Four Lectures on Proof-theoretic Semantics

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Schedule

1. What is Proof-theoretic Semantics (P-tS)?
   - Inferentialism.
   - Consequence.
   - Proof-theoretic Validity (P-tV).
2. Base-extension Semantics (B-eS):
   - B-eS for Intuitionistic Propositional Logic.
   - Naturality, categorically speaking.
   - B-eS and P-tV.

Schedule

3. Reductive Logic, Tactical proof, and Logic Programming:
    - Reductive Logic and P-tV.
    - Tactical Proof.
    - Remarks on Logic Programming and Coalgebra.
4. Modal and Substructural Logics, Resource Semantics, and Modelling:
    - B-eS for Modal Logics.
    - B-eS for Substructural Logics.
    - Resource Semantics and Modelling with B-eS.

Schedule

- Most of what we will introduce will be quite new to most people, with a fairly significant philosophical basis, and with quite a lot of ground to be covered.
- Our approach will mainly be conceptual, with little detailed, formal proof.
- Nevertheless, the formal details of everything we cover are available in books and papers that will be referenced.

Lecture 2: Base-extension Semantics for Intuitionistic Propositional Logic

Mark Twain was an inferentialist:

> *It's not what you don't know that gets you into trouble.*
> *It's what you know for sure that just ain't so.*

This is the lesson of inferentialist epistemic logic.

Base-extension Semantics for Intuitionistic Propositional Logic

- We now turn our attention from the validity of proofs to the validity of formulae.
- Tor Sandqvist — *Base-extension Semantics for Intuitionistic Sentential Logic*, *Logic Journal of the IGPL*, 2015 — has given an elegant B-eS for intuitionistic propositional logic.
- This analysis demonstrates very clearly the basic principles of B-eS, so we'll spend some time today looking at how it works.
- We'll also take a quick look at how the construction of this paper can be set up in categorical logic, and see that everything in the B-eS for IPL is formally natural.

  And we'lll conclude with some brief thoughts on a connection between B-eS and P-tV (from yesterday).

Base-extension Semantics for Intuitionistic Propositional Logic: Background

- In many ways, B-eS will seem very familiar:
  - At its core is a 'support' relation for validity that closely resembles a satisfaction relation in, say, Kripke semantics.
  - We establish familiar-looking soundness and completeness theorems (again, cf. Kripke semantics).
- But the base case of the relation, for atoms, is very different:
  - In Kripke semantics, say, the base case of satisfaction goes something like

    $$w \models p \quad \text{iff} \quad w \in \mathcal{V}(p)$$

    where $\mathcal{V}$ is a 'valuation' of the atoms in the model.
  - In B-eS, however, we have something like

    $$\Vdash_{\mathcal{B}} p \quad \text{iff} \quad \vdash_{\mathcal{B}} p$$

- This difference lies at the core of the nature of the semantics and has profound consequences for the theory.

Base-extension Semantics for Intuitionistic Propositional Logic: Background

There's a backstory to Tor's work on IPL:

- Incompleteness (Piecha and Schroeder-Heister)
- Completeness (Goldfarb, Stafford)
- Tor Sandqvist's completeness theorem

Base-extension Semantics for Intuitionistic Propositional Logic: Background

Base-extension Semantics for Intuitionistic Propositional Logic: Derivability in a Base

- We assume a language containing $\bot$ and a denumerably infinite collection of atomic/ sentences, and closed under the binary sentential connectives $\supset$, $\wedge$, and $\vee$.
- Lower-case italic letters will be used to refer to basic sentences, upper-case italics to finite sets thereof.
- For sentences in general we shall use lower-case Greek letters, and for finite sets of sentences, upper-case Greek letters.
- The usual conventions for suppressing set-theoretic notation will be observed, so that, in the context of symbols such as $\vdash$ or $\Vdash$, P, Q means '$P \cup Q$', 'Φ, ϕ' means $\Phi \cup \{\phi\}$, etc..

Base-extension Semantics for Intuitionistic Propositional Logic: Derivability in a Base

- By a basic rule we mean an ordered pair $\langle Q, r\rangle$, where r is a basic sentence and Q a finite (possibly empty) set of pairs of the form $\langle P, q\rangle$, where q is a basic sentence and P a (possibly empty) set of basic sentences: that is,

$$\frac{\begin{matrix}[P_1] & & [P_n] \\ q_1 & \dots & q_n\end{matrix}}{r}$$

- We write $(P_1 \Rightarrow q_1)$, . . . , $(P_k \Rightarrow q_k) \Rightarrow r$ for $\langle\{\langle P_1, q_1\rangle, \dots, \langle P_k, q_k\rangle\}, r\rangle$.
- Intuitively, the rule is read: Given derivations of q_1 through q_k, to infer r, discharging from the derivations in question premiss sets P_1 through P_k, respectively.
- A base is a set of basic rules.

