Multiagent Bifurcation Logic

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Abstract

We introduce Bifurcation Logic, BL, which combines a basic classical modality with separating conjunction, $*$, together with its naturally associated multiplicative implication, $-\!*$, that is defined using the modal ordering. Specifically, a formula $\phi_1 * \phi_2$, is true at a world w if and only if each ϕ_i holds at worlds w_i that are each above w, on separate branches of the order, and have no common upper bound. We illustrate the use of BL through a model of an access control protocol in which multiple agents must jointly authenticate a key. This example makes essential use of BL' modality, $\Box$, to represent authentication. We also provide a labelled tableaux calculus for BL and establish soundness and completeness relative to its relational semantics. Then we define a multiagent version of BL, called MBL, that is similar to BL, but employs connectives indexed by agents, denoted a. To illustrate modelling with MBL we return to our security modelling example, and show how the agent modalities available in MBL enable a richer analysis of the authentication protocol. In particular, they allow the representation not only the intensional behaviours of the actors present in the modelling scenario, but also the role of extensionally-determined system-policies. Our examples of modelling multiagent access control are quite generic in their form, suggesting many applications. To complete these results we extend the tableaux system for BL, together with its soundness and completeness results, to MBL. Finally the standard finite model property fails for BL and MBL. However, we show that for both BL and MBL in the absence of $-\!*$, but in the presence of $*$, every model has an equivalent finite representation and that this is sufficient to obtain decidability.

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1 Introduction

We begin by introducing Bifurcation Logic, BL, which combines a basic classical modality with separating conjunction, $*$, together with its naturally associated multiplicative implication, $-\!\!*$, that is defined using the modal ordering.[1] We then extend BL to introduce Multiagent Bifurcation Logic MBL, in which separation is obtained through agent-indexed ordering.

We provide a relational semantics and a labelled tableaux calculus for BL and establish soundness and completeness relative to BL's relational semantics. The standard finite model property fails for BL. However, we show that, in the absence of $-\!\!*$, but in the presence of $*$, every model has a finite representation and that this is sufficient to obtain decidability. We illustrate the use of BL in logical modelling through an example of access control.

The key property of interest in BL is the semantics of its multiplicatives, the 'separating' $*$ and $-\!\!*$, which is given in terms of the ordering that is used to define the classical modality. This stands in contrast to the set-up in, say, bunched implications (BI — e.g., [18, 20, 13]) in which specific relational structure is used for their definition — see [14] for a through discussion of BI's semantics. Kamide's account of Kripke semantics for modal substructural logics [16] also employs a binary operation on worlds to give a treatment of the multiplicative conjunction that is similar to that of BI's elementary semantics (e.g., [18, 20, 13, 14]). Galmiche, Kimmel, and Pym [11] consider an epistemic modal extension of boolean BI in which the semantics of the multiplicative conjunction employs a monoidal product on worlds. Došen [7] considers a range of issues in the relationship between modal and substructural logics from the perspective of translations between proof systems, and Ono [19] has also considered the proof theory of modal and substructural logics.

The basic idea in BL is that a formula $\phi_1 * \phi_2$ is true at a world w if and only if each ϕ_i holds at worlds w_i that are each above w, on separate branches of the order, and have no common upper bound — that is, they are bifurcated. Consequently, the semantics of the multiplicative implication has the property that the implicational formula $\phi -\!\!* \psi$ and its subformula ϕ are required to hold at bifurcated worlds above the world at which ψ holds. This use of this feature is illustrated in a substantive modelling example given in Section 3. The semantics of the classical connectives and modality is standard.

In Section 2, we introduce the language of Bifurcation Logic and its models, based on frames with a ternary relation structure for the bifurcation semantics. In Section 3, we give an extended, quite generic — that is, evidently mappable to other settings — example of the use of BL in modelling access control, suggesting wider application in knowledge representation and reasoning. This example, albeit somewhat idealized, illustrates the interaction between the classical modality and the multiplicative connectives, especially the somewhat unusual semantic form of the implication, in a simple and direct way. We also discuss some related work in this section.

In Section 4, we give a system of labelled tableaux for BL. The form of the calculus follows the pattern established in, for example, [13, 11] and allows, in Section 5, soundness and completeness results to be established (cf. [13, 11]).

In Section 6, we introduce a multiagent version of BL, called MBL, that is similar to BL, but employs logical operators that are indexed by agents: $[a]$, $*_a$, and $-\!\!*_a$. In Section 7 we illustrate modelling with MBL by returning to our security modelling example, and show how the agent modalities available in MBL enable a richer analysis of the authentication protocol. In particular, they allow the representation not only the intensional behaviours of the actors present in the

[1] These ideas were first given in [10], presented at Theoretical Aspects of Rationality and Knowledge (TARK), Düsseldorf, 2025.

modelling scenario, but also the role of extensionally-determined system-policies. Our examples of modelling multiagent access control are quite generic in their form, suggesting many applications.

To complete these results, in Section 8 we extend the tableaux system for BL to MBL, and also prove its soundness and completeness results. While the usual finite model property fails for BL, a modified form of it does hold. Section 9 explains, through a counterexample, why the standard form fails and introduces a modified form through the concept of 'model with back links'. Intuitively, BL is a logic with the subformula property — in the sense evident from the tableaux system — and is about paths in finitely branching trees. Back links describe how the paths go back to already-seen configurations (see Section 9 for a formal explanation). Using this modified finite model property, the decidability of BL and also of MBL is obtained. In Section 10 we conclude and propose some perspectives.

2 Bifurcation Logic

In this section, we introduce Bifurcation Logic, BL, by giving a definition in terms of ternary relational semantics in the style of Routley-Meyer [22], with some similarity to the work of Fuhrmann and Mares [9].

Definition 1 (Language)**.** *Let P be a countable set of propositional letters. The* formulae *of BL, the set of which is denoted by Φ, are given by the following grammar:*

$$\phi ::= p\,(\in P) \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid \phi * \phi \mid \phi \mathrel{-\!\!*} \phi$$

The connectives $\vee$, $\rightarrow$, $\leftrightarrow$, $\Diamond$, and the units $\top$, $\bot$ are defined as follows: $\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$, $\phi \rightarrow \psi = \neg\phi \vee \psi$, $\phi \leftrightarrow \psi = (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, $\top = \phi \vee \neg\phi$, $\bot = \phi \wedge \neg\phi$, $\Diamond\phi = \neg\Box\neg\phi$. $\Box$

To minimize the use of parentheses; we use the following strict order of precedence (with right associativity): $\Box, \Diamond, \neg \; > \; * \; > \; \wedge, \vee \; > \; -\!\!* \; > \; \rightarrow \; > \; \leftrightarrow$.

Definition 2 (Tree)**.** *Let $(W, \leq)$ be a partial order. Two elements $w_1, w_2 \in W$ are* separated (or disjoint)*, denoted $w_1 \bot w_2$, if neither $w_1 \leq w_2$ nor $w_2 \leq w_1$. We call $(W, \leq)$* rooted *if there exists $w \in W$ such that $w \leq w'$ for all $w' \in W$. $(W, \leq)$ has the* persistent separation property *if it satisfies the following condition:*

$$(P) \quad \textit{for all } w_1, w_2, w'_1 \in W, \ \textit{if } w_1 \bot w_2 \textit{ and } w_1 \leq w'_1, \ \textit{then } w'_1 \bot w_2.$$

$(W, \leq)$ is called a tree *if it is rooted and has the persistent separation property.* $\Box$

Definition 3 (Frame)**.** *A BL* frame *is a structure $\mathcal{F} = (W, \leq, R)$, where $(W, \leq)$ is a tree of elements called* worlds *and R is the ternary relation on worlds defined as follows:*

$$(B) \quad \textit{for all } w, w_1, w_2 \in W, \ R(w, w_1, w_2) \textit{ iff } w \leq w_1, w \leq w_2 \textit{ and } w_1 \bot w_2.$$

That is, w_1 and w_2 belong to distinct futures of w. $\Box$

Definition 4 (Model)**.** *A BL* model *is a triple $\mathcal{M} = (\mathcal{F}, V, \Vdash)$, where $\mathcal{F}$ is a BL frame and V is a* valuation function *from W to $\wp(P)$. The* satisfaction relation *$\Vdash$ is inductively defined as the* *smallest relation on $W \times \Phi$ such that*

$$
\begin{array}{rll}
\mathcal{M}, w \Vdash p & \textit{iff} & p \in V(w) \\
\mathcal{M}, w \Vdash \neg\phi & \textit{iff} & \mathcal{M}, w \nVdash \phi \\
\mathcal{M}, w \Vdash \phi \wedge \psi & \textit{iff} & \mathcal{M}, w \Vdash \phi \textit{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \Box\phi & \textit{iff} & \textit{for all } w' \in W, \textit{ if } w \leq w' \textit{ then } \mathcal{M}, w' \Vdash \phi \\
\mathcal{M}, w \Vdash \phi * \psi & \textit{iff} & \textit{for some } w_1, w_2 \in W \textit{ such that } R(w, w_1, w_2), \\
& & \mathcal{M}, w_1 \Vdash \phi \textit{ and } \mathcal{M}, w_2 \Vdash \psi \\
\mathcal{M}, w \Vdash \phi \mathrel{-\!\!*} \psi & \textit{iff} & \textit{for all } w_1, w_2 \in W \textit{ such that } R(w_2, w, w_1), \\
& & \textit{if } \mathcal{M}, w_1 \Vdash \phi \textit{ then } \mathcal{M}, w_2 \Vdash \psi
\end{array}
$$

A formula ϕ is satisfied in a model $\mathcal{M}$*, denoted $\mathcal{M} \Vdash \phi$, if $\mathcal{M}, w \Vdash \phi$ for all worlds w in $\mathcal{M}$. We write $w \Vdash \phi$ instead of $\mathcal{M}, w \Vdash \phi$ whenever the model is clear from the context. ϕ is* satisfiable *if it is satisfied in some model $\mathcal{M}$, and* valid*, denoted $\Vdash \phi$, if it is satisfied in all models.* □

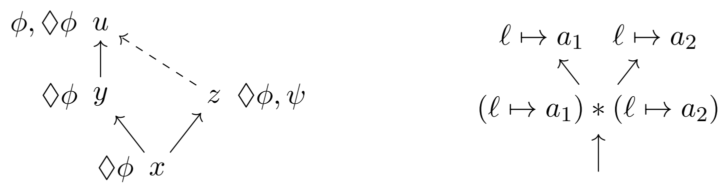


Figure 1: Examples of BL structures

BL uses frames that obey the persistent separation property as we believe that they better correspond to an intuitive understanding of bifurcation.

We define *non-persistent (or lax)* BL as the extension of BL that deals with frames that are not required to obey the persistent separation property.

For example, the formula $\varphi = (\Diamond\phi * \psi) \leftrightarrow (\phi * \psi)$ is valid in BL (a tableau proof is given in Section 4), but it is not valid in lax BL as it can be falsified in the direct acyclic graph (DAG) given on the left-hand side of Figure 1. Indeed, we have $x \Vdash \Diamond\phi * \psi$ because we have $R(x, y, z)$, $y \Vdash \Diamond\phi$ and $z \Vdash \psi$, but we do not have $x \Vdash \phi * \psi$ because the only world satisfying ϕ is u and u does not belong to a distinct future of z, which is the only world satisfying ψ.

Proposition 5. *BL is a conservative extension of the modal logic $S4$.*

Proof. This follows immediately from observing that the only conditions imposed on the order relation are for the purpose of defining $*$ and $\mathrel{-\!\!*}$. □

Although BL is a conservative extension of $S4$, it differs from both bunched and other separating logics. First, we remark that since BL addresses separation as an ordering problem rather than a resource composition problem (as in, for example, bunched logics, where $*$ usually corresponds to a product in a monoid), we do not include a unit $\top^*$ for the multiplicative conjunction $*$. Having the multiplicative unit $\top^*$ would unnecessarily complicate our definition of the bifurcation relation or rule out many partial order structures. Indeed, we would need to satisfy $w \Vdash \phi * \top^*$ iff $w \Vdash \phi$ for all worlds w and all formulae ϕ. Hence, by definition of $*$, we would need worlds w_1, w_2 such that $R(w, w_1, w_2)$, $w_1 \Vdash \phi$ and $w_2 \Vdash \top^*$. In particular, consider a BL frame with only one world w and set $\phi = \top$. We have $w \Vdash \top$, but not $w \Vdash \top * \top^*$ because $R(w, w, w)$ is impossible to achieve as it would imply both $w \leq w$ and $w \not\leq w$ by definition of R.

Second, BL also differs from Separation Logic, as illustrated in the right-hand side of Figure 1. In Separation Logic [15, 21], the built-in points-to predicate $(\ell \mapsto a)$ intuitively denotes a memory heap with only one cell whose location (address) is ℓ and whose value is a. Heaps are defined as partial functions from locations to values and composition of heaps is given by the union of functions with disjoint domains. Therefore, the formula $(\ell \mapsto a_1) * (\ell \mapsto a_2)$ is not satisfiable in Separation Logic as it denotes the disjoint composition of two one-cell heaps that share the same location ℓ. In BL, as $*$ represents bifurcation, a node satisfying $(\ell \mapsto a_1) * (\ell \mapsto a_2)$ simply implies that the location ℓ might have two distinct futures, one in which it points to the value a_1 and the other one in which it points to the value a_2. More interestingly, the formula $\Box(\ell \mapsto a)$, when satisfied by some world w, would imply that in all possible futures of w, the location ℓ should point to the same value a. This would be useful, for example, to state that an interrupt vector always points to the address where its legitimate handler resides.

3 Modelling With Bifurcation Logic

Logics can be used not merely to describe reasoning itself, but also to describe reasoning about systems. This use of logics as modelling tools has delivered significant advances in many areas — too numerous to describe here — including program analysis and verification, with one leading example, making essential use of multiplicative conjunction, being Separation Logic [15, 21]. Another highly effective example in the same spirit is Context Logic [3, 4]. More abstractly, substructural modal logics provide reasoning tools for models of systems in the 'distributed systems metaphor' (e.g., [1, 5, 12]). Demri and Deter [6] have surveyed connections between modality and separation, but they do not consider separating connectives defined through ordering. More detailed connections with Separation Logic, as mentioned above, are beyond the scope of this paper; so, instead of describing how connections with that might work, we give a quite generic example — many examples involving obtaining, representing, and verifying knowledge and information will be very similar — of how BL's modality and multiplicatives interact through an example in the context of multi-agent access control.

