. We therefore quantify over all bases and define $\Gamma \Vdash \varphi$ to mean $\Gamma \Vdash_{\mathcal{B}} \varphi$ for arbitrary \mathcal{B} . Intuitively, this is equivalent to requiring that no specific commitments are assumed—that is, $\Gamma \Vdash_{\emptyset} \varphi$.

Lemma 2.7. *If $\emptyset \in \mathfrak{B}$,*

$$\Gamma \Vdash \varphi \quad \text{iff} \quad \Gamma \Vdash_{\emptyset} \varphi$$

Proof. This follows as in Sandqvist [25] (discussion below Lemma 3.2). □

3. PROOF THEORY

Having now provided a semantic account of second-order logic (S-oL), we turn to its axiomatization. We will show that the choice of base system affects the resulting axiomatization. In particular, we focus on two canonical options:

- $\mathfrak{C} := \{\mathcal{A} \mid \mathcal{A} \text{ is zero- or first-level}\}$
- $\mathfrak{I} := \{\mathcal{A} \mid \mathcal{A} \text{ is zero-, first-, or second-level}\}$

The basis \mathfrak{C} yields a classical S-oL and the basis \mathfrak{I} yields an intuitionistic S-oL (in the sense that it does not accept the law of double-negation elimination). Although this can be shown mathematically, a clear intuitive explanation for why this distinction arises remains elusive.

3.1. Hilbert Calculus. We will use Hilbert calculi as the medium of proof. As they are doubtless familiar, we give a terse account introducing only the mathematically relevant content for this paper.

We will be working with the following sets of axioms:

- HC comprises all the axioms in Figure 3, including dne
- HI comprises all the axioms in Figure 3, excluding dne.

Definition 3.1 (Consequence from an Axiomatization). *Let H be a Hilbert calculus. The H -consequence relation \vdash_H is defined inductively as follows:*

- AXIOM. If $\varphi \in \mathbf{H}$ then $\Gamma \vdash_{\mathbf{H}} \varphi$
- HYPOTHESIS. If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathbf{H}} \varphi$
- MODUS PONENS. If $\Gamma \vdash_{\mathbf{H}} \varphi$ and $\Gamma \vdash_{\mathbf{H}} \varphi \rightarrow \psi$, then $\Gamma \vdash_{\mathbf{H}} \psi$
- FIRST-ORDER GENERALIZATION. If $\Gamma \vdash_{\mathbf{H}} \psi \rightarrow \varphi$ and $x \notin \text{FIV}(\psi, \Gamma)$, then $\Gamma \vdash_{\mathbf{H}} \psi \rightarrow \forall x \varphi$
- SECOND-ORDER GENERALIZATION. If $\Gamma \vdash_{\mathbf{H}} \psi \rightarrow \varphi$ and $X \notin \text{FPV}(\psi, \Gamma)$, then $\Gamma \vdash_{\mathbf{H}} \psi \rightarrow \forall X \varphi$.

Before proceeding, we state some elementary results about these proof systems. In the following, $\mathbf{A} \in \{\mathbf{HI}, \mathbf{HC}\}$:

Proposition 3.2. *If $\Gamma \vdash_{\mathbf{A}} \varphi \rightarrow \psi$ iff $\varphi, \Gamma \vdash_{\mathbf{A}} \psi$*

Proposition 3.3. *If $\Gamma \vdash_{\mathbf{A}} \varphi$ and e is an individual-constant (resp. E is a predicate-constant) that possibly occurs in φ but does not occur in Γ , then $\Gamma \vdash_{\mathbf{A}} \varphi[e \mapsto x]$ (resp. $\Gamma \vdash_{\mathbf{A}} \varphi[E \mapsto X]$).*

Our soundness and completeness results are as follows:

Theorem. *Let $\mathfrak{B} = \mathfrak{C}$ (resp. $\mathfrak{B} = \mathfrak{J}$):*

$$\Gamma \Vdash \varphi \quad \text{iff} \quad \Gamma \vdash_{\mathbf{HC}} \varphi \text{ (resp. } \Gamma \vdash_{\mathbf{HI}} \varphi \text{)}$$

The remainder of this section is concerned with proving this theorem.

3.2. Soundness. We show that, according to the choice of basis, support satisfies the inductive definition of provability in \mathbf{HC} and \mathbf{HI} . Firstly, Lemma 3.4 claims that the common axioms in \mathbf{HC} and \mathbf{HI} hold for both choices of basis \mathfrak{C} and \mathfrak{J} . Secondly, Lemma 3.5 claims that dne holds for \mathfrak{C} . Finally, Lemma 3.6 shows that the inductive definition of consequence (Definition 3.1) holds for both \mathfrak{C} and \mathfrak{J} with the axioms \mathbf{HC} and \mathbf{HI} , respectively.