Base-extension Semantics for Intuitionistic Propositional Logic: Derivability in a Base

- Given a base $\mathcal{B}$, the relation $\vdash_{\mathcal{B}}$ of derivability in $\mathcal{B}$ of a basic sentence from a finite set of basic sentences is inductively generated by the following two clauses:
  - (Ref): $S, p \vdash_{\mathcal{B}} p$
  - (App): If $((P_1 \Rightarrow q_1), \ldots, (P_k \Rightarrow q_k) \Rightarrow r) \in \mathcal{B}$ and $S, P_1 \vdash_{\mathcal{B}} q_1$ and ... and $S, P_k \vdash_{\mathcal{B}} q_k$, then $S \vdash_{\mathcal{B}} r$.
- The relations $\vdash_{\mathcal{B}}$ are central to the semantics.
- Key philosophical point: base rules are *pre-logical* — they do not reference the (object-level) logical constants.

Base-extension Semantics for Intuitionistic Propositional Logic: Derivability in a Base

A couple of lemmas about $\vdash_{\mathcal{B}}$ are needed:

- Lemma (Atomic Weakening). If $P \vdash_{\mathcal{B}} q$, then $U, P \vdash_{\mathcal{B}} q$.
- Lemma (Atomic Base-extension): $T \vdash_{\mathcal{B}} u$ just in case, for every $\mathcal{C} \supseteq \mathcal{B}$, if $\vdash_{\mathcal{C}} t$, for every $t \in T$, then $\vdash_{\mathcal{C}} u$.

Their proofs are straightforward: see Sandqvist's *Base-extension semantic for intuitionistic sentential logic*.

Base-extension Semantics for Intuitionistic Propositional Logic: Semantics and Intuitionistic Derivability

Base-extension Semantics for Intuitionistic Propositional Logic: Semantics and Intuitionistic Derivability

(At)	$\Vdash_{\mathcal{B}} p$	iff	$\vdash_{\mathcal{B}} p$
($\bot$)	$\Vdash_{\mathcal{B}} \bot$	iff	for all atomic p, $\Vdash_{\mathcal{B}} p$
($\vee$)	$\Vdash_{\mathcal{B}} \phi \vee \psi$	iff	for every atomic p and every $\mathcal{C} \supseteq \mathcal{B}$, if $\phi \Vdash_{\mathcal{C}} p$ and $\psi \Vdash_{\mathcal{C}} p$, then $\Vdash_{\mathcal{C}} p$
($\wedge$)	$\Vdash_{\mathcal{B}} \phi \wedge \psi$	iff	$\Vdash_{\mathcal{B}} \phi$ and $\Vdash_{\mathcal{B}} \psi$
($\supset$)	$\Vdash_{\mathcal{B}} \phi \supset \psi$	iff	$\phi \Vdash_{\mathcal{B}} \psi$
(Inf)	for $\Theta \neq \emptyset$, $\Theta \Vdash_{\mathcal{B}} \phi$	iff	for every $\mathcal{C} \supseteq \mathcal{B}$, if $\Vdash_{\mathcal{C}} \theta$, for every $\theta \in \Theta$, then $\Vdash_{\mathcal{C}} \phi$

- The use of base extension (recall Prawitz's justification) transmits to $\supset$ via (Inf)
- Could also use the generalized form for $\wedge$: $\Vdash_{\mathcal{B}} \phi \wedge \psi$ iff for every atomic p and every $\mathcal{C} \supseteq \mathcal{B}$, if $\phi, \psi \Vdash_{\mathcal{C}}$, then $\Vdash_{\mathcal{C}} p$.
- $\Vdash_{\mathcal{B}}$ gives a conservative extension of $\vdash_{\mathcal{B}}$ to the full language:

Base-extension Semantics for Intuitionistic Propositional Logic: Semantics and Intuitionistic Derivability

The clause for $\bot$ may at first sight seem a bit surprising, but note:

- The given form yields the usual intuitionistic introduction and elimination rules for negation, defined as $\neg\phi = \phi \supset \bot$.
- As well as Ex Falso Quodlibet, for any ϕ,

$$\frac{\bot}{\phi}$$

- So

$$\frac{\phi_1 \qquad \phi_2 \qquad \cdots}{\bot}$$

- Recall the set of atoms is assumed to be denumerably infinite.