It is perhaps important to stress some aspects of the methodology of modelling using logical tools. In general, mathematical models of systems and scenarios are constructed in order to answer specific questions about the system or scenario of interest. A few observations about the nature of models — in this sense, not the formal logical notion of model — will perhaps be helpful.

(a) Models in this sense do not represent all of the components of the system or scenario of interest — according to Alfred Korzybski [17], 'the map is not the territory', articulating the view that an abstraction derived from something is not the thing itself.

(b) Modelling is not formal specification or verification.

(c) Borrowing Einstein's principle: in addressing its intended purpose, a model should be as simple as possible and no simpler.

(d) Specific models may involve specific interpretations of the mathematical constructs that are deployed in their description. These interpretations should be expected neither to translate to other specific models nor to have more general validity.

The consequences of these features of modelling are perhaps a little surprising for logicians: some aspects of a model of a system or scenario constructed using logical tools may be left implicit as a result of the abstraction required to make a model conceptually and/or technically tractable (point (a), above). Such a model may seek to capture an essential feature of a system or scenario without specifying the feature formally (point (b), above).

Finally, such a model may omit — that is, abstract away from — some details of the system or scenario of interest for clarity and for simplicity; that is, lack of needless complexity (points (a) and (c), above). We shall remark upon these features of modelling in the examples that follow.

Example 6 (Crimson Tide [24][2] Part 1)**.** *The plot of the film* Crimson Tide *[24] takes place mainly on board a United States Navy Ballistic Missile Submarine (the USS Alabama). During a period of international tension, the submarine receives an order to launch nuclear-armed missiles. For the release of the weapons to be authorized, a composite key must be authenticated:*

1. *An order to release weapons is received in a message that contains a code.*
2. *Two senior officers (neither the Captain nor the Executive Officer) must independently authenticate the message by verifying the code against a local authentication device. Authentication of a key at a world (think 'necessarily correct') is denoted using* $\Box$.
3. *This yields a composite —* $\Box\kappa_1 * \Box\kappa_2$ *in Figure 2 — of authenticated keys that is passed (verbally) to the Executive Officer and the Captain.*
4. *The Executive Officer confirms that the correct protocol has been followed and so authenticates the composite key, thereby yielding* $\Box(\Box\kappa_1 * \Box\kappa_2)$ *in Figure 2.*
5. *The Captain then confirms the authentication and may order the release of the weapons.*

In this modelling set-up, worlds can be considered to have both spatial and temporal properties. For simplicity — see points (a) and (c), above — we just think of them as 'states'. This concludes the first part of our example.

We remark that the tree model is interpreted downwards; that is, we build our intuitive understanding of the example by reading the arrows of the tree from a node to its parent, whereas the purely mathematical/semantical reading of the logical connectives as per Definition 4 goes the opposite way, from a node to its successors. This is in accordance to point (d) above. Therefore, one should not pay too much attention to the semantical reading of the box modality as we just overloaded it so that $\Box\kappa_1$ *means '*κ_1 *has been authenticated'. Let us also note that* $\Box\kappa_1 * \Box\kappa_2$ *does not denote the composition of two authenticated key, which would yield a key (and the definition of structure, such as a monoid, where keys can be composed), but the fact that the two keys have been independently authenticated.* $\Box$

Example 7 (Crimson Tide [24] Part 2)**.** *The use of a protocol can also be represented in Bifurcation Logic. This is illustrated in Figure 3. Here the idea is that a protocol is modelled by an implicational formula that, given an authenticated key, yields a key, which may then need to be authenticated. (Informally, the implicational formula may be thought of as the type of a function that returns a key.)*

We can see how in an inessential, slightly more detailed, variant of the set-up, Figure 4 represents the use of protocols in Crimson Tide *as follows:*

1. *In the discussion based on Figure 2, the role of protocols is suppressed.*

[2] Released in France as 'USS Alabama'.

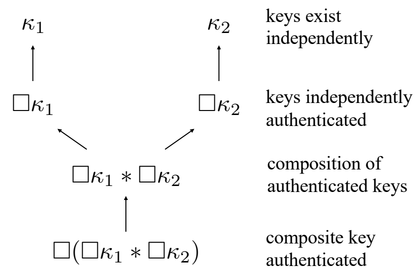


Figure 2: Joint access

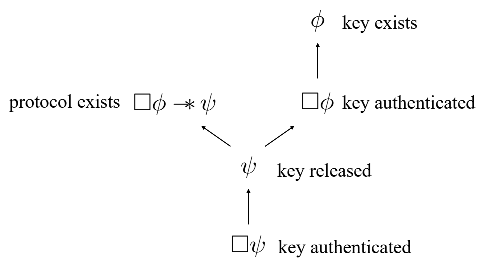


Figure 3: Protocol for obtaining a key

2. *Figure 3 illustrates the use of protocols in general.*
3. *In the case of our example, we represent the Captain's key permitting the release of weapons by ψ. For the key to be authenticated, it must have been released through the protocol.*
4. *The protocol is given, in Figure 4, by the implicational formula $\Box(\Box\phi_1 * \Box\phi_2) -\!\!* \psi$. That is, the authenticated composite key allows access to the Captain's key, ψ, which, when authenticated, allows the weapons to be released.*

Here we use the modality $\Box$ to denote authentication, but what is the role of separation? The role of $*$ should be clear: it enforces the independence of multiple authentications. But what of $-\!\!*$? It ensures that there is no interference between the authentication of the composite key and its use to make available the Captain's key, ψ (which must itself be authenticated for use).

Note the essential use of the semantics of $-\!\!*$: first, the separation between worlds afforded by $-\!\!*$, as opposed to $\rightarrow$, ensures no interference between the existence of the protocol and existence of the (authenticated) key to which it applies; second, the particular semantic form of this implication — in that its application 'looks back' down the ordering — captures exactly the release of the Captain's key, ψ, through access to the authenticated composite key. $\Box$

4 A Tableaux Calculus for BL: $\mathbf{T_{BL}}$

The tableaux calculus for classical propositional logic [8] can be adapted systematically to calculi for many non-classical logics by the addition of labelling. The basic idea is that the structure of a Kripke model for a given non-classical logic is used to define a tableaux calculus for that logic by reflecting its structure in an algebra of labels that is used to impose side-conditions on the tableaux

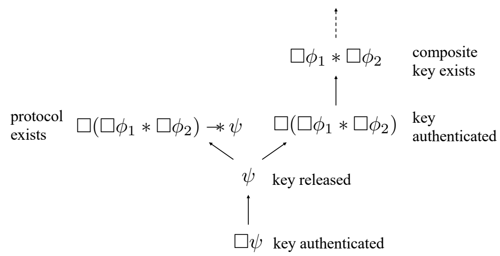


Figure 4: Protocol for obtaining a composite key

rules. Through this mechanism, the basic classical and/or tableaux figures are modified to capture non-classical connectives.

$\mathrm{T_{BL}}$ is presented in Figure 5. $\mathrm{T_{BL}}$ has logical rules that capture the meaning of the connectives, structural rules that capture the properties of BL models, and closure rules (whose conclusion is a cross mark) that capture (logical or structural) inconsistencies. As usual, closure rules put an end to the expansion of a branch. $\mathrm{T_{BL}}$ can address either BL or its lax variant depending on the inclusion or not of the optional persistency rule $\|_P$. We refer to $\mathrm{T_{BL}}$ without the rule $\|_P$ as lax $\mathrm{T_{BL}}$. All the results for BL presented in this section also hold for lax BL by simply ignoring in the proofs all the references to the rule $\|_P$ or to the persistent separation property. Therefore, $\mathrm{T_{BL}}$ is sound and complete w.r.t. BL models, and lax $\mathrm{T_{BL}}$ is sound and complete w.r.t. lax BL models.

Definition 8. *Let L be a countable set of symbols, called* labels*, that contains a specific symbol ϵ. A* labelled formula *is a pair (ϕ, x), written $\phi : x$, where ϕ is a formula and x is a label. A* label constraint *is an expression of the form $x \prec y$, where x, y are labels.* □

Definition 9. *Let Sg be the set $\{T, F\}$ of* signs*. A* signed labelled formula *is a triple (S, ϕ, x), written $S\,\phi : x$, where S is a sign and $\phi : x$ is a labelled formula. Similarly, a* signed label constraint *is a label constraint prefixed with a sign.* □

We define $T\,x \sim y$ as a shorthand for the expression $T\,x \prec y, T\,y \prec x$. Similarly, $T\,x \parallel y$ is a shorthand for $F\,x \prec y, F\,y \prec x$ and $T\,x \prec y \parallel z$ is a shorthand for $T\,x \prec y, T\,x \prec z, T\,y \parallel z$.

Definition 10. *A* tableau for $\phi : x$ *is a finitely branching rooted tree built inductively according to the rules given in Figure 5 and the root node of which is the signed labelled formula $F\,\phi : x$.* □

Definition 11. *A tableau branch $\mathfrak{b}$ is* closed *if it ends with a closure rule. A tableau $\mathfrak{t}$ is* closed *if all of its branches are closed.* □

Definition 12. *Let $\phi : x$ be a labelled formula. A* $\mathrm{T_{BL}}$-proof of $\phi : x$ *is a* closed tableau for $\phi : x$*. We write $\vdash \phi : x$ if $\phi : x$ is* provable in $\mathrm{T_{BL}}$*, that is if there exists a T_{BL}-proof of $\phi : x$. Similarly, a formula ϕ is* provable *in T_{BL}, written $\vdash \phi$, if $\vdash \phi : x$ for some label x.* □

Example 13. *The tableau depicted on the left-hand side of Figure 6 is a T_{BL}-proof of $\phi * \psi \to \Diamond\phi \wedge \Diamond\psi$. Step 3 is only given to improve readability and is not really necessary as it is just the explicit expansion of the shorthand for $T\,x \prec y \parallel z$.* □

Example 14. *A T_{BL}-proof of $\varphi \equiv (\Diamond\phi * \psi) \leftrightarrow (\phi * \psi)$ is given in the tableau below.*

Logical rules

$$\frac{T\,\neg\phi : x}{F\,\phi : x}\;{\scriptstyle T\neg} \qquad \frac{F\,\neg\phi : x}{T\,\phi : x}\;{\scriptstyle F\neg} \qquad \frac{T\,\phi \wedge \psi : x}{\begin{array}{c} T\,\phi : x \\ T\,\psi : x \end{array}}\;{\scriptstyle T\wedge} \qquad \frac{F\,\phi \wedge \psi : x}{F\,\phi : x \;\mid\; F\,\psi : x}\;{\scriptstyle F\wedge} \qquad \frac{\begin{array}{c} T\,\Box\phi : x \\ T\,x \prec y \end{array}}{T\,\phi : y}\;{\scriptstyle T\Box} \qquad \frac{F\,\Box\phi : x}{\begin{array}{c} T\,x \prec u \\ F\,\phi : u \end{array}}\;{\scriptstyle F\Box}$$

$$\frac{T\,\phi * \psi : x}{\begin{array}{c} T\,x \prec u \parallel v \\ T\,\phi : u \\ T\,\psi : v \end{array}}\;{\scriptstyle T*} \qquad \frac{\begin{array}{c} F\,\phi * \psi : x \\ T\,x \prec y \parallel z \end{array}}{F\,\phi : y \;\mid\; F\,\psi : z}\;{\scriptstyle F*} \qquad \frac{\begin{array}{c} T\,\phi \twoheadrightarrow \psi : x \\ T\,z \prec x \parallel y \end{array}}{F\,\phi : y \;\mid\; T\,\psi : z}\;{\scriptstyle T\twoheadrightarrow} \qquad \frac{F\,\phi \twoheadrightarrow \psi : x}{\begin{array}{c} T\,v \prec x \parallel u \\ T\,\phi : u \\ F\,\psi : v \end{array}}\;{\scriptstyle F\twoheadrightarrow} \qquad \frac{\begin{array}{c} S\,\phi : x \\ T\,x \sim y \end{array}}{S\,\phi : y}\;{\scriptstyle S\sim}$$

Structural rules

$$\frac{S\,\phi : x}{T\,x \prec x}\;{\scriptstyle \prec_R} \qquad \frac{T\,x \prec y,\; T\,y \prec z}{T\,x \prec z}\;{\scriptstyle \prec_T} \qquad \frac{S\,\phi : x,\; S\,\psi : y}{T\,x \prec y \;\mid\; T\,y \prec x \;\mid\; T\,x \parallel y}\;{\scriptstyle CD} \qquad \left[\frac{T\,x_1 \parallel x_2,\; T\,x_i \prec y}{T\,x_j \parallel y}\;{\scriptstyle \parallel_P}\right]$$

$$\begin{array}{c} T\,x \parallel z \\ \hline\hline F\,x \prec z,\; F\,z \prec x \end{array}\;{\scriptstyle \parallel} \qquad \begin{array}{c} T\,x \prec y \parallel z \\ \hline\hline T\,x \prec y,\; T\,x \prec z,\; T\,y \parallel z \end{array}\;{\scriptstyle B} \qquad \begin{array}{c} T\,x \prec y,\; T\,y \prec x \\ \hline\hline T\,x \sim y \end{array}\;{\scriptstyle \sim} \qquad \frac{S\,\phi : x}{T\,\epsilon \prec x}\;{\scriptstyle \prec_\epsilon}$$

Closure rules

$$\frac{T\,\phi : x,\; F\,\phi : x}{\times}\;{\scriptstyle \times_\Phi} \qquad \frac{T\,x \prec y,\; F\,x \prec y}{\times}\;{\scriptstyle \times_\prec} \qquad \frac{F\,\epsilon \prec x}{\times}\;{\scriptstyle \times_\epsilon}$$