Lemma 3.4. *If $\mathfrak{B} \in \{\mathfrak{C}, \mathfrak{J}\}$, then $\Vdash \varphi$ for each $\varphi \in \mathbf{HI}$. That is, for any $\varphi, \psi, \chi \in \text{FORM}$:*

- (K) $\Vdash \varphi \rightarrow (\psi \rightarrow \varphi)$
- (S) $\Vdash (\varphi \rightarrow (\psi \rightarrow Z)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
- ($\forall \mathbf{E}$) $\Vdash \forall x \varphi \rightarrow \varphi[x \mapsto a]$ for any $a \in \mathbf{ICON}$
- ($\forall \mathbf{E}$) $\Vdash \forall X \varphi \rightarrow \varphi[X \mapsto P]$ for any $P \in \mathbf{PCONS}$
- ($\neg \mathbf{I}$) $\Vdash (\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg \psi) \rightarrow \neg \varphi)$
- (efq) $\Vdash (\varphi \rightarrow \perp) \rightarrow (\varphi \rightarrow \psi)$

Proof. Except for the cases concerning \forall , the proofs are identical to those in Gheorghiu [8]. Therefore it remains to show $\Vdash \forall X \varphi \rightarrow \varphi[X \mapsto P]$ for any $P \in \mathbf{PCONS}$.

Let $\mathcal{B} \in \mathfrak{B}$ be arbitrary such that $\Vdash_{\mathcal{B}} \forall X \varphi$. By (\forall), $\Vdash_{\mathcal{B}} \varphi[X \mapsto P]$ for any $P \in \mathbf{PCONS}$. Since \mathcal{B} was arbitrary, by (Inf), $\Vdash_{\emptyset} \forall X \varphi \rightarrow \varphi[X \mapsto P]$. The desired result follow from Lemma 2.7. \square

Lemma 3.5. *Let $\mathfrak{B} = \mathfrak{C}$. If $\Gamma \Vdash \neg \neg \varphi$, then $\Gamma \Vdash \varphi$.*

Proof. Let $\mathcal{X} \in \mathfrak{B}$ be such that $\Vdash_{\mathcal{X}} \neg \neg \varphi$. We proceed by induction on φ to show that $\Vdash_{\mathcal{X}} \varphi$:

- $\varphi = P \in \overline{\mathbf{ATOM}}$. We follow the argument by Sandqvist [24]. Let \mathcal{C} be defined as follows:

$$\mathcal{C} := \mathcal{X} \cup \{P \Rightarrow Q \mid Q \in \overline{\mathbf{ATOM}}\}$$

It is easy to see that $\Vdash_{\mathcal{C}} \neg P$. Moreover, since $\mathcal{C} \supseteq_{\mathcal{C}} \mathcal{X}$, we have $\Vdash_{\mathcal{C}} \perp$ by (Inf). It follows by (\perp) that $\Vdash_{\mathcal{C}} P$.

It remains to show that $\Vdash_{\mathcal{X}} P$. Consider how $\Vdash_{\mathcal{C}} P$ obtains. Suppose that it requires a rule in $\mathcal{C} - \mathcal{X}$, then there is a first such instance in the computation. For that instance however, it must be that $\Vdash_{\mathcal{C}} P$ holds without the use of rules in $\mathcal{C} - \mathcal{X}$. Hence, $\Vdash_{\mathcal{C}} P$ obtains only using rules in \mathcal{X} . Whence, $\Vdash_{\mathcal{X}} P$, as required.

- $\varphi = \varphi_1 \rightarrow \varphi_2$. Let $\mathcal{Z} \supseteq_{\mathcal{C}} \mathcal{Y} \supseteq_{\mathcal{C}} \mathcal{X}$ be arbitrary such that:

- (i) $\Vdash_{\mathcal{X}} \neg\neg(\varphi_1 \rightarrow \varphi_2)$
- (ii) $\Vdash_{\mathcal{Y}} \varphi_1$
- (iii) $\varphi_2 \Vdash_{\mathcal{X}} \perp$

From (ii) and (iii), observe $\varphi_1 \rightarrow \varphi_2 \Vdash_{\mathcal{X}} \perp$. Hence, from (i), infer $\Vdash_{\mathcal{X}} \perp$. Thus, by (Inf) and (\rightarrow) on (i), $\Vdash_{\mathcal{Y}} \neg\neg\varphi_2$. By the induction hypothesis (IH), $\Vdash_{\mathcal{Y}} \varphi_2$. Whence, by (Inf) and (\rightarrow) , $\Vdash_{\mathcal{X}} \varphi_1 \rightarrow \varphi_2$, as required.

- $\varphi = \forall x\psi$. Let $\mathcal{Z} \supseteq_{\mathcal{C}} \mathcal{Y} \supseteq_{\mathcal{C}} \mathcal{X}$ be arbitrary such that:

- (i) $\Vdash_{\mathcal{X}} \neg\neg\forall x\psi$
- (ii) $\varphi[x \mapsto a] \Vdash_{\mathcal{Y}} \perp$ for every $a \in \text{ICON}$
- (iii) $\Vdash_{\mathcal{X}} \forall x\psi$

By (\forall) on (iii), $\Vdash_{\mathcal{X}} \varphi[x \mapsto a]$ for every $a \in \text{ICON}$. Thus, by (ii), $\Vdash_{\mathcal{X}} \perp$. Hence, by (Inf), $\forall x\psi \Vdash_{\mathcal{Y}} \perp$. Whence, by (i), $\Vdash_{\mathcal{Y}} \perp$. It follows by (Inf) and (ii) that $\Vdash_{\mathcal{X}} \neg\neg\varphi[x \mapsto a]$ for all $a \in \text{ICON}$. Where, by the IH, we infer $\Vdash_{\mathcal{X}} \varphi[x \mapsto a]$ for all $a \in \text{ICON}$. Finally, from (\forall) , we obtain $\Vdash_{\mathcal{X}} \forall x\psi$, as required.