See Dummett's *The Logical Basis of Metaphysics* — where the identification of $\bot$ with the conjunction of all atoms is explained through 'harmony' — Sandqvist's B-es for IPL, and his notes at `https://sites.google.com/view/pts-symposium-uk/schedule`.

Base-extension Semantics for Intuitionistic Propositional Logic: Semantics and Intuitionistic Derivability

And what about the clause for disjunction?

- Why not just something analogous the Kripke-style clause, say

  $\Vdash_{\mathcal{B}} \phi \vee \psi$ iff $\Vdash_{\mathcal{B}} \phi$ or $\Vdash_{\mathcal{B}} \psi$?

- Technical reason: the given clause works, giving completeness, whereas the Kripke clause does not. See Piecha and Schroeder-Heister, Sandqvist, and so on.

Base-extension Semantics for Intuitionistic Propositional Logic: Semantics and Intuitionistic Derivability

- But why, conceptually?
  - Semantics is based on proofs, and this is the proof-theoretic form (see NJ):

$$\dfrac{\phi \vee \psi \qquad \begin{matrix}[\phi] \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \chi\end{matrix}}{\chi} \; \vee E$$

    Hence the technical result, essentially.
  - Which corresponds to the 2nd-order definition of the connectives.
  - And we seek to ground in *atoms*, and note that implication is handle as pure consequence, via (Inf).
- Can argue that conjunction should also be given in this form. It can, and it works. See various papers by Gheorghiu/Gu/Pym.

Generalized $\wedge$

- Could also use the generalized form for $\wedge$:

  $\Vdash_{\mathcal{B}} \phi \wedge \psi$ iff for every atomic *p* and every $\mathcal{C} \supseteq \mathcal{B}$, if $\phi, \psi \Vdash_{\mathcal{C}}$ *p*, then $\Vdash_{\mathcal{C}}$ *p*.

-

$$\frac{\phi \wedge \psi \qquad \begin{array}{c}[\phi, \psi] \\ \chi\end{array}}{\chi} \; \wedge E$$

Base-extension Semantics for Intuitionistic Propositional Logic: Semantics and Intuitionistic Derivability

- Theorem (Atomic completeness). $T \Vdash_{\mathcal{B}} q$ iff $T \vdash_{\mathcal{B}} q$.

  The proof uses the atomic base-extension lemma.
- Lemma (Base extension and transitivity properties)
  (a) If $\Theta \Vdash_{\mathcal{B}} \phi$ and $\mathcal{B} \subseteq \mathcal{C}$, then $\Theta \Vdash_{\mathcal{C}} \phi$.
  (b) $\Theta \Vdash_{\mathcal{B}} \phi$ iff, for every $\mathcal{C} \supseteq \mathcal{B}$, if $\Vdash_{\mathcal{C}} \theta$, for every $\theta \in \Theta$, then $\Theta \Vdash_{\mathcal{C}} \phi$.
  (c) $\Theta \Vdash_{\mathcal{B}} \phi$, for every $\phi \in \Phi$ and, moreover, $\Phi \Vdash_{\mathcal{B}} \psi$, then $\Theta \Vdash_{\mathcal{B}} \psi$.

When $\Vdash_{\mathcal{B}} \phi$, we say that $\mathcal{B}$ supports ϕ. If every base supporting all members of Θ supports ϕ, we write $\Theta \Vdash \phi$ and call the inference from to Θ to ϕ valid. An individual sentence ϕ is called valid if $\Vdash \phi$. By (b) of the lemma above, we have that $\Theta \Vdash \phi$ just in case $\Theta \Vdash_{\emptyset} \phi$.