Derivable rules

$$\frac{T\,\phi \vee \psi : x}{T\,\phi : x \;\mid\; T\,\psi : x}\;{\scriptstyle T\vee} \qquad \frac{F\,\phi \vee \psi : x}{\begin{array}{c} F\,\phi : x \\ F\,\psi : x \end{array}}\;{\scriptstyle F\vee} \qquad \frac{T\,\phi \to \psi : x}{F\,\phi : x \;\mid\; T\,\psi : x}\;{\scriptstyle T\to} \qquad \frac{F\,\phi \to \psi : x}{\begin{array}{c} T\,\phi : x \\ F\,\psi : x \end{array}}\;{\scriptstyle F\to}$$

$$\frac{T\,\phi \leftrightarrow \psi : x}{\begin{array}{c} T\,\phi : x \\ T\,\psi : x \end{array} \;\middle|\; \begin{array}{c} F\,\phi : x \\ F\,\psi : x \end{array}}\;{\scriptstyle T\leftrightarrow} \qquad \frac{F\,\phi \leftrightarrow \psi : x}{\begin{array}{c} F\,\phi : x \\ T\,\psi : x \end{array} \;\middle|\; \begin{array}{c} T\,\phi : x \\ F\,\psi : x \end{array}}\;{\scriptstyle F\leftrightarrow} \qquad \frac{T\,\Diamond\phi : x}{\begin{array}{c} T\,x \prec u \\ T\,\phi : u \end{array}}\;{\scriptstyle T\Diamond} \qquad \frac{\begin{array}{c} F\,\Diamond\phi : x \\ T\,x \prec y \end{array}}{F\,\phi : y}\;{\scriptstyle F\Diamond} \qquad \frac{T\,x \parallel y}{T\,y \parallel x}\;{\scriptstyle \parallel_C}$$

$$\frac{T\,\bot : x}{\times}\;{\scriptstyle \times_\bot} \qquad \frac{F\,\top : x}{\times}\;{\scriptstyle \times_\top} \qquad \frac{\begin{array}{c} T\,x_1 \parallel x_2 \\ T\,x_i \prec x_j \end{array}}{\times}\;{\scriptstyle \times_\parallel} \qquad \frac{\begin{array}{c} T\,x \prec y_1 \parallel y_2 \\ F\,x \prec y_i \end{array}}{\times}\;{\scriptstyle \times_{BA}} \qquad \frac{\begin{array}{c} T\,x \prec y_1 \parallel y_2 \\ T\,y_i \prec y_j \end{array}}{\times}\;{\scriptstyle \times_{BS}}$$

Side-conditions and comments

- In rules introducing u and/or v, u and v are distinct labels that are fresh in the branch.
- In rules involving i and/or j, $i \in \{1, 2\}$ and $j = 3 - i$.
- Double lines indicate rules that can be used bottom-up or top-down.
- The brackets indicate that the rule is optional (included for BL, excluded for lax BL).

Figure 5: Rules of the T_{BL} calculus

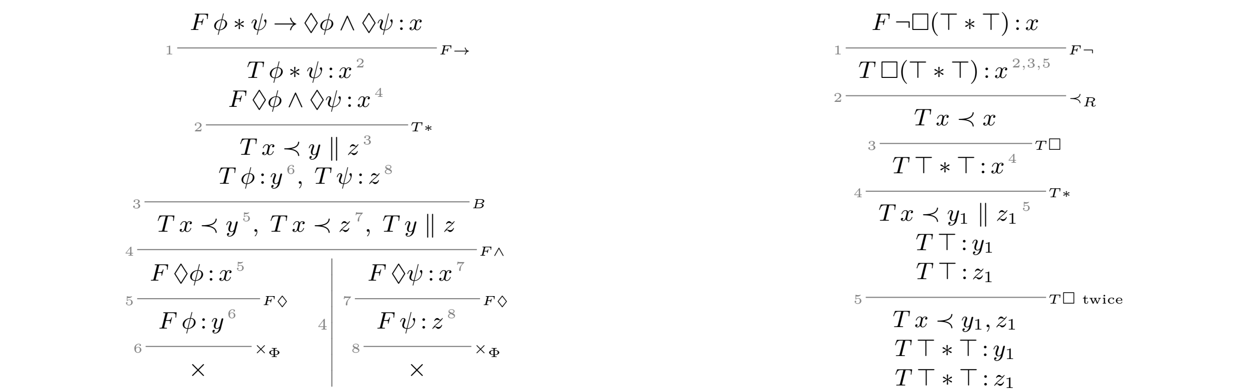
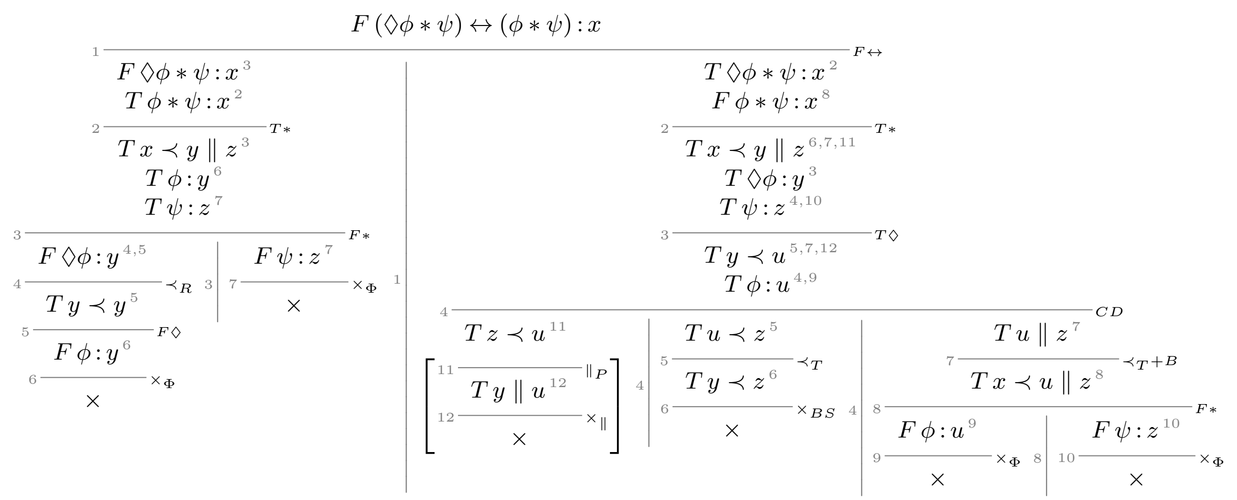


Figure 6: Tableaux examples



After Step 1, the tableau splits into two parts, the interesting one being the second one (on the right-hand side of the first vertical rule) that eventually leads to four closed branches. Step 4 is an example of the case distinction rule. We remark that the first case only leads to a closed branch when the optional rule $\|_P$ is used (resulting in the optional steps Step 11 *and Step* 12*). Indeed, without persistence of separation, one can build a countermodel of φ as illustrated in Figure 6. Hence, φ is a formula that distinguishes persistent from non-persistent BL.* $\square$

Example 15. *The tableau depicted on the right-hand side of Figure 6 is an example of an infinite open tableau for $\neg\square(\top * \top)$. The signed formula $T\,\square(\top * \top) : x$ introduced in Step 1 must be expanded for all labels u such that $T\, x \prec u$ occurs in the branch. The first such label is x, so that $T\ \top * \top : x$ is introduced in Step 3.*

The expansion of $T\ \top * \top : x$ in Step 4 generates two new successors of x, namely y_1, z_1, and then induces Step 5, where two new expansions of $T\,\square(\top * \top) : x$ are performed (for simplicity, in one step for both y_1 and z_1). Step 5 results in the introduction of $T\ \top * \top : y_1$ and $T\ \top * \top : z_1$, the expansion of which creates four new successors of x, two of y_1 and two of z_1.

The whole process described previously then repeats itself infinitely often as the BL models such that $x \Vdash \square(\top * \top)$ are the ones in which all successors of x have at least two distinct successors. This point is further discussed at the beginning of Section 9. $\square$

5 Soundness and Completeness of $\mathbf{T_{BL}}$

Definition 16. *The* domain of a tableau branch $\mathfrak{b}$*, denoted* $D(\mathfrak{b})$*, is the set* $\{\, x \mid (S\,\phi : x) \in \mathfrak{b} \,\}$ *of all labels occurring in* $\mathfrak{b}$*.* □

Definition 17 (Realization)**.** *Let* $\mathfrak{b}$ *be a tableau branch. A* realization *of* $\mathfrak{b}$ *in a BL model* $\mathcal{M} = (W, \leq, R, V, \Vdash)$ *is a function* ρ *from* $D(\mathfrak{b})$ *to* W *such that*

- *if* $(T\,\phi : x) \in \mathfrak{b}$*, then* $\rho(x) \Vdash \phi$*, and if* $(F\,\phi : x) \in \mathfrak{b}$*, then* $\rho(x) \nVdash \phi$*,*
- *if* $(T\,x \prec y) \in \mathfrak{b}$*, then* $\rho(x) \leq \rho(y)$*, and if* $(F\,x \prec y) \in \mathfrak{b}$*, then* $\rho(x) \nleq \rho(y)$*,*
- *if* $\epsilon \in D(\mathfrak{b})$*, then* $\rho(\epsilon) = \omega$*, where* ω *is the root in* $\mathcal{M}$*.*

A tableau branch is realizable *if it has at least one realization in some BL model. A tableau is* realizable *if it has at least one realizable branch.* □

Lemma 18. *If a tableau branch is closed, then it is not realizable.*

Proof. Let $\mathfrak{b}$ be a closed tableau branch. Assume that $\mathfrak{b}$ is realizable. Then, there exists a realization ρ of $\mathfrak{b}$ in some BL model.

- If $\mathfrak{b}$ is closed because both $(T\,\phi : x) \in \mathfrak{b}$ and $(F\,\phi : x) \in \mathfrak{b}$, then by definition of a realization, we have both $\rho(x) \Vdash \phi$ and $\rho(x) \nVdash \phi$, which is a contradiction.
- If $\mathfrak{b}$ is closed because both $(T\,x \prec y) \in \mathfrak{b}$ and $(F\,x \prec y) \in \mathfrak{b}$, then by definition of a realization, we have both $\rho(x) \leq \rho(y)$ and $\rho(x) \nleq \rho(y)$, which is a contradiction.
- If $\mathfrak{b}$ is closed because $(F\,\epsilon \prec x) \in \mathfrak{b}$, then, by definition of a realization, we have $\rho(\epsilon) \nleq \rho(x)$, which is a contradiction since $\rho(\epsilon) = \omega$ and ω is the root in $\mathcal{M}$.

Therefore, $\mathfrak{b}$ cannot be realizable. □

Lemma 19. *If a tableau* $\mathfrak{t}$ *is realizable, then expanding* $\mathfrak{t}$ *using one of the tableau expansion rules given in Figure 5 results in a realizable tableau* $\mathfrak{t}'$*.*

Proof. Suppose that $\mathfrak{t}$ is realizable. Then, it has at least one realizable branch $\mathfrak{b}$. If $\mathfrak{t}'$ is obtained from $\mathfrak{t}$ by expanding a branch that is distinct from $\mathfrak{b}$ then $\mathfrak{t}'$ remains realizable since it still contains the unchanged realizable branch $\mathfrak{b}$. Otherwise, $\mathfrak{t}'$ is obtained by expanding $\mathfrak{b}$ into $\mathfrak{b}'$. We then proceed by case analysis on the tableau rule expanding $\mathfrak{b}$. Let ρ be a realization of $\mathfrak{b}$ is some BL model.

We consider just a few illustrative cases, the others being similar.

- Case $T{\twoheadrightarrow}$:

  If $(T\,z \prec x \parallel y) \in \mathfrak{b}$ and $(T\,\phi \twoheadrightarrow \psi : x) \in \mathfrak{b}$, then by Definition 17, we get $R(\rho(z), \rho(x), \rho(y))$ and $\rho(x) \Vdash \phi \twoheadrightarrow \psi$. It then follows from Definition 4 that $\rho(y) \Vdash \phi$ and $\rho(z) \nVdash \psi$. Therefore, ρ realizes $\mathfrak{b}'$.

- Case $F{\twoheadrightarrow}$:

  If $(F\,\phi \twoheadrightarrow \psi : x) \in \mathfrak{b}$, then by Definition 17, we get $\rho(x) \nVdash \phi \twoheadrightarrow \psi$. By Definition 4, there exist $w_1, w_2 \in W$ such that $R(w_2, \rho(x), w_1)$, $w_1 \Vdash \phi$ and $w_2 \nVdash \psi$. Since $\mathfrak{b}'$ extends $\mathfrak{b}$ with $(T\,v \prec x \parallel u)$, $(T\,\phi : u)$ and $(F\,\psi : v)$, where u and b are distinct fresh labels, we can extend ρ into a realization of $\mathfrak{b}'$ by setting $\rho(u) = w_1$ and $\rho(v) = w_2$.