- $\varphi = \forall X\psi$. This follows *mutatis mutandis* on the preceding case.

This completes the induction. \square

It is easy to see that this lemma fails for \mathcal{J} . It suffices to find a base \mathcal{A} such that $\Vdash_{\mathcal{A}} \neg\neg A$ but not $\Vdash_{\mathcal{A}} A$ for an atom A . Let \mathcal{A} comprise the rules

$$\frac{[A]}{\frac{B}{B} \quad \frac{B}{C}}$$

for some $B \in \overline{\text{ATOM}}$ any $C \in \overline{\text{ATOM}}$. The atom B essentially represents \perp in \mathcal{A} — for any $\mathcal{B} \supseteq \mathcal{A}$, we have $\Vdash_{\mathcal{B}} B$ iff $\Vdash_{\mathcal{B}} \perp$. Observe that this counterexample relies essentially on having second-level rules.

It remains to show that support admits all the rules in Definition 3.1.

Lemma 3.6. *The following hold for $\mathfrak{B} \in \{\mathcal{C}, \mathcal{J}\}$:*

- (HYPOTHESIS). *If $\varphi \in \Gamma$, then $\Gamma \Vdash \varphi$.*
- (MODUS PONENS). *If $\Gamma \Vdash \varphi$ and $\Gamma \Vdash \varphi \rightarrow \psi$, then $\Gamma \Vdash \psi$.*
- (FIRST-ORDER GENERALIZATION). *If $\Gamma \Vdash \psi \rightarrow \varphi$ and $x \notin \text{FIV}(\psi, \Gamma)$, then $\Gamma \Vdash \psi \rightarrow \forall x\varphi$.*
- (SECOND-ORDER GENERALIZATION). *If $\Gamma \Vdash \psi \rightarrow \varphi$ and $X \notin \text{FPV}(\psi, \Gamma)$, then $\Gamma \Vdash \psi \rightarrow \forall X\varphi$.*

Proof. Except for the cases concerning \forall , the proofs are identical to those in Gheorghiu [8]. Therefore, it remains to show: If $\Gamma \Vdash \psi \rightarrow \varphi$ and $X \notin \text{FPV}(\psi, \Gamma)$, then $\Gamma \Vdash \psi \rightarrow \forall X\varphi$.

Suppose $\Gamma \Vdash \psi \rightarrow \varphi$. Let $\mathcal{B} \in \mathfrak{B}$ be arbitrary such that $\Vdash_{\mathcal{B}} \chi$ for $\chi \in \psi, \Gamma$. By (Inf) and (\rightarrow) , we have $\Vdash_{\mathcal{B}} \varphi$. We have two cases, either $X \in \text{FPV}(\varphi)$ or

$X \notin \text{FPV}(\varphi)$. In the first case, we have $\Vdash \forall X\varphi$ by (wff). In the second case, we $\Vdash \forall X\varphi$ by (\forall) as $\varphi[X \mapsto P] = \varphi$ for any $P \in \text{PCONS}$. \square

The following is an immediate corollaries of the lemmas above:

Theorem 3.7 (Soundness). *If $\Gamma \vdash_{\text{HC}} \varphi$ (resp. $\Gamma \vdash_{\text{HI}} \varphi$), then $\Gamma \Vdash \varphi$ when $\mathfrak{B} = \mathfrak{C}$ (resp. $\mathfrak{B} = \mathfrak{I}$).*

3.3. Completeness. We show that HC and HI are complete for support (\Vdash) under basis \mathfrak{C} and \mathfrak{I} , respectively. To this end, we follow the technique developed by Sandqvist [25] for intuitionistic propositional logic. We briefly outline it here and give the formal details below.

This construction is quite simple in both in principle and execution. It has been deployed to handle substructural logics with ease and apparently echoes some fundamental results on proof-search by Mints given some decades earlier — see Gheorghiu et al. [9, 10, 11, 6]. As Sandqvist [25] remarks:

‘The mathematical resources required for the purpose are quite elementary; there will be no need to invoke canonical models, König’s lemma, or even bar induction. The proof proceeds, instead, by *simulating* an intuitionistic deduction using atomic formulae within a base specifically tailored to the inference at hand.’

The key idea underlying this simulation technique is the systematic translation of formulas into atomic formulae.