Base-extension Semantics for Intuitionistic Propositional Logic: Semantics and Intuitionistic Derivability

NJ derivability ($\vdash$):

$$
\begin{array}{rl}
(R) & \Theta, \phi \vdash \phi \\
(\supset I) & \text{if } \Theta, \phi \vdash \psi, \text{ then } \Theta \vdash \phi \supset \psi \\
(\supset E) & \text{if } \Theta \vdash \phi \text{ and } \Theta \vdash \phi \supset \psi, \text{ then } \Theta \vdash \psi \\
(\wedge I) & \text{if } \Theta \vdash \phi \text{ and } \Theta \vdash \psi, \text{ then } \Theta \vdash \phi \wedge \psi \\
(\wedge E) & \text{if } \Theta \vdash \phi \wedge \psi, \text{ then } \Theta \vdash \phi \text{ and } \Theta \vdash \psi \\
(\vee I) & \text{if } \Theta \vdash \phi \text{ or } \Theta \vdash \psi, \text{ then } \Theta \vdash \phi \vee \psi \\
(\vee E) & \text{if } \Theta \vdash \phi \vee \psi \text{ and } \Theta, \phi \vdash \chi \text{ and } \Theta, \psi \vdash \chi, \text{ then } \Theta \vdash \chi \\
(\bot E) & \text{if } \Theta \vdash \bot, \text{ then } \Theta \vdash \phi
\end{array}
$$

Base-extension semantics for Intuitionistic Propositional Logic: Soundness

Theorem (Soundness)

If $\Xi \vdash_{NJ} \xi$*, then* $\Xi \Vdash \xi$

Proof.

The set-up (see the lemma above) ensures that $\Vdash$ is transitive in the sense that if $\Theta \Vdash \phi$, for every $\phi \in \Phi$, and if $\Phi \Vdash \psi$, then $\Theta \Vdash \psi$.

Base-extension semantics for Intuitionistic Propositional Logic: Soundness

By the inductive definition of $\vdash$, it is sufficient to prove the following:

$(R)'$	$\Theta, \phi \Vdash \phi$
$(\supset I)'$	if $\Theta, \phi \Vdash \psi$, then $\Theta \Vdash \phi \supset \psi$
$(\supset E)'$	if $\Theta \Vdash \phi$ $\Theta \Vdash \phi \supset \psi$, then ψ
$(\wedge I)'$	if $\Theta \Vdash \phi$ and $\Theta \Vdash \psi$, then $\Theta \Vdash \phi \wedge \psi$
$(\wedge E)'$	if $\Theta \Vdash \phi \wedge \psi$, then $\Theta \Vdash \phi$ and $\Theta \Vdash \psi$
$(\vee I)'$	if $\Theta \Vdash \phi$ or $\Theta \Vdash \psi$, then $\Theta \Vdash \phi \vee \psi$
$(\vee E)'$	if $\Theta \Vdash \phi \vee \psi$ and $\Theta, \phi \Vdash \chi$ and $\Theta, \psi \Vdash \chi$, then $\Theta \Vdash \chi$
$(\bot E)'$	if $\Theta \Vdash \bot$, then $\Theta \Vdash \phi$

Base-extension semantics for Intuitionistic Propositional Logic: Soundness

These are proved by induction on the structure of the cases, requiring the transitivity property mentioned above.

Most of the cases are straightforward, but $(\vee E)$' is a bit more delicate thna the others. See Sandqvist.

Base-extension semantics for intuitionistic propositional logic: Completeness

Theorem (Completeness)

If $\Xi \Vdash \xi$*, then* $\Xi \vdash_{NJ} \xi$*.*

- The proof of completeness requires the construction of a 'special base' that contains exactly all of the atomic instances of the rules of NJ.
- This is *weakly* analogous to the construction of a term model in the proof of completeness for NJ and Kripke models.

Base-extension Semantics for Intuitionistic Propositional Logic: Completeness

- Although conceptually delicate, the proof of completeness is, in comparison perhaps to model-theoretic completeness theorems, technically pleasingly elementary — but delicate.
- The strategy is to simulate an NJ proof using basic sentences in a ‘special base’ that captures the specific inference.

Special Base

- To see the idea, suppose that $\Theta \Vdash \zeta$, and that a member of Θ contains as a subformula the conjunction $p \wedge q$. Corresponding to the natural deduction rules allowing inference from p and q to $p \wedge q$, from $p \wedge q$ to p, and from $p \wedge q$ to q, the specially tailored base $\mathcal{N}$ will contain, for a basic sentence r arbitrarily selected to represent $p \wedge q$, the rules, where r, representing $p \wedge q$, is fresh:

  '$\wedge I$' $(\Rightarrow p)$, $(\Rightarrow q) \Rightarrow r$

  '$\wedge E$' $(\Rightarrow r) \Rightarrow p$ and $(\Rightarrow r) \Rightarrow q$.

- The key step is the construction of the 'special base'.

Base-extension Semantics for Intuitionistic Propositional Logic: Completeness

How is the 'special base' constructed?