The other cases are similar. $\square$

Theorem 20 (Soundness)**.** *If $\vdash\phi$, then $\vDash\phi$.*

Proof. If $\vdash\phi$ then we have a closed tableau $\mathfrak{t}$ for $\phi : x$ for some label x. Assume that $\nvDash\phi$. Any tableau construction procedure that results in $\mathfrak{t}$ begins with the tableau $\mathfrak{t}_0$ that consists in the single node $F\,\phi : x$. Since $\mathfrak{t}_0$ is realizable, Lemma 19 implies that $\mathfrak{t}$ should also be realizable and should therefore contain at least one realizable branch $\mathfrak{b}$. By Lemma 18, $\mathfrak{b}$ cannot be closed. It then follows that $\mathfrak{t}$ is open, which is a contradiction. Thus, we have $\vDash\phi$. $\square$

Definition 21. *A tableau branch $\mathfrak{b}$ is* saturated *if it satisfies all of the following conditions:*

1. *if $(T\,\neg\phi : x) \in \mathfrak{b}$, then $(F\,\phi : x) \in \mathfrak{b}$*
2. *if $(F\,\neg\phi : x) \in \mathfrak{b}$, then $(T\,\phi : x) \in \mathfrak{b}$*
3. *if $(T\,\phi \wedge \psi : x) \in \mathfrak{b}$, then $(T\,\phi : x) \in \mathfrak{b}$ and $(T\,\psi : x) \in \mathfrak{b}$*
4. *if $(F\,\phi \wedge \psi : x) \in \mathfrak{b}$, then $(F\,\phi : x) \in \mathfrak{b}$ or $(F\,\psi : x) \in \mathfrak{b}$*
5. *if $(T\,\Box\phi : x) \in \mathfrak{b}$, then, for all $y \in D(\mathfrak{b})$, if $(T\,x \prec y) \in \mathfrak{b}$, then $(T\,\phi : y) \in \mathfrak{b}$*
6. *if $(F\,\Box\phi : x) \in \mathfrak{b}$, then, for some $y \in D(\mathfrak{b})$, $(T\,x \prec y) \in \mathfrak{b}$ and $(F\,\phi : y) \in \mathfrak{b}$*
7. *if $(T\,\phi * \psi : x) \in \mathfrak{b}$, then, for some $y, z \in D(\mathfrak{b})$, $(T\,x \prec y \parallel z) \in \mathfrak{b}$, $(T\,\phi : y) \in \mathfrak{b}$, and $(T\,\psi : z) \in \mathfrak{b}$*
8. *if $(F\,\phi*\psi : x) \in \mathfrak{b}$, then, for all $y, z \in D(\mathfrak{b})$, if $(T\,x \prec y \parallel z) \in \mathfrak{b}$, then $(F\,\phi : y) \in \mathfrak{b}$ or $(F\,\psi : z) \in \mathfrak{b}$*
9. *if $(T\,\phi \mathrel{-\!\!*} \psi : x) \in \mathfrak{b}$, then, for all $y, z \in D(\mathfrak{b})$, if $(T\,z \prec x \parallel y) \in \mathfrak{b}$, then $(F\,\phi : y) \in \mathfrak{b}$ or $(T\,\psi : z) \in \mathfrak{b}$*
10. *if $(F\,\phi \mathrel{-\!\!*} \psi : x) \in \mathfrak{b}$, then, for some $y, z \in D(\mathfrak{b})$, $(T\,z \prec x \parallel y) \in \mathfrak{b}$, $(T\,\phi : y) \in \mathfrak{b}$, and $(F\,\psi : z) \in \mathfrak{b}$*
11. *if $(T\,x \sim y) \in \mathfrak{b}$ and $(S\,\phi : x) \in \mathfrak{b}$, then $(S\,\phi : y) \in \mathfrak{b}$*
12. *if $(S\,\phi : x) \in \mathfrak{b}$, then $(T\,x \prec x) \in \mathfrak{b}$*
13. *if $(T\,x \prec y) \in \mathfrak{b}$ and $(T\,y \prec z) \in \mathfrak{b}$, then $(T\,x \prec z) \in \mathfrak{b}$*
14. *if $(S\,\phi : x) \in \mathfrak{b}$ and $(S\,\psi : y) \in \mathfrak{b}$, then $(T\,x \prec y) \in \mathfrak{b}$ or $(T\,y \prec x) \in \mathfrak{b}$ or $(T\,x \parallel y) \in \mathfrak{b}$*
15. *if $(T\,x \parallel y) \in \mathfrak{b}$ and $(T\,x \prec z) \in \mathfrak{b}$, then $(T\,z \parallel y) \in \mathfrak{b}$*
16. *if $(S\,\phi : x) \in \mathfrak{b}$, then $(T\,\epsilon \prec x) \in \mathfrak{b}$.* $\square$

Lemma 22. *Let $\mathfrak{b}$ be a saturated open tableau branch. The binary relation $\prec^{\mathfrak{b}}$ over $D(\mathfrak{b})$ defined as $x \prec^{\mathfrak{b}} y$ iff $(T\,x \prec y) \in \mathfrak{b}$ is a quasi-order over $D(\mathfrak{b})$ such that if $(F\,x \prec y) \in \mathfrak{b}$ then $x \not\prec^{\mathfrak{b}} y$.*

Proof. Transitivity and reflexivity of $\prec^{\mathfrak{b}}$ clearly follow from Conditions 12 and 13 of Definition 21. Now, if $(F\,x \prec y) \in \mathfrak{b}$, then $(T\,x \prec y) \notin \mathfrak{b}$ because $\mathfrak{b}$ is open. Hence, $x \not\prec^{\mathfrak{b}} y$ by definition of $\prec^{\mathfrak{b}}$. $\square$

Lemma 23. *Let $\mathfrak{b}$ be a saturated open branch. Let $\sim^{\mathfrak{b}}$ be the equivalence relation over $D(\mathfrak{b})$ defined as $x \sim^{\mathfrak{b}} y$ iff $x \prec^{\mathfrak{b}} y$ and $y \prec^{\mathfrak{b}} x$. For all $x \in D(\mathfrak{b})$, let $[x] = \{\, y \in D(\mathfrak{b}) \mid x \sim^{\mathfrak{b}} y \,\}$ denote the equivalence class of x under $\sim^{\mathfrak{b}}$ and let $W^{\mathfrak{b}}$ denote the quotient $D(\mathfrak{b})/\sim^{\mathfrak{b}}$. Then, the binary relation $\leq^{\mathfrak{b}}$ over $W^{\mathfrak{b}}$ defined as $[x] \leq^{\mathfrak{b}} [y]$ iff $x \prec^{\mathfrak{b}} y$ is a partial order over $W^{\mathfrak{b}}$ that satisfies the persistent separation property.*

Proof. First, we show that $\leq^{\mathfrak{b}}$ is well defined by showing that the following conditions are equivalent for all $[u], [v] \in W^{\mathfrak{b}}$:

(*a*) $u \prec^{\mathfrak{b}} v$

(b) $x \prec^{\mathfrak{b}} y$ for all $x \in [u]$ and all $y \in [v]$

(c) $x \prec^{\mathfrak{b}} y$ for some $x \in [u]$ and some $y \in [v]$

It is clear that (a) implies (c) and that (b) implies (c) (and (a)). Therefore, we only have to show that (c) implies (b). Assume $x \prec^{\mathfrak{b}} y$ for some $x \in [u]$ and some $y \in [v]$. Then, $x \sim^{\mathfrak{b}} u$ implies $u \prec^{\mathfrak{b}} x$ and $y \sim^{\mathfrak{b}} v$ implies $y \prec^{\mathfrak{b}} v$. Pick an arbitrary $x' \in [u]$, then $x' \sim^{\mathfrak{b}} u$ implies $x' \prec^{\mathfrak{b}} u$. Since $u \prec^{\mathfrak{b}} x$, we get $x' \prec^{\mathfrak{b}} x$. Pick an arbitrary $y' \in [v]$, then $y' \sim^{\mathfrak{b}} v$ implies $v \prec^{\mathfrak{b}} y'$ Since $y \prec^{\mathfrak{b}} v$, we get $y \prec^{\mathfrak{b}} y'$. Finally, since we assumed $x \prec^{\mathfrak{b}} y$, $x' \prec^{\mathfrak{b}} x$ and $y \prec^{\mathfrak{b}} y'$ imply $x' \prec^{\mathfrak{b}} y'$.

Second, we show that $\leq^{\mathfrak{b}}$ is a partial order over $W^{\mathfrak{b}}$. Since $\prec^{\mathfrak{b}}$ is a quasi-order over $D(\mathfrak{b})$ by Lemma 22, it immediately follows that $\leq^{\mathfrak{b}}$ is both reflexive and transitive. It remains to show that $\leq^{\mathfrak{b}}$ is anti-symmetric. Assume $[x] \leq^{\mathfrak{b}} [y]$ and $[y] \leq^{\mathfrak{b}} [x]$, then, by definition of $\leq^{\mathfrak{b}}$, we have $x \prec^{\mathfrak{b}} y$ and $y \prec^{\mathfrak{b}} x$, from which we get $x \sim^{\mathfrak{b}} y$ by definition of $\sim^{\mathfrak{b}}$. Hence, $[x] = [y]$.

Last, we show that $\leq^{\mathfrak{b}}$ satisfies the separation persistence property stated in Definition 2. We pick arbitrary $[x], [y], [z] \in W^{\mathfrak{b}}$ such that $[x] \perp [y]$ and $[x] \leq^{\mathfrak{b}} [z]$ and show that $[z] \perp [y]$. We have the following facts:

(i) By definition of $\perp$, $[x] \perp [y]$ implies $[x] \not\leq^{\mathfrak{b}} [y]$ and $[y] \not\leq^{\mathfrak{b}} [x]$.

(ii) By definition of $\leq^{\mathfrak{b}}$, $[x] \leq^{\mathfrak{b}} [z]$ implies $x \prec^{\mathfrak{b}} z$.

Assume that $[z] \leq^{\mathfrak{b}} [y]$. Then, $z \prec^{\mathfrak{b}} y$ by definition of $\leq^{\mathfrak{b}}$. Since $x \prec^{\mathfrak{b}} z$ and $z \prec^{\mathfrak{b}} y$ imply $x \prec^{\mathfrak{b}} y$, we get $[x] \leq^{\mathfrak{b}} [y]$ by definition of $\leq^{\mathfrak{b}}$, which contradicts $[x] \perp [y]$ (i). Hence, $[z] \not\leq^{\mathfrak{b}} [y]$.
Assume that $[y] \leq^{\mathfrak{b}} [z]$. Then, $y \prec^{\mathfrak{b}} z$ by definition of $\leq^{\mathfrak{b}}$, from which we get $(T\, y \prec z) \in \mathfrak{b}$ by definition of $\prec^{\mathfrak{b}}$. Besides, $[x] \not\leq^{\mathfrak{b}} [y]$ and $[y] \not\leq^{\mathfrak{b}} [x]$ imply $x \not\prec^{\mathfrak{b}} y$ and $y \not\prec^{\mathfrak{b}} x$ by definition of $\leq^{\mathfrak{b}}$. By definition of $\prec^{\mathfrak{b}}$, we get $(T\, x \prec y) \notin \mathfrak{b}$ and $(T\, y \prec x) \notin \mathfrak{b}$. Since $\mathfrak{b}$ is saturated, $(T\, x \parallel y) \in \mathfrak{b}$ then follows from Condition 14 of Definition 21. In turn, $(T\, x \parallel y) \in \mathfrak{b}$ and $(T\, y \prec z) \in \mathfrak{b}$ imply $(T\, x \parallel z) \in \mathfrak{b}$ by Condition 15 of Definition 21. $(T\, x \parallel z) \in \mathfrak{b}$ implies $(F\, x \prec z) \in \mathfrak{b}$ by definition of $\parallel$, but from $(F\, x \prec z) \in \mathfrak{b}$, Lemma 22 implies $x \not\prec^{\mathfrak{b}} z$, which contradicts $x \prec^{\mathfrak{b}} z$ (ii). Hence, $[y] \not\leq^{\mathfrak{b}} [z]$.
From, $[z] \not\leq^{\mathfrak{b}} [y]$ and $[y] \not\leq^{\mathfrak{b}} [z]$, we conclude $[y] \perp [z]$ by definition of $\perp$. $\square$

Lemma 24 (Model existence)**.** *Let $\mathfrak{b}$ be a saturated open tableau branch. Then, $\mathfrak{b}$ induces a BL model $\mathcal{M}^{\mathfrak{b}} = (W, \leq, R, V, \Vdash)$, where*

- *$W = W^{\mathfrak{b}}$ and $\leq = \leq^{\mathfrak{b}}$ as per Lemma 23,*
- *W has a root $\omega = [\epsilon]$,*
- *R is the ternary relation induced by $\leq^{\mathfrak{b}}$ as per Definition 3, and,*
- *for all worlds $[x] \in W^{\mathfrak{b}}$, $V([x]) = \{\, p \mid (T\, p : x) \in \mathfrak{b} \,\}$.*

Moreover, $\mathcal{M}^{\mathfrak{b}}$ is such that if $(T\, \phi : x) \in \mathfrak{b}$ then $[x] \Vdash \phi$ and if $(F\, \phi : x) \in \mathfrak{b}$ then $[x] \nVdash \phi$.

Proof. By mutual induction on the structure of ϕ for all labels x.

Base case $\phi = p$:

If $(T\, p : x) \in \mathfrak{b}$, then $p \in V([x])$ by definition of V, which implies $[x] \Vdash p$ by Definition 4.

If $(F\, p : x) \in \mathfrak{b}$, then since $\mathfrak{b}$ is open, we have $(T\, p : x) \notin \mathfrak{b}$. Hence, we get $p \notin V([x])$ by definition of V, which implies $[x] \nVdash p$ by Definition 4.

Case $\phi = \phi_1 \mathrel{-\!\!*} \phi_2$**:**

If $(F\,\phi_1 \mathrel{-\!\!*} \phi_2 : x) \in \mathfrak{b}$, then for some $y, z \in D(\mathfrak{b})$, we have $(T\,z \prec x \parallel y) \in \mathfrak{b}$, $(T\,\phi_1 : y) \in \mathfrak{b}$ and $(F\,\phi_2 : z) \in \mathfrak{b}$. $(T\,z \prec x \parallel y) \in \mathfrak{b}$ implies $R([z],[x],[y])$ by Lemma 23. Moreover, by induction hypothesis, $(T\,\phi_1 : y) \in \mathfrak{b}$ and $(F\,\phi_2 : z) \in \mathfrak{b}$ imply $[y] \Vdash \phi_1$ and $[z] \nVdash \phi_2$. Hence, by Definition 4, we have $[x] \nVdash \phi_1 \mathrel{-\!\!*} \phi_2$.