Fix $\mathfrak{B} = \mathfrak{C}$ (resp. $\mathfrak{B} = \mathfrak{I}$) and let φ be a formula such that $\Vdash \varphi$. Suppose φ has a subformula $A \wedge B$ in which $A, B \in \text{PCONS}_0 \subseteq \text{ATOM}$. Then we include the following rules in the ‘specifically tailored’ base \mathcal{N} for φ :

$$\frac{A \quad B}{C} \quad \frac{C}{A} \quad \frac{C}{B}$$

where $C \in \text{PCONS}$ is a fresh atomic formula representing the conjunction $A \wedge B$. This means that C behaves in \mathcal{N} as $A \wedge B$ behaves in HC (resp. HI). More generally, each subformula χ of φ is assigned a corresponding basic counterpart χ^b — for example, $(A \wedge B)^b := C$.

The major work of the completeness proof is establishing the equivalence of χ and χ^b within \mathcal{N} :

$$\chi \Vdash_{\mathcal{N}} \chi^b \quad \text{and} \quad \chi^b \Vdash_{\mathcal{N}} \chi.$$

Since we assume $\Vdash \varphi$, it follows that $\Vdash \varphi^b$, and given that every rule in \mathcal{N} corresponds to an intuitionistic natural deduction rule, we conclude $\vdash \varphi$, as required. We now proceed to the detailed account of this strategy.

To begin, suppose we have a set of formulae Γ and a formula γ . Let Π be set of predicates that occur either in Γ or γ . Let **EIND** and **EPRED** be the set of individual- and predicate-constants, respectively, that do *not* occur in Γ or γ — we call these elements *eigen*-constants and *eigen*-predicates. Distinguishing these sets will enable us to handle the quantifiers.

Let Ξ be the set of atoms over Π using any terms or predicates, including the eigen-constants and eigen-predicates. Let Ξ^* be the second-order language over those atoms; that is, all formulae generated by the grammar:

$$\varphi ::= A \in \Xi \mid \varphi \rightarrow \varphi \mid \forall x\varphi \mid \forall X\varphi$$

$(\varphi \rightarrow (\psi \rightarrow \varphi))^b$	(K)
$((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)))^b$	(S)
$(\forall x \varphi \rightarrow \varphi[x \mapsto t])^b$	($\forall E$)
$(\forall X \varphi \rightarrow \varphi[X \mapsto P])^b$	($\forall E$)
$((\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg \psi) \rightarrow \neg \varphi))^b$	($\neg I$)
$((\varphi \rightarrow \perp) \rightarrow (\varphi \rightarrow \psi))^b$	(efq)
$\varphi^b, (\varphi \rightarrow \psi)^b \Rightarrow \psi^b$	(mp)
$(\varphi \rightarrow \psi)^b \Rightarrow (\varphi \rightarrow \forall x \psi[e \mapsto x])^b$	($\forall I$)
$(\varphi \rightarrow \psi)^b \Rightarrow (\varphi \rightarrow \forall X \psi[E \mapsto X])^b$	($\forall I$)
.....	
$(\neg \neg \varphi \rightarrow \varphi)^b$	(dne)

FIGURE 4. Simulation Base for Second-order Logic

Intuitively, Ξ^* captures all the kinds of complex formulae that we may encounter as Γ and γ unfold in the semantics.

Fix an injection $(-)^b : \Xi^* \rightarrow \text{PCONS}_0$ that is the identity on $\Xi^* \cap \text{PCONS}_0$ — that is, $P^b = P$ for $P \in \Xi^* \cap \text{PCONS}_0$. This *flattening* operator will enable us to bridge support and provability.

As $(-)^b$ is an injection, we have a left-inverse $(-)^{\natural} : \text{PCONS}_0 \rightarrow \Xi^*$ such that $P^{\natural} = P$ for $P \notin \Xi^* \cap \text{PCONS}_0$. We extend both $(-)^b$ and $(-)^{\natural}$ to sets point-wise,

$$\Gamma^b = \{\gamma^b \mid \gamma \in \Gamma\} \quad \text{and} \quad \mathbb{P}^{\natural} := \{P^{\natural} \mid P \in \mathbb{P}\}$$

We consider two bases:

- The classical base \mathcal{K} is given by all instances of the atomic rules in Figure 4 *including* $(\text{dne})^b$
- The intuitionistic base \mathcal{J} is given by all instances of the atomic rules in Figure 4 *excluding* $(\text{dne})^b$

By all instances, we mean that φ, ψ, χ, ξ range over Ξ^* , x ranges over IVAR , and a ranges over ICON , X ranges over PVAR , P ranges over PCONS , e ranges over EIND , and E ranges over EPRED . In the case of $\forall I$ and $\forall I$, we restrict such that $x \notin \text{FIV}(\varphi)$ and e does not occur in φ , and $X \notin \text{FPV}(\varphi)$ and E does not occur in φ .

We now show some essential technical results about $\mathcal{N} \in \{\mathcal{K}, \mathcal{J}\}$ that collectively deliver the desired completeness theorem(s). The first shows that provability in \mathcal{N} obeys the semantic clauses of Figure 2. The second shows that φ^b and φ are semantically equivalent relative to \mathcal{N} . The third shows that provability in \mathcal{K} (resp. \mathcal{J}) corresponds to provability in HC (resp. HI).