- Let Γ be the set containing all members of $\Xi \cup \xi$ and their subsentences. With every non-basic $\gamma \in \Gamma$ associate a basic sentence $\gamma^\flat$ such that if $\gamma_1 \neq \gamma_2$, then $\gamma_1^\flat \neq \gamma_2^\flat$.
- Also, for every basic $g \in \Gamma$, set $g^\flat = g$.
- Conversely, with every basic p associate a sentence $p^\natural$ such that, for every γ, $(\gamma^\flat)^\natural = \gamma$ (basic or not) in Γ. If p is not in the range of $-^\flat$, set $p^\natural = p$ — so that $-^\natural$ is an extension of the inverse of $-^\flat$, defined for all basic sentences.
- For any Φ and P, write $\Phi^\flat = \{\phi^\flat \mid \phi \in \Phi\}$ (and $P^\flat = \{p^\flat \mid p \in P\}$).

Base-extension Semantics for Intuitionistic Propositional Logic: Completeness

So the strategy is to construct a base $\mathcal{N}$ mimicking the rules of natural deduction by way of $-^{\flat}$ such that

(a) for every $\gamma \in \Gamma$ and every $\mathcal{B} \supseteq \mathcal{N}$, $\Vdash_{\mathcal{B}} \gamma^{\flat}$ iff $\Vdash_{\mathcal{B}} \gamma$

(b) for any P and q, if $P \vdash_{\mathcal{N}} q$, then $P^{\natural} \vdash q^{\natural}$.

Base-extension Semantics for Intuitionistic Propositional Logic: Completeness

These properties of $\mathcal{N}$ will yield the completeness result in the following way:

- First, from our hypothesis that $\Theta \Vdash \zeta$, it follows that

  $$\Theta^\flat \Vdash_{\mathcal{N}} \zeta^\flat$$

  because, if $\mathcal{B} \supseteq \mathcal{N}$ and $\Vdash_{\mathcal{B}} \xi^\flat$, for every $\xi^\flat \in \Theta^\flat$, then by (a), $\Vdash_{\mathcal{B}} \xi$ for every $\xi \in \Theta$;

  so, because $\Theta \Vdash \zeta$, $\Vdash_{\mathcal{B}} \zeta$, so that $\Vdash_{\mathcal{B}} \zeta^\flat$, by (a).
- Then, by the earlier theorem that $T \Vdash_{\mathcal{B}} q$ iff $T \vdash_{\mathcal{B}} q$, we have $\Theta^\flat \vdash_{\mathcal{N}} \zeta^\beta$, so that by (b), $(\Theta^\flat)^\natural \vdash (\zeta^\flat)^\natural$, which is just that $\Theta \vdash \zeta$, as desired.

Base-extension Semantics for Intuitionistic Propositional Logic: Completeness

$\mathcal{N}$ is defined as the base containing all and only rules of the following forms (and the representing atoms):

(1) $(\phi^\flat \Rightarrow \psi^\flat) \Rightarrow (\phi \supset \psi)^\flat$

(2) $(\Rightarrow (\phi \supset \psi)^\flat), (\Rightarrow \phi^\flat) \Rightarrow \psi^\flat$

(3) $(\Rightarrow \phi^\flat), (\Rightarrow \psi^\flat) \Rightarrow (\phi \wedge \psi)^\flat$

(4) $(\Rightarrow (\phi \wedge \psi)^\flat) \Rightarrow \phi^\flat$

(5) $(\Rightarrow (\phi \wedge \psi)^\flat) \Rightarrow \psi^\flat$

(6) $(\Rightarrow \phi^\flat) \Rightarrow (\phi \vee \psi)^\flat$

(7) $(\Rightarrow \psi^\flat) \Rightarrow (\phi \vee \psi)^\flat$

(8) $(\Rightarrow (\phi \vee \psi)^\flat), (\phi^\flat \Rightarrow p), (\psi^\flat \Rightarrow p) \Rightarrow p$

(9) $(\Rightarrow \bot^\flat) \Rightarrow p$

Base-extension Semantics for Intuitionistic Propositional Logic: Completeness

- The remainder of the proof is a slightly intricate argument by induction on the structure of everything in sight to establish properties (a) and (b), as stated above.
- See Sandqvist's *Base-extension semantics for intuitionistic sentential calculus* for the details.