If $(T\,\phi_1 \mathrel{-\!\!*} \phi_2 : x) \in \mathfrak{b}$, then for all $y, z \in D(\mathfrak{b})$, if $(T\,z \prec x \parallel y) \in \mathfrak{b}$ then $(F\,\phi_1 : y) \in \mathfrak{b}$ or $(T\,\phi_2 : z) \in \mathfrak{b}$. Pick arbitrary $[u],[v] \in W^{\mathfrak{b}}$ such that $R([v],[x],[u])$ and $[u] \Vdash \phi_1$. Since $[v] \leq^{\mathfrak{b}} [x]$ and $[v] \leq^{\mathfrak{b}} [u]$, we have $(T\,v \prec x) \in \mathfrak{b}$ and $(T\,v \prec u) \in \mathfrak{b}$ by definition of $\leq^{\mathfrak{b}}$. Similarly, since $[x] \nleq^{\mathfrak{b}} [u]$ and $[u] \nleq^{\mathfrak{b}} [x]$, we have $(T\,x \prec u) \notin \mathfrak{b}$ and $(T\,u \prec x) \notin \mathfrak{b}$, which by Condition 14 implies $(T\,x \parallel u) \in \mathfrak{b}$. It then follows that we have $(T\,v \prec x \parallel u) \in \mathfrak{b}$, which implies $(F\,\phi_1 : u) \in \mathfrak{b}$ or $(T\,\phi_2 : v) \in \mathfrak{b}$. By the induction hypothesis, we then get $[u] \nVdash \phi_1$ or $[v] \Vdash \phi_2$. Since we assume $[u] \Vdash \phi_1$, we necessarily have $[v] \Vdash \phi_2$. Therefore, $[x] \Vdash \phi_1 \mathrel{-\!\!*} \phi_2$.

The other cases are similar. □

Corollary 25. *If $\mathfrak{b}$ is a saturated open tableau branch in a tableau for $\phi : x$, then the induced model $\mathcal{M}^{\mathfrak{b}} = (W^{\mathfrak{b}}, \leq^{\mathfrak{b}}, R, V, \Vdash)$ is such that $[x] \nVdash \phi$.*

Proof. Since $(F\,\phi : x)$ is the root of any tableau for $\phi : x$, The concluding property of Lemma 24 implies that $x \nVdash \phi$. □

Theorem 26 (Completeness)**.** *If $\Vdash \phi$, then $\vdash \phi$.*

Proof. It is standard to define a tableau construction procedure that applies the rules given in Figure 5 with a fair strategy. Such a procedure will either result in a finite closed tableau for ϕ, in which case we get a $\mathrm{T_{BL}}$-proof of ϕ, or build at least one (possibly infinite) complete open branch $\mathfrak{b}$, in which case $\mathfrak{b}$ gives rise to a BL model $\mathcal{M}^{\mathfrak{b}}$ such that $\mathcal{M}^{\mathfrak{b}} \nvDash \phi$ by Corollary 25. □

For simplicity, BL frames were introduced in Definition 3 as rooted trees, which justifies the presence of the two rules $\prec_\epsilon$ and $\times_\epsilon$ in $\mathrm{T_{BL}}$. Let us call rootless $\mathrm{T_{BL}}$ and rootless lax $\mathrm{T_{BL}}$ the tableau systems obtained respectively from $\mathrm{T_{BL}}$ and lax $\mathrm{T_{BL}}$ by dropping the rules $\prec_\epsilon$ and $\times_\epsilon$. A careful inspection of the tableau rules given in Figure 5 then shows that rootless $\mathrm{T_{BL}}$ and lax $\mathrm{T_{BL}}$ can only build partial orders that correspond to undirected graphs with a single connected component. More precisely, rootless $\mathrm{T_{BL}}$ builds partial orders that actually correspond to polytrees, which are DAGs whose underlying undirected graph is a tree. Polytrees are particular cases of multitrees, which are DAGs where the subgraph that can be reached from any single node forms a tree, or equivalently, diamond free partial orders. Similarly, rootless lax $\mathrm{T_{BL}}$ builds partial orders that correspond to connected DAGs. As a consequence, we get that rootless $\mathrm{T_{BL}}$ is sound and complete w.r.t. BL frames where rooted trees are replaced with polytrees, and that rootless lax $\mathrm{T_{BL}}$ is sound and complete w.r.t. BL frames where rooted trees are replaced with connected DAGs.

6 Multiagent Bifurcation Logic

Definition 27 (Language)**.** *Let A be a countable set of agents containing a distinguished element, denoted t, and called the* global agent*. The* formulae *of multiagent BL, or MBL, the set of which is denoted by Φ, are defined like the formulas of BL only that the connectives $\Box$, $*$ and $\mathrel{-\!\!*}$ are now indexed with agents $a \in A$, like so: $[a], *_a, \mathrel{-\!\!*}_a$. We also define $\langle a \rangle \phi := \neg [a] \neg \phi$.* □

Definition 28 (Slice)**.** *Let $\mathcal{T} = (W, \leq)$ be a partial order. $\mathcal{T}_0 = (W_0, \leq_0)$ is a* sub partial order *of $\mathcal{T}$ if $W_0 \subseteq W$ and (SP) for all w_1, $w_2 \in W_0$, $w_1 \leq_0 w_2$ if and only if $w_1 \leq w_2$. Additionally, $\mathcal{T}_0$ is* tight *if it is such that (ST) for all w_1, w_2, $w_3 \in W$, if $w_1 \leq w_2$, $w_2 \leq w_3$, and $w_1 \leq_0 w_3$, then $w_1 \leq_0 w_2$ and $w_2 \leq_0 w_3$. A tight sub partial order of $\mathcal{T}$ is called a* slice *of $\mathcal{T}$.* □

Definition 29. *Let W be a set of elements and let I be a function from $W \times W$ to $\wp(A)$. For all $a \in A$, define $\leq_a = \{ (w_1, w_2) \mid a \in I(w_1, w_2) \}$ and $W_a = \{ w \mid w \leq_a w'$ or $w' \leq_a w$, for some $w' \}$. I is an* agent interpretation *if and only if $(W_t, \leq_t)$ is a tree, $W_t = W$ and for all $a \in A$, $(W_a, \leq_a)$ is a slice of $(W_t, \leq_t)$.* □

Let us note that $w_1 \not\leq_a w_2$ holds whenever $w_1 \notin W_a$ or $w_2 \notin W_a$. We then require $w_1 \perp_a w_2$ to hold if and only if w_1, $w_2 \in W_a$, $w_1 \not\leq_a w_2$, and $w_2 \not\leq_a w_1$. We write R_a for the bifurcation relation induced on W_a by $\leq_a$ as per Definition 3.

Definition 30 (Frame)**.** *An MBL* frame *is a structure $\mathcal{F} = (W, I)$, where W is a set of elements, called* worlds*, and I is an* agent interpretation*.* □

Given a MBL frame $\mathcal{F}$ and two agents a and b, we say that *a is subsumed by b* (in $\mathcal{F}$), whenever $\leq_a \subseteq \leq_b$. The following lemma states interesting properties of agent subsumption.

Lemma 31. *Let $\mathcal{F} = (W, I)$ be a MBL frame. Then, the following propositions hold for all agents a, $b \in A$ such that $\leq_a \subseteq \leq_b$:*

1. *$(W_a, \leq_a)$ is a slice of $(W_b, \leq_b)$*
2. *if $w_1 \perp_a w_2$, then $w_1 \perp_b w_2$*
3. *if $R_a(w, w_1, w_2)$, then $R_b(w, w_1, w_2)$.*

Proof. For the first proposition, it is clear that $\leq_a \subseteq \leq_b$ implies $W_a \subseteq W_b$. It remains to show that $(W_a, \leq_a)$ satisfies Conditions (SP) and (ST) of Definition 28 w.r.t. $(W_b, \leq_b)$. For Condition (SP), pick arbitrary w_1, $w_2 \in W_a$. First, since $\leq_a \subseteq \leq_b$, $w_1 \leq_a w_2$ implies $w_1 \leq_b w_2$. Second, if $w_1 \leq_b w_2$, then $w_1 \leq_t w_2$ because $(W_b, \leq_b)$ is a sub partial order of $(W_t, \leq_t)$. From $w_1 \leq_t w_2$, we get $w_1 \leq_a w_2$ since w_1, $w_2 \in W_a$ and $(W_a, \leq_a)$ is a sub partial order of $(W_t, \leq_t)$. Hence, for all w_1, $w_2 \in W_a$, $w_1 \leq_a w_2$ iff $w_1 \leq_b w_2$. For Condition (ST), suppose that $w_1 \leq_b w_2$, $w_2 \leq_b w_3$, and $w_1 \leq_a w_3$ hold for arbitrary w_1, w_2, $w_3 \in W_b$. Since $(W_b, \leq_b)$ is a sub partial order of $(W_t, \leq_t)$, $w_1 \leq_b w_2$ and $w_2 \leq_b w_3$ imply $w_1 \leq_t w_2$ and $w_2 \leq_t w_3$. Since $(W_a, \leq_a)$ is a slice of $(W_t, \leq_t)$, it satisfies Condition (ST) w.r.t. $(W_t, \leq_t)$ by Definition 28. Consequently, $w_1 \leq_t w_2$, $w_2 \leq_t w_3$ and $w_1 \leq_a w_3$ imply $w_1 \leq_a w_2$ and $w_2 \leq_a w_3$. Hence, for all w_1, w_2, $w_3 \in W_b$, if $w_1 \leq_b w_2$, $w_2 \leq_b w_3$, and $w_1 \leq_a w_3$, then $w_1 \leq_a w_2$ and $w_2 \leq_a w_3$.

For the second proposition, suppose that $w_1 \perp_a w_2$ holds. Then, w_1, $w_2 \in W_a$, $w_1 \not\leq_a w_2$, and $w_2 \not\leq_a w_1$ by definition. Assume that $w_1 \perp_b w_2$ does not hold. Since $\leq_a \subseteq \leq_b$, we get w_1, $w_2 \in W_b$. Hence, we must have either $w_1 \leq_b w_2$, or $w_2 \leq_b w_1$. If $w_1 \leq_b w_2$, then, the contradiction $w_1 \leq_a w_2$ follows from the first proposition since $(W_a, \leq_a)$ is a sub partial order of $(W_b, \leq_b)$ and should therefore satisfy Condition (SP) of Definition 28. Similarly, if $w_2 \leq_b w_1$, we get the contradiction that $w_2 \leq_a w_1$. Hence, $w_1 \perp_b w_2$ holds.

For the third proposition, suppose that $R_a(w, w_1, w_2)$ holds. Then, we have $w \leq_a w_1$, $w \leq_a w_2$, and $w_1 \perp_a w_2$ by definition. Since $\leq_a \subseteq \leq_b$, $w \leq_a w_1$ and $w \leq_a w_2$ imply $w \leq_b w_1$ and $w \leq_b w_2$. Moreover, $w_1 \perp_a w_2$ implies $w_1 \perp_b w_2$ by the second proposition. Hence, we have $R_b(w, w_1, w_2)$. □

Definition 32 (Model)**.** *An* MBL model *is a triple* $\mathcal{M} = (\mathcal{F}, V, \Vdash)$*, where* $\mathcal{F}$ *is an MBL frame and* V *is a* valuation function *from* W *to* $\wp(P)$*. The* satisfaction relation $\Vdash$ *is defined similar as in BL, with the following new clauses for indexed connectives:*

$$\begin{array}{rll}
\mathcal{M}, w \Vdash [a]\phi & \textit{iff} & \textit{for all } w' \in W, \textit{ if } w \leq_a w' \textit{ then } \mathcal{M}, w' \Vdash \phi \\
\mathcal{M}, w \Vdash \phi *_a \psi & \textit{iff} & \textit{for some } w_1, w_2 \in W \textit{ such that } R_a(w, w_1, w_2), \\
& & \mathcal{M}, w_1 \Vdash \phi \textit{ and } \mathcal{M}, w_2 \Vdash \psi \\
\mathcal{M}, w \Vdash \phi \mathrel{-\!\!*}_a \psi & \textit{iff} & \textit{for all } w_1, w_2 \in W \textit{ such that } R_a(w_2, w, w_1), \\
& & \textit{if } \mathcal{M}, w_1 \Vdash \phi \textit{ then } \mathcal{M}, w_2 \Vdash \psi
\end{array}$$

□

We remark that the connectives of multiagent BL that are indexed with t correspond to the non-indexed connectives of mono-modal BL. More precisely, $\Box = [t]$, $* = *_t$ and $\mathrel{-\!\!*} = \mathrel{-\!\!*}_t$.

We now prove two lemmas that exhibit interesting properties of agent subsumption from the logical point of view of the agent-based connectives. Lemma 34 is particularly interesting as it shows that agent subsumption can be witnessed by the satisfaction of the formula $[a]\langle b\rangle\top$ in every world of a MBL model.

Lemma 33. *Let* $\mathcal{M} = (\mathcal{F}, V, \Vdash)$ *be an MBL model. Let* a *and* b *be two agents such that* $\leq_a \subseteq \leq_b$*. The following propositions hold for all worlds* w*:*

1. *if* $\mathcal{M}, w \Vdash [b]\phi$*, then* $\mathcal{M}, w \Vdash [a]\phi$*,*
2. *if* $\mathcal{M}, w \Vdash \phi *_a \psi$*, then* $\mathcal{M}, w \Vdash \phi *_b \psi$*,*
3. *if* $\mathcal{M}, w \Vdash \phi \mathrel{-\!\!*}_b \psi$*, then* $\mathcal{M}, w \Vdash \phi \mathrel{-\!\!*}_a \psi$*.*

Proof. The first two propositions are a direct consequence of $\leq_a \subseteq \leq_b$. For the third one, suppose that $\mathcal{M}, w \Vdash \phi \mathrel{-\!\!*}_b \psi$ for some arbitrary world w. Let w_1, w_2 be two arbitrary worlds such that $\mathcal{M}, w_1 \Vdash \phi$ and $R_a(w_2, w, w_1)$. We need to show that $\mathcal{M}, w_2 \Vdash \psi$, but since $\leq_a \subseteq \leq_b$, $R_a(w_2, w, w_1)$ implies $R_b(w_2, w, w_1)$ by Lemma 31. Therefore, we have $\mathcal{M}, w \Vdash \phi \mathrel{-\!\!*}_b \psi$, $\mathcal{M}, w_1 \Vdash \phi$ and $R_b(w_2, w, w_1)$. Hence, $\mathcal{M}, w_2 \Vdash \psi$ by Definition 32. □

Lemma 34. *Let* $\mathcal{M} = (\mathcal{F}, V, \Vdash)$ *be an MBL model. Let* a *and* b *be two agents. Then,* $\mathcal{M} \Vdash [a]\langle b\rangle\top$ *if and only if* $\leq_a \subseteq \leq_b$*.*

Proof. For the if direction, assume $\leq_a \subseteq \leq_b$ and pick arbitrary $w_1, w_2 \in W$ such that $w_1 \leq_a w_2$. From the assumption, $w_1 \leq_a w_2$ implies $w_1 \leq_b w_2$. Since $\leq_b$ is a partial order, we have $w_2 \leq_b w_2$, which implies $\mathcal{M}, w_2 \Vdash \langle b\rangle\top$. Hence, $\mathcal{M}, w_1 \Vdash [a]\langle b\rangle\top$.