Lemma 3.8. *Let $\mathfrak{B} = \mathfrak{C}$ (resp. $\mathfrak{B} = \mathfrak{J}$) and $\mathcal{N} = \mathcal{K}$ (resp. $\mathcal{N} = \mathcal{J}$). The following hold for any $\mathcal{N}' \supseteq_{\mathfrak{B}} \mathcal{N}$:*

- $\vdash_{\mathcal{N}'} (\varphi \rightarrow \psi)^b$ iff $\varphi^b \vdash_{\mathcal{N}'} \psi^b$
- $\vdash_{\mathcal{N}'} (\forall x \varphi)^b$ iff $\vdash_{\mathcal{N}'} (\varphi[x \mapsto c])^b$ for any $c \in \text{ICON}$.

- $\vdash_{\mathcal{N}'} (\forall X\varphi)^b$ iff $\vdash_{\mathcal{N}'} (\varphi[X \mapsto P])^b$ for any appropriate $P \in \text{PCONS}$.
- $\mathbb{P} \vdash_{\mathcal{N}'} P$ iff, for any $\mathcal{N}'' \supseteq_{\mathfrak{B}} \mathcal{N}'$, if $\vdash_{\mathcal{N}''} Q$ for $Q \in \mathbb{P}$, then $\vdash_{\mathcal{N}''} P$.

Proof. We show each one in turn:

- First, suppose $\vdash_{\mathcal{N}'} (\varphi \rightarrow \psi)^b$. From this, it is easy to see that $\varphi^b \vdash_{\mathcal{N}'} (\varphi \rightarrow \psi)^b$. Simultaneously, by REF, From this, it is easy to see that $\varphi^b \vdash_{\mathcal{N}'} \varphi^b$. Hence, by APP using **mp**, we have $\varphi^b \vdash_{\mathcal{N}'} \psi^b$.
Second, suppose $\varphi^b \vdash_{\mathcal{N}'} \psi^b$. Showing $\vdash_{\mathcal{N}'} (\varphi \rightarrow \psi)^b$ follows from standard approaches to the Deduction Theorem for classical logic — see, for example, Herbrand [14].
- First, suppose $\vdash_{\mathcal{N}'} (\forall x\varphi)^b$. By APP using $\forall E$ and **mp**^b, we have $\vdash_{\mathcal{N}'} (\varphi[x \mapsto a])^b$ for any $a \in \text{ICON}$.
Second, suppose $\vdash_{\mathcal{N}'} (\varphi[x \mapsto a])^b$ for any $a \in \text{ICON}$. Let $\top \in \Xi^*$ such that $\vdash_{\mathcal{N}'} \top^b$ (e.g., $\top = P \rightarrow P$ for any $P \in \Xi$) and let $e \in \text{EIND} \subseteq \text{ICON}$. From the assumption, it is easy to see that $\top^b \vdash_{\mathcal{N}'} (\varphi[x \mapsto e])^b$. Hence, by (i), $\vdash_{\mathcal{N}'} (\top \rightarrow \varphi[x \mapsto e])^b$. Whence, $\vdash_{\mathcal{N}'} (\top \rightarrow \forall x\varphi[x \mapsto e][e \mapsto x])^b$. Since $\varphi[x \mapsto e][e \mapsto x] = \varphi$, this says $\vdash_{\mathcal{N}'} (\top \rightarrow \forall x\varphi)^b$. Whence, by (i) and the definition of \top , conclude $\vdash_{\mathcal{N}'} \forall x\varphi^b$.
- This argument follows *mutatis mutandis* on the previous case.
- This is identical to Sandqvist [25] (Lemma 2.2).

This completes the proof. \square

Lemma 3.9. *If $\mathfrak{B} = \mathfrak{C}$ (resp. $\mathfrak{B} = \mathfrak{J}$) and $\mathcal{N} = \mathcal{K}$ (resp. $\mathcal{N} = \mathcal{J}$), then*

$$\Vdash_{\mathcal{N}'} \varphi^b \quad \text{iff} \quad \Vdash_{\mathcal{N}'} \varphi$$

Proof. This proceeds by structural induction on φ using Lemma 3.8. We illustrate the case where $\varphi = \forall X\psi$, the others being similar:

$$\begin{aligned} \Vdash_{\mathcal{A}'} (\forall X\psi)^b & \quad \text{iff} \quad \vdash_{\mathcal{A}'} (\forall X\psi)^b & (\text{At}) \\ & \quad \text{iff} \quad \vdash_{\mathcal{A}'} (\psi[X \mapsto P])^b \text{ for any appropriate } P \in \text{PCONS} \\ & & (\text{Lemma 3.8}) \\ & \quad \text{iff} \quad \Vdash_{\mathcal{A}'} (\psi[X \mapsto P]) \text{ for any appropriate } P \in \text{PCONS} \\ & & (\text{Induction Hypothesis}) \\ & \quad \text{iff} \quad \Vdash_{\mathcal{A}'} \forall X\psi & (\forall) \end{aligned}$$

The other cases have essentially been given by Gheorghiu [8]. \square

Lemma 3.10. *The following hold:*

- If $\mathbb{P} \vdash_{\mathcal{J}} P$, then $\mathbb{P}^b \vdash_{\text{HI}} P^b$.
- If $\mathbb{P} \vdash_{\mathcal{K}} P$, then $\mathbb{P}^b \vdash_{\text{HC}} P^b$.