Base-extension Semantics for Intuitionistic Propositional Logic: Disjunction

Just a few remarks to reflect upon, before we move on:

- Our (Sandqvist's) use of the clause

$$(\vee) \quad \Vdash_{\mathcal{B}} \phi \vee \psi \quad \text{iff} \quad \begin{array}{l} \text{for every atomic } p \text{ and every} \\ \mathcal{C} \supseteq \mathcal{B}, \text{ if } \phi \Vdash_{\mathcal{C}} p \text{ and } \psi \Vdash_{\mathcal{C}} p, \\ \text{then } \Vdash_{\mathcal{C}} p \end{array}$$

  corresponds to the $\vee E$ rule in NJ. So, given our construction, is completeness surprising? Kripke models get lucky.
- It also corresponds to Beth's treatment of disjunction in model-theoretic semantics:

$$w \models \phi \vee \psi \quad \text{iff} \quad u \models \phi \text{ and } v \models \psi, \text{ where } w = u + v$$

  See Lambek and Scott for a discussion.
- It also corresponds to the second-order definition of disjunction (see, for example, Troelstra and Schwichtenberg for a discussion).

Base-extension Semantics for Intuitionistic Propositional Logic: Disjunction

According to Sandqvist, 'If $\Vdash_{\mathcal{B}}$ is taken to signify hypothetical acceptance on the basis of $\mathcal{B}$, is it intuitively reasonable to require that $\Vdash_{\mathcal{B}} \phi$ or $\Vdash_{\mathcal{B}} \psi$ whenever $\Vdash_{\mathcal{B}} \phi \vee \psi$? In the view of this author, no: one may perfectly well take it as hypothetically given that at least one of ϕ and ψ holds good without committing oneself specifically to the one or the other. What must be acknowledged in such a state is merely that whatever follows from ϕ as well as from ψ must be accepted outright—albeit, as always, conditionally on whatever basic rules have been adopted. And this, of course, is just the idea underlying our clause $(\vee)$.'

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

- There's a serious tradition of capturing logic, both proof theory and model theory, in the language of category theory.
- I include at the end some introductory references — a biased collection, I'm afraid.
- The connections between P-tV and BHK suggest some things will hang together there.
- But a categorical treatment of B-eS (for IPL) is quite informative.
- I give a brief introduction/summary, based on *Categorical Proof-theoretic Semantics*, by Pym, Ritter, and Robinson, *Studia Logica*, 2024.

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

- The main observation is perhaps that the soundness and completeness results are characterized by natural transformations in a category of presheaves.
- The status of disjunction is nicely illuminated — it is not a coproduct, but rather is constructed naturally according to the 2nd-order definition.
- Connections with continuation semantics are exposed. (We con't get to this today.)

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

- We work with judgements of the form

$$x_1 : \phi_1, \ldots, x_i : \phi_i, \ldots, x_m : \phi_m \vdash \Phi(x_1, \ldots, x_m) : \phi$$

  read as: if the x_is are witnesses for proofs of the ϕ_is, then $\Phi(x_1, \ldots, x_m)$ denotes a proof of ϕ constructed using the rules of NJ.

- If Φ_i is a specific proof of ϕ_i, then it can be substituted for x_i throughout this judgement to give

$$x_1 : \phi_1, \ldots, x_m : \phi_m \vdash \Phi(x_1, \ldots, x_m)[\Phi_i / x_i] : \phi$$

  where the assumption $x_i : \phi_i$ has been removed and the occurrence of x_i in Φ has been replaced by Φ_i.

- We are concerned in the first instance with derivations that are restricted to the rules of a base.

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

We introduce terms for derivations in a base as

$$\Phi ::= x \mid \Phi_{\mathcal{R}}(\Phi_1, \ldots, \Phi_m)$$

where we work with base rules

$$\frac{\begin{matrix}[P_1] & & [P_n] \\ q_1 & \ldots & q_n\end{matrix}}{r}\ \mathcal{R}$$

$$(\mathrm{Ref}) \quad \frac{}{(X : P), x : p \vdash_{\mathcal{B}} x : p}$$

$$(\mathrm{App}_{\mathcal{R}}) \quad \frac{(X : P), (X_i : P_i) \vdash_{\mathcal{B}} \Phi_i : q_i \quad i = 1, \ldots, n}{(X : P) \vdash_{\mathcal{B}} \Phi_{\mathcal{R}}(\Phi_1, \ldots, \Phi_n) : r}$$

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

The categorical framework:

- We interpret formulae in presheaves over a base category of 'worlds': $\mathbf{Set}^{\mathcal{W}^{op}}$
- This category is 'cartesian closed' — it has products (conjunctions) and exponentials (function spaces, implications).
- The interpretation a formula ϕ is a functor $[\![\phi]\!] : \mathcal{W}^{op} \to \mathbf{Set}$.
- The category of worlds is constructed from bases and proofs in bases.