For the only if direction, assume $\mathcal{M} \Vdash [a]\langle b\rangle\top$ and pick arbitrary $w_1, w_2 \in W$ such that $w_1 \leq_a w_2$. First, since $\leq_a$ is a sub partial order of $\leq_t$, $w_1 \leq_a w_2$ implies $w_1 \leq_t w_2$. Second, $\mathcal{M} \Vdash [a]\langle b\rangle\top$ implies both $\mathcal{M}, w_1 \Vdash [a]\langle b\rangle\top$, and $\mathcal{M}, w_2 \Vdash [a]\langle b\rangle\top$. Hence, we have both $w_1 \leq_b w_1'$ for some $w_1' \in W$, and $w_2 \leq_b w_2'$ for some $w_2' \in W$. Thus, both w_1 and w_2 are defined for $\leq_b$, that is, we have $w_1, w_2 \in W_b$. Since $\leq_b$ is sub partial order of $\leq_t$, $w_1 \leq_b w_2$ immediately follows from $w_1 \leq_t w_2$ by definition of a sub partial order. □

7 Modelling With Multiagent Bifurcation Logic

We revisit the example based on the film 'Crimson Tide' [24] from Section 3 employing multiple agents in its model.

The agents that we'll employ are the following: two officers of the line, a_1 and a_2, the Executive Officer, b, and the Captain, c. These are the key players in the weapons-release scenario. One might also add agent for the weapons officer, d, referred to in the film [24] as 'Weaps'.

In addition to these intensional agents, that are internal to the modelling scenario, an extensional 'policy' agent is required. This represents the intended policy for the submarine's operations as defined by its command authority (i.e., the Pentagon, as implementing the will of the government) that is external to the modelling scenario. We denote this policy by the agent p. Formally, p has the same semantic structure — ordering $\leq_p$ and ternary relation R_p on worlds — as other agents. This use of intensional and extensional agents within the context of MBL seems to offer a quite generic modelling tool, just as does the interaction between multiplicatives $*$ and $-\!*$ and the modality $\Box$ in the underlying BL.

Example 35 (Crimson Tide [24] Part 1, revisited)**.** *The following agents denote the boat's key personnel in the weapons-release protocol:*

- a_1 = *first Line Officer*
- a_2 = *second Line Officer*
- b = *Executive Officer*
- c = *Captain*
- d = *Weaps*

and

- p = *Pentagon policy.*

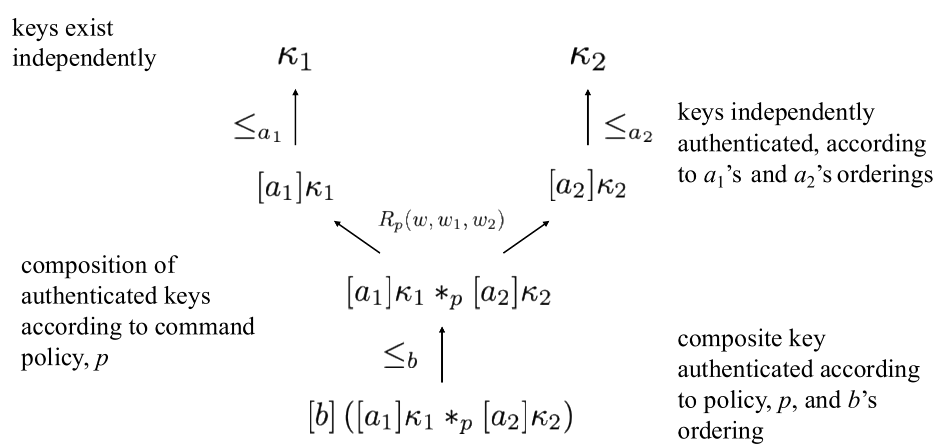


Figure 7: Multiagent joint access

The multiagent set-up of joint access and authentication is depicted in Figure 7. The keys κ_1 and κ_2 are independently authenticated by the line officers a_1 and a_2, employing their orderings $\leq_{a_1}$ and $\leq_{a_2}$, respectively — this should be read simply as recording each a_is use of its authentication rights/process at world w_i.

The extensional protocol p, here specified by the Pentagon, now permits the composition, using $*_p$, of the authentication keys at world w according to the relation $R(w, w_1, w_2)$. The composite key is then authenticated by the Executive Officer, agent b, employing his ordering $\leq_b$, yielding $[b]([a_1]\kappa_1 *_p [a_2]\kappa_2)$.

The policy protocol p now permits the key to be released for final authentication by the Captain, agent c. This is considered in Figure 8, which we discuss in Example 36. □

While we seek to elide as much unnecessary formal detail as possible, note that the use of ternary relations and orderings both indexed by agents allows considerable expressive flexibility in capturing the 'Crimson Tide' system and scenario, and this flexibility is implicitly used in setting up the example.

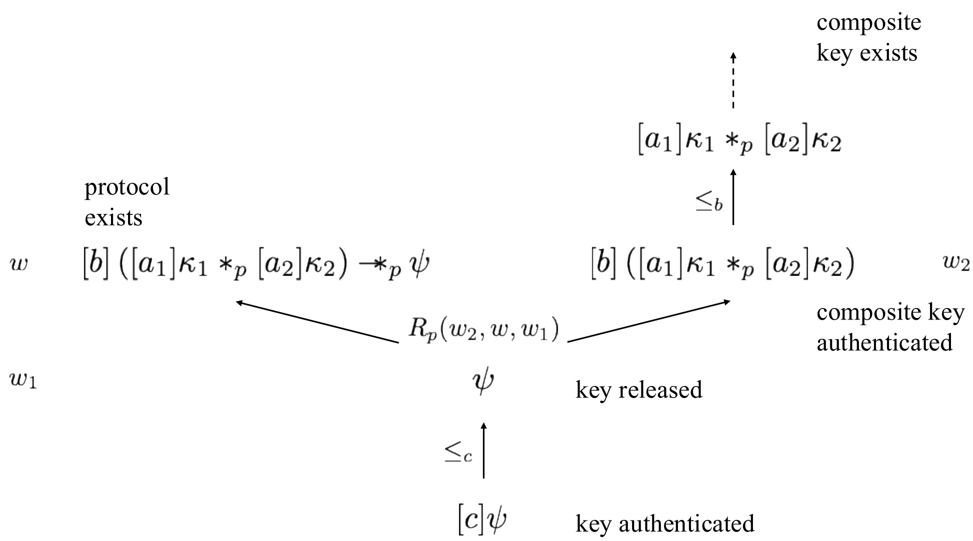


Figure 8: Multiagent joint access protocol

In addition to enriching the basic example along these lines, we might also consider steps in the protocol such as the receiving of the message, the request and granting of permission to authenticate, the detail of the authentication — local key on boat is known to base but known to the crew only when access for comparison with the key received in the message — and, finally, release of metal keys.

Example 36 (Crimson Tide [24] Part 2, revisited)**.** *Figure 8 depicts how the protocol p permits the release of the key, authenticated as described in Example 35 to the Captain. The protocol p takes the composite key as authenticated by the Executive Officer and release the composite key to the Captain — here ψ denotes 'authenticated key released' — according to $R_p(w_2, w, w_1)$. The Captain, agent c, then authenticates according to $\leq_c$.* □

At this point, the protocol permits the release of the key to $Weaps$, agent d, who may deploy the key without further authentication. Expressing this last step is left as an exercise for the reader.

In the film Crimson Tide [24], one group of officers — led by the Executive Officer — disagrees with the Captain's operational choices. They seize control of the submarine and implement their own version of the protocol for the release of weapons.

Such a protocol will vary the worlds or system states that are encountered, but some aspects of the execution of a protocol for a fixed purpose — in this example, the release of weapons — must be invariant because they involve interaction with the boat's hardware. For example, depending on the details of the boat's design and construction, it may be that two physical keys must be released and inserted into physical locks in order for access to specific mechanisms to be accessed. Consequently, the dissenters' protocol must respect this requirement.

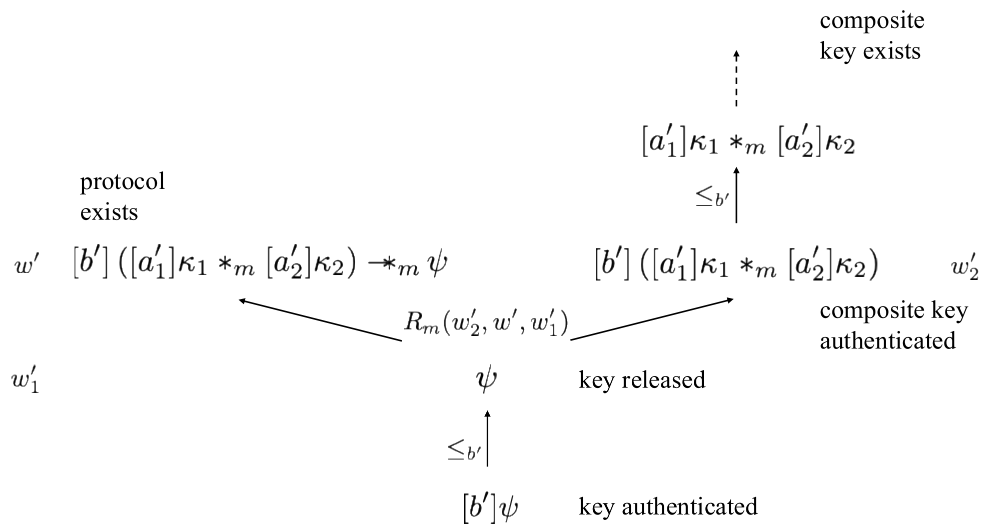


Figure 9: Dissenters' multiagent joint access protocol

Example 37 (Dissent)**.** *The protocol followed by the dissenters*[3] *is depicted in Figure 9. In this protocol, each of the agents a_1, a_2, and b is replaced by their dissenting alter ego, a'_1, a'_2, and b', but the dissenters would otherwise follow a protocol m that is essentially the same as p — since m is constrained by the same hardware — though relative to the relation R_m being instantiated at different worlds, w'_1, w'_2, and w'. In these worlds, the possibility of authentication is different, and may not occur — in the narrative of the film [24], the dissenters interpret differently from the Captain the information that can be inferred from the orders that the submarine has only partially received (because of a transmission failure). Note that, in this version of the protocol, the final step involves not the Captain, who is detained in his cabin, but the (alter ego of) the Executive Officer.* □

It seems that requirements of this kind will typically amount to relationships between the relations R_a, B_b, ... for different agents a, b, ... For example, in the examples depicted in Figures 8 and 9, one possibility (among many) might be that $R_m \subseteq R_p$, with the dissenters accepting fewer circumstances in which to permit the release of the key; that is, for example, we may have $R_p(w_2, w, w_1)$, but not $R_m(w'_2, w', w'_1)$.

[3] Or 'mutineers', depending on your particular intensional perspective, but not, as it turns out, according to the extensional protocol. In the narrative of [24], it transpires that protocol p inadequately specifies the relationship between agents b and c, and perhaps other key actors, in the event of a disagreement.

8 A Tableaux System for MBL: $\mathrm{T_{MBL}}$

In this section we adapt and extend the tableau system for BL to the multiagent setting. The rules of $\mathrm{T_{MBL}}$ are given in Figure 10. Most of them are the rules of $\mathrm{T_{BL}}$, where the agent-based connectives ($\Box$, $\Diamond$, $*$, $-\!\!*$) and relations ($\prec$, $\sim$, $\|$) are now indexed with an appropriate agent (including all of the derivable rules of $\mathrm{T_{BL}}$ given in Figure 5, which, for conciseness, are not repeated in Figure 10).

In $\mathrm{T_{BL}}$, labels represent worlds in a quasi-ordered structure given by $(D(\mathfrak{b}), \prec)$, where the domain $D(\mathfrak{b})$ of a branch $\mathfrak{b}$ is the set of all labels occurring as the label of a sign formula in $\mathfrak{b}$. This remains true in $\mathrm{T_{MBL}}$ provided that $\prec$ now corresponds to $\prec_t$, the quasi-order described by the global agent t. For all other agents a, the relation $\prec_a$ is only supposed to represent a sub quasi-order of $\prec_t$, as semantically, $(W_a, \leq_a)$ is required by Definition 30 to be a sub partial-order of $(W_t, \leq_t)$. Therefore, given a world $w \in W_t$, $w \leq_a w$ holds if and only if $w \in W_a$, which justifies that, in $\mathrm{T_{MBL}}$, we are going to interpret the presence of a reflexive constraint $T\, x \prec_a x$ (in a tableau branch $\mathfrak{b}$) as a witness of the definedness of the label x for the agent a (in $\mathfrak{b}$).

The previous discussion has several consequences. First, we modify the premisses of the case distinction rule CD according to the new notion of definedness in terms of reflexivity constraints. Then, we change the reflexivity rule $\prec_R$ so that it can now also conclude $F\, x \prec_a x$, as x might not necessarily be defined for the agent a. In turn, changing $\prec_R$ requires us to add a new closure rule $\times_R$ to derive a contradiction from the presence of $F\, x \prec_t x$ in a tableau branch, since $w \leq_t w$ holds for all worlds w, and closure from the rule $\times_\Phi$ can no longer be guaranteed (except for the left conclusion of $\prec_R$ which derives $T\, x \prec_t x$).