Proof. We proceed by induction on provability in a base:

- (REF). It must be that $P \in \mathbb{P}$. Hence, $\mathbb{P}^b \vdash_{\text{NK}} P^b$ by HYPOTHESIS.
- (APP). We proceed by case analysis on the last atomic rule used:
 - For the axioms $(K)^b$, $(S)^b$, $(\forall E)^b$, $(\forall \mathcal{E})^b$, \neg^b , $\text{efq}^b \in \mathcal{J} \subseteq \mathcal{K}$, the result is immediate by AXIOM using (K) , (S) , $(\forall E)$, $(\forall \mathcal{E})$, \neg , $\text{efq} \in \text{HI} \subseteq \text{HC}$, respectively.
 - For the axiom $(\text{dne})^b \in \mathcal{K}$, the result is immediate by $\text{dne} \in \text{HC}$.

- If the last rule used was $(\text{mp})^b \in \mathcal{A} = \mathcal{J}$ (resp. $\mathcal{A} = \mathcal{K}$), then $P = \psi^b$ for some $\psi \in \Xi^*$, and there is $\varphi^b \in \Xi^*$ such that $\mathbb{P} \vdash_{\mathcal{A}} (\varphi \rightarrow \psi)^b$ and $\mathbb{P} \vdash_{\mathcal{A}} \varphi^b$. By the induction hypothesis (IH), $\mathbb{P}^{\natural} \vdash_A \varphi \rightarrow \psi$ and $\mathbb{P}^{\natural} \vdash_A \varphi$ for $A = \text{HI}$ (resp. $A = \text{HI}$). Observe that $P^{\natural} = \varphi$. The desired result follows by MODUS PONENS.
- If the last rule used was $(\forall I)^b \in \mathcal{A} = \mathcal{J}$ (resp. $\mathcal{A} = \mathcal{K}$), then $P = (\varphi \rightarrow \forall x\psi[e \mapsto x])^b$ for some $\varphi, \psi \in \Xi^*$, $e \in \text{EIND}$ such that e does not occur in φ , and $x \in \text{IVAR}$ such that $x \notin \text{FIV}(\varphi)$. Moreover, $\mathbb{P} \vdash_{\mathcal{A}} (\varphi \rightarrow \psi)^b$. By the IH, $\mathbb{P}^{\natural} \vdash_A \varphi \rightarrow \psi$ for $A = \text{HI}$ (resp. $A = \text{HC}$). By Proposition 3.3, it follows that $\mathbb{P}^{\natural} \vdash_A (\varphi \rightarrow \psi)[e \mapsto x]$. The desired result follows by FIRST-ORDER GENERALIZATION.
- If the last rule used was $(\forall I)^b \in \mathcal{A} = \mathcal{J}$ (resp. $\mathcal{A} = \mathcal{K}$), the result follows *mutatis mutandis* on the previous case.

This completes the case analysis.

This completes the induction. \square

Theorem 3.11 (Completeness). *Let $\mathfrak{B} = \mathfrak{C}$ (resp. $\mathfrak{B} = \mathfrak{J}$) and $\mathbf{N} = \text{HC}$ (resp. $\mathbf{N} = \text{HI}$). Let Γ be a finite set of sentences and φ a sentence. If $\Gamma \Vdash \varphi$, then $\Gamma \vdash_{\mathbf{N}} \varphi$.*

Proof. Assume $\Gamma \neq \emptyset$, the case $\Gamma = \emptyset$ being similar. Let \mathcal{N} be its simulation base. We reason as follows:

$$\begin{array}{lll}
\Gamma \Vdash \varphi & \text{implies} & \text{for any } \mathcal{N}' \supseteq_{\mathfrak{B}} \mathcal{N}, \text{ if } \Vdash_{\mathcal{N}'} \psi \text{ for } \psi \in \Gamma, \text{ then } \Vdash_{\mathcal{N}'} \varphi \text{ (Inf)} \\
& \text{implies} & \text{for any } \mathcal{N}' \supseteq_{\mathfrak{B}} \mathcal{N}', \text{ if } \vdash_{\mathcal{N}'} P \text{ for } P \in \Gamma^b, \text{ then } \vdash_{\mathcal{N}'} \varphi^b \\
& & \text{(Lemma 3.9)} \\
& \text{implies} & \Gamma^b \vdash_{\mathcal{N}} \varphi^b \text{ (Lemma 3.8)} \\
& \text{implies} & (\Gamma^b)^{\natural} \vdash_{\mathbf{N}} (\varphi^b)^{\natural} \text{ (Lemma 3.10)} \\
& \text{implies} & \Gamma \vdash_{\mathbf{N}} \varphi
\end{array}$$

The last step follows from the definition of $(-)^{\natural}$ as a left-inverse of $(-)^b$. \square

4. EQUIVALENCE WITH HENKIN'S SECOND-ORDER LOGIC

Henkin's S-oL [13] can be characterized by the natural deduction system **NC** (Figure 5 including **dne**). Let **NI** denote the intuitionistic counterpart consisting of all the rules in Figure 5 excluding **dne**. Of course, for $\forall I$ and $\forall I$ it is important that x and X do not occur in any of the hypotheses of the sub-derivation for φ .