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

- Base category: The interpretation of an atomic proposition p in $\mathbf{Set}^{\mathcal{W}^{op}}$ is the functor whose value at 'world' $(\mathcal{B}, (X : P))$ is the set of derivations of p in $\mathcal{B}$ from hypotheses $(X : P)$. The action on morphisms of $\mathcal{W}$ is given by substitution.
  We define a category $\mathcal{W}$ as follows:
  - Objects of $\mathcal{W}$ are pairs $(\mathcal{B}, (X : P))$, where $\mathcal{B}$ is a base and $(X : P)$ is a context
  - A morphism from $(\mathcal{B}, (X : P))$ to $(\mathcal{C}, (Y : Q))$ is given by an inclusion of the base $\mathcal{C}$ into $\mathcal{B}$ and a set of derivations $X : P \vdash_{\mathcal{B}} \Phi_i : q_i$, where $Q = \{q_1, \ldots, q_m\}$. We write such a morphism as $(\Phi_1, \ldots, \Phi_m)$
  - Identity and composition straightforward.

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

Define a functor $[\![\phi]\!] : \mathcal{W}^{op} \to \mathbf{Set}$ by induction over ϕ as follows:

- $[\![p]\!](\mathcal{B}, (X : P))$ is the set of derivations $(X : P) \vdash_{\mathcal{B}} \Phi : p$. Any morphism $(\Phi_1, \ldots, \Phi_m)$ from $(\mathcal{B}, (X : P))$ to $(\mathcal{C}, (Y : Q))$ maps a derivation $(Y : Q) \vdash_{\mathcal{C}} \Phi : p$, which is also a derivation $(Y : Q) \vdash_{\mathcal{B}} \Phi : p$, to the derivation $(X : P) \vdash_{\mathcal{B}} \Phi[\Phi_1/x_1, \ldots, \Phi_n/x_n] : p$.
- $[\![\phi \wedge \psi]\!]$ is the product of the functors $[\![\phi]\!]$ and $[\![\psi]\!]$
- $[\![\phi \supset \psi]\!]$ is defined as $[\![\phi]\!] \supset [\![\psi]\!]$ (the exponential functor)
- $[\![\phi \vee \psi]\!]$ is defined as follows: let $F = [\![\phi]\!]$, $G = [\![\psi]\!]$, and $K((\mathcal{B}, (X : P)), p) = (F \supset [\![p]\!]) \supset ((G \supset [\![p]\!]) \supset [\![p]\!])(\mathcal{B}, (X : P))$. This can be extended to a functor $\mathcal{W}^{op} \times \mathcal{A} \to \mathbf{Set}$. Then $[\![\phi \vee \psi]\!]$ is defined as $\forall_{\mathcal{A}} K$ (a construction that handles the form of the $\vee$-clause in the category)
- $[\![\bot]\!]$ is defined as follows: let $K((\mathcal{B}, (X : P)), p) = [\![p]\!](\mathcal{B}, (X : P))$. This can be extended to a functor $\mathcal{W}^{op} \times \mathcal{A} \to \mathbf{Set}$. Then $[\![\bot]\!]$ is defined as $\forall_{\mathcal{A}} K$.

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

If F and G are functors between categories C and D, then a natural transformation η between F and G is family of morphisms that satisfies the following:

- η must associate to every object x in C an arrow $\eta_x : F(x) \to G(x)$
- for every $f : x \to y$ in C, $\eta_y \circ F(f) = G(f) \circ \eta_x$, where $\circ$ denotes composition of morphisms.

Informally, the notion of a natural transformation captures that a given map between functors can be done consistently over an entire category. In the situation above, we refer to the structure being 'natural in x'.

Base-extension Semantics for Intuitionistic Propositional Logic: A Category-theoretic View

- Since formula ϕ (and so contexts Γ) are interpreted as presheaves, consequences $\Phi : \Gamma \vdash \phi$ are interpreted as maps between functors.
- It turns out that everything is formally natural.
- Lemma: Algebraic Soundness. Suppose $\Gamma \Vdash_{\mathcal{B}} \phi$. Let $\mathcal{W}'$ be the category $\mathcal{W}_{\mathcal{B}}$. Then there exists a natural transformation $\eta_{\mathcal{B}} : [\![\Gamma]\!]^{\mathcal{W}'} \to [\![\phi]\!]^{\mathcal{W}'}$.
- Lemma: Algebraic Completeness. Consider any base $\mathcal{B}$. Let $\mathcal{W}'$ be the category $\mathcal{W}_{\mathcal{B}}$. If there exists a natural transformation $\eta_{\mathcal{B}} : [\![\Gamma]\!]^{\mathcal{W}'} \to [\![\phi]\!]^{\mathcal{W}'}$, then $\Gamma \Vdash_{\mathcal{B}} \phi$.