We finally introduce five new structural rules, namely $T \prec$, $F \prec$, SL, SP and ST, to account for the fact that $(W_a, \leq_a)$ has to be a slice of $(W_t, \leq_t)$ for all agents a. The rules $T \prec$ and $F \prec$ ensure that $\prec_a$ is contained in $\prec_t$. The rule SL ensures that if $x \prec_a y$ holds, then both x and y are defined for the agent a. The rules ST and SP ensure that $\prec_a$ represents a tight sub partial order of $\prec_t$ (once quotiented by $\sim_a$ as explained later in Lemma 44).

An example of $\mathrm{T_{MBL}}$-proof is given in Figure 11.

The proof that $\mathrm{T_{MBL}}$ is sound w.r.t. MBL models follows the same lines as the one for $\mathrm{T_{BL}}$.

Definition 38 (Realization). *Let $\mathfrak{b}$ be a tableau branch. A* realization *of $\mathfrak{b}$ in an MBL model $\mathcal{M} = (W, I, t, V, \Vdash)$ is a function ρ from $D(\mathfrak{b})$ to W such that*

- *if $(T\, \phi : x) \in \mathfrak{b}$, then $\rho(x) \Vdash \phi$, and if $(F\, \phi : x) \in \mathfrak{b}$, then $\rho(x) \nVdash \phi$,*
- *if $(T\, x \prec_a y) \in \mathfrak{b}$, then $\rho(x) \leq_a \rho(y)$, and if $(F\, x \prec_a y) \in \mathfrak{b}$, then $\rho(x) \nleq_a \rho(y)$,*
- *if $\epsilon \in D(\mathfrak{b})$, then $\rho(\epsilon) = \omega$, where ω is the root in $\mathcal{M}$.*

A tableau branch is realizable *if it has at least one realization in some MBL model. A tableau is* realizable *if it has at least one realizable branch.* $\Box$

Lemma 39. *If a tableau branch is closed, then it is not realizable.*

Proof. Let $\mathfrak{b}$ be a closed tableau branch. Assume that $\mathfrak{b}$ is realizable. Then, there exists a realization ρ of $\mathfrak{b}$ in some MBL model.

- If $\mathfrak{b}$ is closed because both $(T\, \phi : x) \in \mathfrak{b}$ and $(F\, \phi : x) \in \mathfrak{b}$, then by definition of a realization, we have both $\rho(x) \Vdash \phi$ and $\rho(x) \nVdash \phi$, which is a contradiction.

- If $\mathfrak{b}$ is closed because both $(T\, x \prec_a y) \in \mathfrak{b}$ and $(F\, x \prec_a y) \in \mathfrak{b}$, then by definition of a realization, we have both $\rho(x) \leq_a \rho(y)$ and $\rho(x) \nleq_a \rho(y)$, which is a contradiction.

Logical rules

$$\frac{T\,\neg\phi : x}{F\,\phi : x}\;{\scriptstyle T\neg} \qquad \frac{F\,\neg\phi : x}{T\,\phi : x}\;{\scriptstyle F\neg} \qquad \frac{T\,\phi \wedge \psi : x}{\begin{array}{c} T\,\phi : x \\ T\,\psi : x \end{array}}\;{\scriptstyle T\wedge} \qquad \frac{F\,\phi \wedge \psi : x}{F\,\phi : x \;\mid\; F\,\psi : x}\;{\scriptstyle F\wedge} \qquad \frac{\begin{array}{c} T\,[a]\phi : x \\ T\,x \prec_a y \end{array}}{T\,\phi : y}\;{\scriptstyle T\Box} \qquad \frac{F\,[a]\phi : x}{\begin{array}{c} T\,x \prec_a u \\ F\,\phi : u \end{array}}\;{\scriptstyle F\Box}$$

$$\frac{T\,\phi *_a \psi : x}{\begin{array}{c} T\,x \prec_a u \parallel_a v \\ T\,\phi : u \\ T\,\psi : v \end{array}}\;{\scriptstyle T*} \qquad \frac{\begin{array}{c} F\,\phi *_a \psi : x \\ T\,x \prec_a y \parallel_a z \end{array}}{F\,\phi : y \;\mid\; F\,\psi : z}\;{\scriptstyle F*} \qquad \frac{\begin{array}{c} T\,\phi \mathrel{-\!\!*}_a \psi : x \\ T\,z \prec_a x \parallel_a y \end{array}}{F\,\phi : y \;\mid\; T\,\psi : z}\;{\scriptstyle T-\!*} \qquad \frac{F\,\phi \mathrel{-\!\!*}_a \psi : x}{\begin{array}{c} T\,v \prec_a x \parallel_a u \\ T\,\phi : u \\ F\,\psi : v \end{array}}\;{\scriptstyle F-\!*} \qquad \frac{\begin{array}{c} S\,\phi : x \\ T\,x \sim_a y \end{array}}{S\,\phi : y}\;{\scriptstyle S\sim}$$

Structural rules

$$\frac{T\,x \prec_a y}{T\,x \prec_t y}\;{\scriptstyle T\prec} \qquad \frac{F\,x \prec_t y}{F\,x \prec_a y}\;{\scriptstyle F\prec} \qquad \frac{T\,x \prec_a y}{T\,x \prec_a x,\; T\,y \prec_a y}\;{\scriptstyle SL} \qquad \left[\;\frac{T\,x_1 \parallel_a x_2,\; T\,x_i \prec_a y}{T\,x_j \parallel_a y}\;{\scriptstyle \parallel_P}\right]$$

$$\frac{T\,x \prec_a x,\; T\,y \prec_a y,\; T\,x \prec_t y}{T\,x \prec_a y}\;{\scriptstyle SP} \qquad \frac{T\,x \prec_t y,\; T\,y \prec_t z,\; T\,x \prec_a z}{T\,x \prec_a y,\; T\,y \prec_a z}\;{\scriptstyle ST} \qquad \frac{S\,\phi : x}{T\,\epsilon \prec_t x}\;{\scriptstyle \prec_\epsilon}$$

$$\frac{S\,\phi : x}{T\,x \prec_a x \;\mid\; F\,x \prec_a x}\;{\scriptstyle \prec_R} \qquad \frac{T\,x \prec_a y,\; T\,y \prec_a z}{T\,x \prec_a z}\;{\scriptstyle \prec_T} \qquad \frac{T\,x \prec_a x,\; T\,y \prec_a y}{T\,x \prec_a y \;\mid\; T\,y \prec_a x \;\mid\; T\,x \parallel_a y}\;{\scriptstyle CD}$$

$$\genfrac{}{}{1.5pt}{}{T\,x \parallel_a z}{F\,x \prec_a z,\; F\,z \prec_a x}\;{\scriptstyle \parallel} \qquad \genfrac{}{}{1.5pt}{}{T\,x \prec_a y \parallel_a z}{T\,x \prec_a y,\; T\,x \prec_a z,\; T\,y \parallel_a z}\;{\scriptstyle B} \qquad \genfrac{}{}{1.5pt}{}{T\,x \prec_a y,\; T\,y \prec_a x}{T\,x \sim_a y}\;{\scriptstyle \sim}$$

Closure rules

$$\frac{T\,\phi : x,\; F\,\phi : x}{\times}\;{\scriptstyle \times_\Phi} \qquad \frac{T\,x \prec_a y,\; F\,x \prec_a y}{\times}\;{\scriptstyle \times_\prec} \qquad \frac{F\,\epsilon \prec_t x}{\times}\;{\scriptstyle \times_\epsilon} \qquad \frac{F\,x \prec_t x}{\times}\;{\scriptstyle \times_R}$$

Side-conditions and comments

- In rules introducing u and/or v, u and v are distinct labels that are fresh in the branch.
- In rules involving i and/or j, $i \in \{1, 2\}$ and $j = 3 - i$.
- Double lines indicate rules that can be used bottom-up or top-down.
- The brackets indicate that the rule is optional (included for MBL, excluded for lax MBL).

Figure 10: Rules of the $\mathrm{T}_{\mathrm{MBL}}$ calculus

$$
\begin{array}{c}
F\,[a]\langle b\rangle\top \to ((\psi_1 *_a \psi_2) \to (\psi_1 *_b \psi_2)) : x \\
{}_{1}\overline{\hspace{18em}}{}_{F\to} \\
T\,[a]\langle b\rangle\top : x^{\,5} \\
F\,(\psi_1 *_a \psi_2) \to (\psi_1 *_b \psi_2) : x^{\,2} \\
{}_{2}\overline{\hspace{14em}}{}_{F\to} \\
T\,\psi_1 *_a \psi_2 : x^{\,3} \\
F\,\psi_1 *_b \psi_2 : x^{\,21} \\
{}_{3}\overline{\hspace{14em}}{}_{T*} \\
T\,x \prec_a y_1 \parallel_a y_2^{\,4,5,8,13-15,17-19} \\
T\,\psi_1 : y_1^{\,22},\; T\,\psi_2 : y_2^{\,23} \\
{}_{4}\overline{\hspace{14em}}{}_{SL} \\
T\,x \prec_a x^{\,5},\; T\,y_1 \prec_a y_1 \\
{}_{5}\overline{\hspace{18em}}{}_{T\Box} \\
T\,\langle b\rangle\top : x^{\,6},\; T\,\langle b\rangle\top : y_1^{\,6},\; T\,\langle b\rangle\top : y_2^{\,6} \\
{}_{6}\overline{\hspace{18em}}{}_{T\Diamond} \\
T\,x \prec_b z^{\,7},\; T\,y_1 \prec_b z_1^{\,9},\; T\,y_2 \prec_b z_2^{\,9} \\
T\top : x,\; T\top : y_1,\; T\top : y_2 \\
{}_{7}\overline{\hspace{18em}}{}_{SL} \\
T\,x \prec_b x^{\,10},\; T\,z \prec_b z \\
{}_{8}\overline{\hspace{10em}}{}_{T\prec\text{ twice}} \\
T\,x \prec_t y_1^{\,10} \\
T\,x \prec_t y_2^{\,10} \\
{}_{9}\overline{\hspace{14em}}{}_{SL\text{ twice}} \\
T\,y_1 \prec_b y_1^{\,10,11},\; T\,z_1 \prec_t z_1 \\
T\,y_2 \prec_b y_2^{\,10,11},\; T\,z_2 \prec_t z_2 \\
{}_{10}\overline{\hspace{14em}}{}_{SP\text{ twice}} \\
T\,x \prec_b y_1^{\,20} \\
T\,x \prec_b y_2^{\,20} \\
{}_{11}\overline{\hspace{32em}}{}_{CD} \\
\begin{array}{c|c|c}
T\,y_1 \prec_b y_2^{\,12} & T\,y_2 \prec_b y_1^{\,16} & T\,y_1 \parallel_b y_2^{\,20} \\
{}_{12}\overline{\hspace{6em}}{}_{T\prec} & {}_{16}\overline{\hspace{6em}}{}_{T\prec} & {}_{20}\overline{\hspace{8em}}{}_{B} \\
T\,y_1 \prec_t y_2^{\,14} & T\,y_2 \prec_t y_1^{\,18} & T\,x \prec_b y_1 \parallel_b y_2^{\,21} \\
{}_{13}\overline{\hspace{6em}}{}_{T\prec} & {}_{17}\overline{\hspace{6em}}{}_{T\prec} & {}_{21}\overline{\hspace{14em}}{}_{F*} \\
T\,x \prec_t y_1^{\,14} & T\,x \prec_t y_2^{\,18} & \begin{array}{c|c} F\,\psi_1 : y_1^{\,22} & F\,\psi_2 : y_2^{\,23} \\ {}_{22}\overline{\hspace{5em}}{}_{\times_\Phi} & {}_{23}\overline{\hspace{5em}}{}_{\times_\Phi} \\ \times & \times \end{array} \\
{}_{14}\overline{\hspace{6em}}{}_{ST} & {}_{18}\overline{\hspace{6em}}{}_{ST} & \\
T\,y_1 \prec_a y_2^{\,15} & T\,y_2 \prec_a y_1^{\,19} & \\
{}_{15}\overline{\hspace{6em}}{}_{\times_\parallel} & {}_{19}\overline{\hspace{6em}}{}_{\times_\parallel} & \\
\times & \times &
\end{array}
\end{array}
$$

Figure 11: $\mathrm{T_{MBL}}$-proof of $[a]\langle b\rangle\top \to ((\psi_1 *_a \psi_2) \to (\psi_1 *_b \psi_2))$.

- If $\mathfrak{b}$ is closed because $(F\,\epsilon \prec_t x) \in \mathfrak{b}$, then, by definition of a realization, we have $\rho(\epsilon) \not\leq_t \rho(x)$, which is a contradiction since $\rho(\epsilon) = \omega$ and ω is the root in $\mathcal{M}$.
- If $\mathfrak{b}$ is closed because $(F\,x \prec_t x) \in \mathfrak{b}$, then by definition of a realization, we have $\rho(x) \not\leq_t \rho(x)$, which is a contradiction since $\leq_t$ is a partial order.

Therefore, $\mathfrak{b}$ cannot be realizable. □

Lemma 40. *If a tableau $\mathfrak{t}$ is realizable, then expanding $\mathfrak{t}$ using one of the tableau expansion rules given in Figure 10 results in a realizable tableau $\mathfrak{t}'$.*

Proof. The proof is similar to the one given in Lemma 19, indexing all label constraints and their corresponding realization with the appropriate agent. □

Theorem 41 (Soundness)**.** *If $\vdash\phi$, then $\vDash\phi$.*

Proof. The proof is similar to the one given in Lemma 20. □

The proof of completeness of T_{MBL} w.r.t. MBL models also follows the same pattern as the one given for T_{BL}. The main difficulty is to adapt the model existence lemma so as to handle the notion of agent interpretation.