In this section, we will show that derivability in **NI** and **NC** corresponds to derivability in **HI** and **HC**, respectively. In Section 3, we showed that **HI** and **HC** were previously shown to axiomatize the B-eS for S-oL given by \mathfrak{C} and \mathfrak{K} , respectively. Therefore, this section shows that that the P-tS in this paper recovers Henkin's S-oL.

Proposition 4.1. *The natural deduction systems are at least as expressive as the Hilbert calculi:*

- If $\Gamma \vdash_{\text{HI}} \varphi$, then $\Gamma \vdash_{\text{NI}} \varphi$.
- If $\Gamma \vdash_{\text{HC}} \varphi$, then $\Gamma \vdash_{\text{NC}} \varphi$.

Proof. Let **H** be either **HI** or **HC**. We proceed by induction on derivability in **H** to show the result:

$$\begin{array}{c}
[A] \\
\vdots \\
\frac{B}{A \rightarrow B} \rightarrow I \quad \frac{\varphi}{\forall x \varphi} \forall I \quad \frac{\varphi}{\forall X \varphi} \forall I \\
\frac{A \rightarrow B \quad A}{B} \rightarrow E \quad \frac{\forall x \varphi}{\varphi[x \mapsto a]} \forall E \quad \frac{\varphi[X \mapsto P]}{\forall X \varphi} \forall E \quad \frac{\perp}{\varphi} \text{efq} \\
\hline
\frac{\neg \neg \varphi}{\varphi}
\end{array}$$

FIGURE 5. Natural Deduction System(s) NI and NC

- AXIOM. As $\text{NI} \subseteq \text{NC}$, it suffices to show $\vdash_{\text{NI}} \alpha$ for $\alpha \in \text{HI}$ and $\vdash_{\text{NC}} \neg \neg \varphi \rightarrow \varphi$. The latter is immediate by $\text{dne}, \rightarrow I \in \text{NC}$, so we concentrate on the former, which we show by case analysis:

- (K). We have the following NI-derivation:

$$\begin{array}{c}
\frac{[\varphi] \quad [\psi]}{\psi \rightarrow \varphi} \rightarrow I \\
\frac{\psi \rightarrow \varphi}{\varphi \rightarrow (\psi \rightarrow \varphi)} \rightarrow I
\end{array}$$

- (S). We have the following NI-derivation:

$$\begin{array}{c}
\frac{[\varphi \rightarrow (\psi \rightarrow \chi)] \quad [\varphi]}{\psi \rightarrow \chi} \rightarrow E \quad \frac{[\varphi \rightarrow \psi] \quad [\varphi]}{\psi} \rightarrow E \\
\frac{\chi}{\varphi \rightarrow \chi} \rightarrow I \\
\frac{\varphi \rightarrow \chi}{(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)} \rightarrow I \\
\frac{(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)}{(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))} \rightarrow I
\end{array}$$

- ($\forall E$). Immediate by $\forall E, \rightarrow I \in \text{NI}$.

- ($\forall E$). Immediate by $\forall E, \rightarrow I \in \text{NI}$.

- ($\neg I$). We have the following NI-derivation:

$$\begin{array}{c}
\frac{[\varphi \rightarrow \neg \psi] \quad [\varphi]}{\neg \psi} \rightarrow E \quad \frac{[\varphi \rightarrow \psi] \quad [\varphi]}{\psi} \rightarrow E \\
\frac{\perp}{\neg \neg \varphi} \rightarrow I \\
\frac{\neg \neg \varphi}{(\varphi \rightarrow \neg \psi) \rightarrow \neg \varphi} \rightarrow I \\
\frac{(\varphi \rightarrow \neg \psi) \rightarrow \neg \varphi}{(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg \psi) \rightarrow \neg \varphi)} \rightarrow I
\end{array}$$

- (efq). We have the following NI-derivation:

$$\begin{array}{c}
\frac{[\varphi \rightarrow \perp] \quad [\varphi]}{\perp} \rightarrow E \\
\frac{\perp}{\psi} \text{efq} \\
\frac{\psi}{\varphi \rightarrow \psi} \rightarrow I \\
\frac{\varphi \rightarrow \psi}{(\varphi \rightarrow \perp) \rightarrow (\varphi \rightarrow \psi)} \rightarrow I
\end{array}$$

- HYPOTHESIS. We require to show that if $\varphi \in \Gamma$, then $\Gamma \vdash_{\text{NI}} \varphi$. This is witnessed by the tree of one node labelled φ .