The usual logical statements of soundness and completeness follow. The latter employs the 'special base', as previously described.

Base-extension Semantics for Intuitionistic Propositional Logic: B-eS and P-tV; B-eS and Logic Programming

A few pointers:

- B-eS and P-tV: Gheorghiu and Pym, *Studia Logica*, 2025, have shown that, for IPL, its P-tV semantics with canonical proofs based on elimination rules can be recovered from its B-eS, so giving a partial resolution of Prawitz's conjecture.
- B-eS and Logic Programming: Gheorghiu and Pym, *Bulletin of the Section of Logic*, 2023, have shown that the least-fixed point construction on the Herbrand universe that gives the semantics of logic programs (defined, following Miller, through hereditary Harrop formulae) can be used to reconstruct the metatheory of IPL's B-eS. Too much to discuss here, but a lot is going on there, and the constructions are informative.
- Tor Sandqvist, Base-extension Semantics as Meaning Theory. 5th P-tS Symposium, London, February 2025. Manuscript: `https://sites.google.com/view/pts-symposium-uk/schedule`

Additional References

- Warren Goldfard. On Dummett's "Proof-theoretic Justifications of Logical Laws". In *Advances in Proof-Theoretic Semantics*.
- Will Stafford and Victor Nascimento. Following All the Rules: Intuitionistic Completeness for Generalized Proof-Theoretic Validity. *Analysis*, 10.1093/analys/anac100, 2024.
- Tor Sandqvist. Atomic bases and the validity of Peirce's law. Presented at 'The meaning of proofs: Celebrating the World Logic Day', 2022:https://sites.google.com/view/wdl-ucl2022/home.
- Tor Sandqvist. Base-extension Semantics as Meaning Theory: Some philosophical reflections on negation, disjunction, and quantification. Available from https://sites.google.com/view/pts-symposium-uk/schedule.
- J. Lambek and P. Scott. Introduction to higher-order categorical logic. Cambridge University Press, 1986.

Additional References

- D. Pym, E. Ritter, and E. Robinson. Categorical Proof-theoretic Semantics. *Studia Logica*, https://doi.org/10.1007/s11225-024-10101-9, 2024.
- A. Gheorghiu and D. Pym. Definite formulae, Negation-as-Failure, and the Base-extension Semantics of Intuitionistic Propositional Logic. *Bulletin of the Section of Logic, 2023*. Manuscript: http://www.cs.ucl.ac.uk/staff/D.Pym/NaFP-tS.pdf.
- S. Mac Lane and I. Moerdijk. Sheaves in Geometry and Logic. Springer, 1994.
- R. Seely. Hyperdoctrines, Natural Deduction, and the Beck Condition. *Math. Log. Quarterly* 1983. https://doi.org/10.1002/malq.19830291005.
- T. Streicher. PhD thesis, Passau, 1988.
- T. Streicher. *Semantics of Type Theory: Correctness, Completeness and Independence Results*. Birkhauser, 2012.

Schedule

1. What is Proof-theoretic Semantics (P-tS)?
   - Inferentialism.
   - Consequence.
   - Proof-theoretic Validity (P-tV).
2. Base-extension Semantics (B-eS):
   - B-eS for Intuitionistic Propositional Logic.
   - Naturality, categorically speaking.
   - B-eS and P-tV.

Schedule

3. Reductive Logic, Tactical Proof, and Logic Programming:
   - Reductive Logic and P-tV.
   - Tactical Proof.
   - Remarks on Logic Programming and Coalgebra.
4. Modal and Substructural Logics, Resource Semantics, and Modelling:
   - B-eS for Modal Logics.
   - B-eS for Substructural Logics.
   - Resource Semantics and Modelling with B-eS.

Schedule

- Most of what we will introduce will be quite new to most people, with a fairly significant philosophical basis, and with quite a lot of ground to be covered.
- Our approach will mainly be conceptual, with little detailed, formal proof.
- Nevertheless, the formal details of everything we cover are available in books and papers that will be referenced.