Definition 42. *A tableau branch $\mathfrak{b}$ is* saturated *if it satisfies all of the following conditions:*

1. *if $(T\,\neg\phi : x) \in \mathfrak{b}$, then $(F\,\phi : x) \in \mathfrak{b}$*
2. *if $(F\,\neg\phi : x) \in \mathfrak{b}$, then $(T\,\phi : x) \in \mathfrak{b}$*
3. *if $(T\,\phi \wedge \psi : x) \in \mathfrak{b}$, then $(T\,\phi : x) \in \mathfrak{b}$ and $(T\,\psi : x) \in \mathfrak{b}$*
4. *if $(F\,\phi \wedge \psi : x) \in \mathfrak{b}$, then $(F\,\phi : x) \in \mathfrak{b}$ or $(F\,\psi : x) \in \mathfrak{b}$*
5. *if $(T\,[a]\phi : x) \in \mathfrak{b}$, then, for all $y \in D(\mathfrak{b})$, if $(T\,x \prec_a y) \in \mathfrak{b}$, then $(T\,\phi : y) \in \mathfrak{b}$*
6. *if $(F\,[a]\phi : x) \in \mathfrak{b}$, then, for some $y \in D(\mathfrak{b})$, $(T\,x \prec_a y) \in \mathfrak{b}$ and $(F\,\phi : y) \in \mathfrak{b}$*
7. *if $(T\,\phi *_a \psi : x) \in \mathfrak{b}$, then, for some $y, z \in D(\mathfrak{b})$, $(T\,x \prec_a y \parallel_a z) \in \mathfrak{b}$, $(T\,\phi : y) \in \mathfrak{b}$, and $(T\,\psi : z) \in \mathfrak{b}$*
8. *if $(F\,\phi *_a \psi : x) \in \mathfrak{b}$, then, for all $y, z \in D(\mathfrak{b})$,*
   if $(T\,x \prec_a y \parallel_a z) \in \mathfrak{b}$, then $(F\,\phi : y) \in \mathfrak{b}$ or $(F\,\psi : z) \in \mathfrak{b}$
9. *if $(T\,\phi \mathrel{-\!\!*}_a \psi : x) \in \mathfrak{b}$, then, for all $y, z \in D(\mathfrak{b})$,*
   if $(T\,z \prec_a x \parallel_a y) \in \mathfrak{b}$, then $(F\,\phi : y) \in \mathfrak{b}$ or $(T\,\psi : z) \in \mathfrak{b}$
10. *if $(F\,\phi \mathrel{-\!\!*}_a \psi : x) \in \mathfrak{b}$, then, for some $y, z \in D(\mathfrak{b})$, $(T\,z \prec_a x \parallel_a y) \in \mathfrak{b}$, $(T\,\phi : y) \in \mathfrak{b}$, and $(F\,\psi : z) \in \mathfrak{b}$*
11. *if $(T\,x \sim_a y) \in \mathfrak{b}$ and $(S\,\phi : x) \in \mathfrak{b}$, then $(S\,\phi : y) \in \mathfrak{b}$*
12. *if $(S\,\phi : x) \in \mathfrak{b}$, then $(T\,x \prec_t x) \in \mathfrak{b}$*
13. *if $(T\,x \prec_a y) \in \mathfrak{b}$ and $(T\,y \prec_a z) \in \mathfrak{b}$, then $(T\,x \prec_a z) \in \mathfrak{b}$*
14. *if $(T\,x \prec_a x) \in \mathfrak{b}$ and $(T\,y \prec_a y) \in \mathfrak{b}$, then $(T\,x \prec_a y) \in \mathfrak{b}$ or $(T\,y \prec_a x) \in \mathfrak{b}$ or $(T\,x \parallel_a y) \in \mathfrak{b}$*
15. *if $(T\,x \parallel_a y) \in \mathfrak{b}$ and $(T\,x \prec_a z) \in \mathfrak{b}$, then $(T\,z \parallel_a y) \in \mathfrak{b}$*
16. *if $(S\,\phi : x) \in \mathfrak{b}$, then $(T\,\epsilon \prec_t x) \in \mathfrak{b}$*
17. *if $(T\,x \prec_a y) \in \mathfrak{b}$, then $(T\,x \prec_t y) \in \mathfrak{b}$*
18. *if $(F\,x \prec_t y) \in \mathfrak{b}$, then $(F\,x \prec_a y) \in \mathfrak{b}$*
19. *if $(T\,x \prec_a y) \in \mathfrak{b}$, then $(T\,x \prec_a x) \in \mathfrak{b}$ and $(T\,y \prec_a y) \in \mathfrak{b}$*
20. *if $(T\,x \prec_a x) \in \mathfrak{b}$, $(T\,y \prec_a y) \in \mathfrak{b}$, and $(T\,x \prec_t y) \in \mathfrak{b}$, then $(T\,x \prec_a y) \in \mathfrak{b}$*
21. *if $(T\,x \prec_t y) \in \mathfrak{b}$, $(T\,y \prec_t z) \in \mathfrak{b}$, and $(T\,x \prec_a z) \in \mathfrak{b}$, then $(T\,x \prec_a y) \in \mathfrak{b}$ and $(T\,y \prec_a z) \in \mathfrak{b}$.* □

Lemma 43. *Let $\mathfrak{b}$ be a saturated open tableau branch. The binary relation $\prec_a^{\mathfrak{b}}$ over $D(\mathfrak{b})$ defined as $x \prec_a^{\mathfrak{b}} y$ iff $(T\,x \prec_a y) \in \mathfrak{b}$ is a quasi-order over $D(\mathfrak{b})$ such that if $(F\,x \prec_a y) \in \mathfrak{b}$ then $x \not\prec_a^{\mathfrak{b}} y$.*

Proof. Transitivity and reflexivity of $\prec_a^{\mathfrak{b}}$ clearly follow from Conditions 12 and 13 of Definition 42. Now, if $(F\,x \prec_a y) \in \mathfrak{b}$, then $(T\,x \prec_a y) \notin \mathfrak{b}$ because $\mathfrak{b}$ is open. Hence, $x \not\prec_a^{\mathfrak{b}} y$ by the definition of $\prec_a^{\mathfrak{b}}$. □

Lemma 44. *Let $\mathfrak{b}$ be a saturated open branch. Let $\sim_a^{\mathfrak{b}}$ be the equivalence relation over $D(\mathfrak{b})$ defined as $x \sim_a^{\mathfrak{b}} y$ iff $x \prec_a^{\mathfrak{b}} y$ and $y \prec_a^{\mathfrak{b}} x$. For all $x \in D(\mathfrak{b})$, let $[x]_a = \{\, y \in D(\mathfrak{b}) \mid x \sim_a^{\mathfrak{b}} y \,\}$ denote the equivalence class of x under $\sim_a^{\mathfrak{b}}$ and let $W_a^{\mathfrak{b}}$ denote the quotient $D(\mathfrak{b})/\sim_a^{\mathfrak{b}}$. Then, the binary relation $\leq_a^{\mathfrak{b}}$ over $W_a^{\mathfrak{b}}$ defined as $[x]_a \leq_a^{\mathfrak{b}} [y]_a$ iff $x \prec_a^{\mathfrak{b}} y$ is a partial order over $W_a^{\mathfrak{b}}$ that satisfies the persistent separation property.*

Proof. Similar to the one given in Lemma 23 using Lemma 43 instead of Lemma 22. □

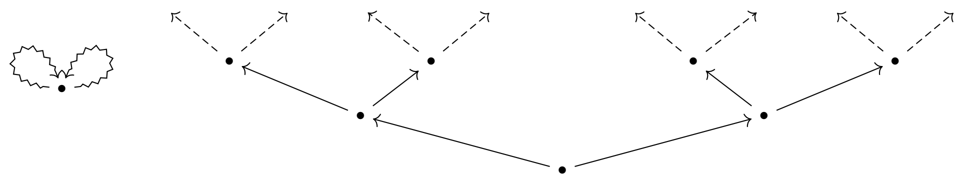


Figure 12: A finite model with backlinks (left) and its infinite unfolding (right)

Lemma 45 (Model existence)**.** *Let $\mathfrak{b}$ be a saturated open tableau branch. Then, $\mathfrak{b}$ induces an MBL model $\mathcal{M}^{\mathfrak{b}} = (W, I, V, \Vdash)$, where*

- $W = W_t^{\mathfrak{b}}$,
- *for all worlds* $[x]_t, [y]_t \in W_t^{\mathfrak{b}}$, $I([x]_t, [y]_t) = \{\, a \mid [x]_a \leq_a^{\mathfrak{b}} [y]_a \,\}$, *and,*
- *for all worlds* $[x]_t \in W_t^{\mathfrak{b}}$, $V([x]_t) = \{\, p \mid (T\, p : x) \in \mathfrak{b} \,\}$.

Moreover, $\mathcal{M}^{\mathfrak{b}}$ is such that if $(T\,\phi : x) \in \mathfrak{b}$ then $[x]_t \Vdash \phi$ and if $(F\,\phi : x) \in \mathfrak{b}$ then $[x]_t \nVdash \phi$.

Proof. We need to show that I is an actual agent interpretation in the sense of Definition 29. First, for all agents $a \in A$, it is easy to check that we have both $[x]_t \leq_a [y]_t$ iff $[x]_a \leq_a^{\mathfrak{b}} [y]_a$, and $[x]_t \in W_a$ iff $[x]_a \in W_a^{\mathfrak{b}}$. Second, Lemma 44 ensures that $(W_t, \leq_t)$ is a partial order, which by the rule $\prec_\epsilon$ and Condition 16 of Definition 42 admits $[\epsilon]_t$ as its root. Hence, $(W_t, \leq_t)$ is a tree. Third, the rules $T\prec$, $F\prec$ and SL ensure using Conditions 17, 18 and 19 of Definition 42 that, for all agents $a \in A$, $(W_a, \leq_a)$ is contained in $(W_t, \leq_t)$. It then follows from Lemma 44 by the rules SP and ST and the corresponding Conditions 20 and 21 that, for all agents $a \in A$, $(W_a, \leq_a)$ is a tight sub partial order, that is, a slice of $(W_t^{\mathfrak{b}}, \prec_t^{\mathfrak{b}})$. Hence, I is a agent interpretation.

The rest of the proof is similar to the one given in Theorem 24. ☐

Theorem 46 (Completeness)**.** *If $\Vdash \phi$, then $\vdash \phi$.*

Proof. The proof is similar to the one given in Theorem 26 ☐

9 A Finite Model Property and Decidability for MBL

The formula $\Box(\top * \top)$ enforces the existence of infinitely many branching points in the tree order and can, therefore, only be satisfied in infinite models like the one shown in Figure 12 (right). Consequently, the finite model property in its traditional formulation fails for BL and MBL.

However, models such as the one in Figure 12 (right) are sufficiently regular to allow for a finite schematic representation. Consider, for example, the scheme in Figure 12 (left). Here, each squiggly edge represents a *backlink*, indicating that a copy of the graph originating from its endpoint should be attached to its starting point via a single edge. By unfolding such a schematic model, we obtain precisely the full binary tree shown in Figure 12 (right).

For simplicity, we only present an informal description of *models with backlinks*. These models are trees augmented with additional backlinks — edges that point backward in the tree order. In the MBL case, backlinks are labelled with agents. Formulas in (M)BL can be interpreted on such models by first unfolding them into (infinite) trees. Using the well-foundedness of the subformula relation the following is easy to see:

Lemma 47. *It is decidable whether a finite model with backlinks satisfies a formula of MBL.*

One can then establish the following result:

Lemma 48. *If φ is $-\!\!*$-free and satisfiable in MBL, then it has a finite model with backlinks.*

Proof sketch. Assume φ is satisfied in some MBL model with root v, and let Σ denote the set of all subformulas of φ. To each world w assign its Σ-signature: The set of all $\psi \in \Sigma$ that are satisfied in it, together with the set of all agent labels appearing in Σ such that $u \leq_a w$ and t is an immediate predecessor of w. Note that there are only finitely many Σ-signatures.

If we consider a class of immediate successors of v with the same signature and remove all but two of them together along with the branches stemming from them, we obtain a different model that still satisfies φ at the root. Note that we need to retain two nodes — unlike in standard modal logic where one representative suffices — as the root might satisfy a statement $\psi_1 *_a \psi_2$ that enforces the existence of two *different* successors, even with equal theories. Applying this idea repeatedly we can create a model of φ that is finitely branching up to any given height.

However, on every sufficiently long branch in the model we will encounter worlds $w < v$ with the same Σ-signature. In this case, we can remove the branch stemming from v and instead create a back link from the immediate predecessor of v to w, labelled with the agent labels indicated in the signature. This too preserves truth of φ at the root. In the end, we obtain a finitely branching model with backlinks that is of bounded height, and therefore finite. □

The proof is a modification of a standard filtration and pruning argument used in other modal logics [2]. These arguments rely on the fact that the truth of a formula depends only on the truth of its subformulae at successor nodes — a property that fails once one considers backwards-looking modalities. This explains the restriction on $-\!\!*$.

Corollary 49. *It is decidable whether a $-\!\!*$-free formula is valid in MBL.*

Proof. By Theorem 46, the $-\!\!*$-free valid formulas of BL are computably enumerable.

Furthermore, by Lemma 48 and Lemma 47 we can also enumerate the non-valid formulas. These are the formulas whose negation is satisfiable, and we obtain them by systematically generating finite models with backlinks and checking which formulas hold at their root. □

10 Summary and Further Work

In this paper, we have introduced Bifurcation Logic, BL, which combines a basic classical modality with separating conjunction, $*$, together with its naturally associated multiplicative implication, $-\!\!*$, that is defined using the modal ordering. We have illustrated the use of BL through a model of an access control protocol. We have provided a labelled tableaux calculus for BL, with soundness and completeness relative to BL's relational semantics.

We have also defined a multiagent version of BL, called MBL, that employs modalities and connectives indexed by agents. We have revisited our modelling example in order to illustrate the interest and use of the agent modalities. To conclude the story, we have extended the tableaux system for BL, together with its soundness and completeness results, to MBL. Finally, we have shown that, for both BL and MBL, we can obtain decidability provided $-\!\!*$ be excluded from formulae. For further work, we might consider a few interesting outstanding challenges, among many others, for both BL and MBL:

– to give (Hilbert-type) axiomatizations;
– to give (sound and complete) natural deduction and sequent calculus presentations;
– to establish the complexity of deciding the logics;
– to explore the addition to the logics of epistemic modalities as well as quantifiers, so extending their potential as modelling tools (cf. [11], for example).

The question of whether there is can be interesting intuitionistic versions of BL and MBL seems quite challenging: it would combine the difficulties of intuitionistic modal logics [23] with the need to handle the multiplicatives in a coherent way.

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