- FIRST-ORDER GENERALIZATION. Assume $\Gamma \vdash_{\mathbf{H}} \psi \rightarrow \varphi$ such that $x \notin \text{FIV}(\varphi, \Gamma)$. We require to show $\Gamma \vdash_{\mathbf{NI}} \psi \rightarrow \forall x\varphi$. Let \mathbf{L} be an \mathbf{NI} -derivation witnessing the assumption, then we show the desired result by the following:

$$\frac{\frac{\frac{\mathbf{L}}{\psi \rightarrow \varphi} \quad \psi}{\varphi} \rightarrow \mathbf{E}}{\frac{\varphi}{\forall x\varphi} \rightarrow \mathbf{I}} \rightarrow \mathbf{I}$$

- SECOND-ORDER GENERALIZATION. This follows *mutatis mutandis* on the previous case.

□

Proposition 4.2. *The Hilbert calculi are at least as expressive as the natural deduction systems:*

- If $\Gamma \vdash_{\mathbf{NI}} \varphi$, then $\Gamma \vdash_{\mathbf{HI}} \varphi$.
- If $\Gamma \vdash_{\mathbf{NC}} \varphi$, then $\Gamma \vdash_{\mathbf{HC}} \varphi$.

Proof. By Theorem 3.11, it suffices to show that the natural deduction are sound for the B-eS of S-oL:

- If $\Gamma \vdash_{\mathbf{NI}} \varphi$, then $\Gamma \Vdash \varphi$ over the basis \mathfrak{J} .
- If $\Gamma \vdash_{\mathbf{NC}} \varphi$, then $\Gamma \Vdash \varphi$ over the basis \mathfrak{C} .

To this end, it suffices to show that \Vdash respects the rules of \mathbf{NI} and \mathbf{NC} under basis \mathfrak{J} and \mathfrak{C} , respectively. This has already be down by Sandqvist [24, 25] (see also Gheorghiu [8]) for $\rightarrow \mathbf{I}$, $\rightarrow \mathbf{E}$, $\forall \mathbf{I}$, $\forall \mathbf{E}$ over both \mathfrak{J} and \mathfrak{C} , and for \mathbf{dne} over \mathfrak{C} . The remaining cases of showing that \Vdash admits $\forall \mathbf{I}$ and $\forall \mathbf{E}$ follows *mutatis mutandis* on the treatment of $\forall \mathbf{I}$ and $\forall \mathbf{E}$. □

This completes the equivalence to Henkin’s account of second-order logic.

5. DISCUSSION

todo: conPA paper

We have developed a proof-theoretic semantics for second-order logic in which both first- and second-order quantification are defined by substitution. The semantics is grounded in atomic systems — pre-logical sets of inference rules — that can be understood as encoding beliefs about the inferential relationships between concepts in the language. For example, if one accepts that ‘Tammy is a vixen’ ($V(t)$), one may infer that ‘Tammy is female’ ($Fe(t)$) and ‘Tammy is a fox’ ($Fo(t)$), among possibly other consequences:

$$\frac{V(t)}{Fe(t)} \quad \frac{V(t)}{Fo(t)}$$

We have shown that, depending on the chosen notion of atomic system, this framework recovers different logics. In particular, we examined two such notions corresponding to classical and intuitionistic versions of Henkin’s second-order logic. This raises a question: Is there a suitable choice of atomic system that would recover full second-order logic? We leave this as an open direction for future work.

At present, we may ask how to interpret the fact that our framework recovers Henkin semantics. Väänänen [31] has argued that despite some formal differences in their model-theoretic treatments, once second-order logic is formalized (as it must

be for foundational purposes), the distinction between Henkin and full semantics effectively vanishes. As he puts it: ‘If two people started using second-order logic for formalizing mathematical proofs, person F with the full second-order logic and person H with the Henkin second-order logic, we would not be able to see any difference.’ At the level of formal reasoning, then, there is no principled basis for preferring one over the other, making the appearance of Henkin semantics in our system entirely acceptable from the perspective of foundational logic.

Nonetheless, the distinction between Henkin and full semantics remains relevant for the broader programme of proof-theoretic semantics. In its two forms, second-order logic is connected in markedly different ways to model-theoretic and set-theoretic foundations of mathematics. These relationships remain largely unexplored within the proof-theoretic tradition. The account of second-order logic presented here offers a promising setting in which such connections can be examined more closely, drawing on both the existing meta-theory and the simplicity and flexibility of our semantic framework.

While B-eS provides an intuitively constructive semantic account of logical consequence, a clear understanding of its computational content, together with a systematic account of its relationship with P-tV for a given logic, remains to be established. The close relationship between intuitionistic second-order logic and the polymorphic lambda-calculus (see, for example, [1]) suggests that B-eS for S-oL may provide a useful starting point for investigating these questions.

Second-order arithmetic provides a compelling setting for formal metamathematics (see, e.g., [31]). We conjecture that an inferentialist analysis of second-order arithmetic, induction principles, and the definability of concepts may provide a computational basis for a formal metamathematics grounded in the structure of atomic systems. This includes, for example, defining real numbers and analysis. Gheorghiu [7] has shown that first-order proof-theoretic semantics gives a simple argument for the consistency of Peano Arithmetic.

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