Four Lectures on Proof-theoretic Semantics

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Schedule

1. What is Proof-theoretic Semantics (P-tS)?
   - Inferentialism.
   - Consequence.
   - Proof-theoretic Validity (P-tV).
2. Base-extension Semantics (B-eS):
   - B-eS for Intuitionistic Propositional Logic.
   - Naturality, categorically speaking.
   - B-eS and P-tV.

Schedule

3. Reductive Logic, Tactical Proof, and Logic Programming:
   - Reductive Logic and P-tV.
   - Tactical Proof.
   - Remarks on Logic Programming, B-eS, and Coalgebra.
4. Modal and Substructural Logics, Resource Semantics, and Modelling:
   - B-eS for Modal Logics.
   - B-eS for Substructural Logics.
   - Resource Semantics and Modelling with B-eS.

Schedule

- Most of what we will introduce will be quite new to most people, with a fairly significant philosophical basis, and with quite a lot of ground to be covered.
- Our approach will mainly be conceptual, with little detailed, formal proof.
- Nevertheless, the formal details of everything we cover are available in books and papers that will be referenced.

Lecture 1: What is Proof-theoretic Semantics?

This first lecture will be high level and informal.

We may not cover all the slides.

More text than normal on slides to help as notes.

Introduction

- *Inferentialism* is a philosophical/metaphysical position that seeks to explain meaning not in terms of denotation but rather in terms of inferential connections between sentences and the role of terms in those sentences — here we conceive of sentences and terms in a quite general sense — in explaining those connections.
- In modern times, it's most prominent advocate is Robert Brandom:
  - *Making it Explicit*, Harvard 1994
  - *Articulation Reasons*, Harvard 2001
  - *Reasons for Logic, Logic for Reasons*, with Ulf Hlobil, Routledge 2024.
- See also Yaroslav Peregrin, *Inferentialism: Why Rules Matter*, Manmillan 2014.
- There is a substantial and quite lively literature.
- Prehistory includes Frege, Carnap, and Wittgenstein, and more.

Introduction

- Significant connections with the philosophy of language.
- And with natural language semantics
- See Nissim Francez's book, *Proof-theoretic semantics*, College Publication 2015.
- We won't explore this aspect any further in these lectures.

Introduction

Proof-theoretic semantics (P-tS) can, in modern terms, be seen as a logical realization of inferentialism.

- Some key figures, stressing our focus in these lectures: Prawitz Dummett, Martin-Löf, Kowalski, Peter Schroeder-Heister and Thomas Piecha, and the Tübingen school, Tor Sandqvist.
- Other longstanding key figures include Sundholm, Wansing, Pereira, Fereira, Fereira, Goldfarb, Steinberger, and many other fine contributors. In recent years, joined by people such as Ayhan, Piccolomini d'Aragona, Stafford, Tranchini, Gheorghiu, Gu, Eckhardt, Buzoku, Pimentel, Ritter, Robinson, and many other fine contributors.

For a snapshot of the community:

`https://sites.google.com/view/ptsnetwork/pts-seminars`

Sincere apologies for brevity and omissions.

Logical Consequence

Model-theoretic Semantics:

- Logical consequence explained in terms of models.
- Standard definition — of *validity* — as articulated by Tarski:

$$\Gamma \models \phi \quad \textit{iff} \quad \text{for all models } \mathcal{M}\text{, if, for all } \psi \in \Gamma,\ \models_{\mathcal{M}} \psi, \text{ then } \models_{\mathcal{M}} \phi$$

  where $\mathcal{M}$ denotes a model and Γ a set of formulae.
- Also write $\Gamma \models_{\mathcal{M}} \phi \ldots$.
- So, consequence amounts to transmission — in the sense of implication in the (usually classical) metatheory — of truth.
- As Peter S-H points out, the categorical concept of truth is prior to the hypothetical concept of consequence.

Logical Consequence

- Proof-theoretic consequence is usually expressed in terms of derivability in a formal system, such as
  - Hilbert-type axiomatic systems,
  - Gentzen-type natural deduction systems, L- and N-type, and
  - tableaux systems, often labelled.
- A consequence $\Gamma \vdash_{\mathcal{K}} \phi$ is derivable in a formal system if it can be generated from elements of Γ using the rules, including axioms, of $\mathcal{K}$.
- The justification of inference or deduction in $\mathcal{K}$ is achieved by showing that the rules of $\mathcal{K}$ are correct, so that a derivation in $\mathcal{K}$ establishes a valid consequence; that is,

$$\Gamma \vdash_{\mathcal{K}} \phi \text{ implies } \Gamma \models \phi$$

- This is usually called *soundness*.
- A few things to note here.

Logical Consequence

- If the converse of ($\Gamma \vdash_{\mathcal{K}} \phi$ implies $\Gamma \models \phi$) also holds, so that validity implies provability — usually called *completeness* — then the proof-theoretic consequence relation matches the model-theoretic one.
- Establishing completeness often involves constructing a *term model* in which provability and model-theoretic consequence coincide exactly.
- Establishing completeness for validity then involves a contrapositive against all models. See, for example, van Dalen's *Logic and Structure*. (Though see Gheorghiu and Pym, Semantical Analysis of the Logic of Bunched Implications. *Studia Logica* (2023) for a different approach to completeness, beyond our scope here.)

Logical Consequence

Before moving on, let's note a few things:

- In this discussion so far we have presumed a notion of 'model'.
- And we have taken truth as the fundamental foundation of consequence and therefore the basis of the justification of inference.
- Defining models can be a significant undertaking — e.g., Kripke models of intuitionistic predicate logic; see, again, van Dalen's *Logic and Structure* — and they can of course be quite big.
- As Wilfrid Hodges has pointed out very carefully, model-theoretic semantics and model theory are really quite different things.
- Proofs are not 'just syntax' as is often implied. Rather, proofs are mathematical objects with static and dynamic structure, just like other mathematical objects.

Logical Consequence

- Prawitz explains in *Logical Consequence from a Constructivist Point of View*, M-tS conflates the meaning of the logical constants with the *meaning of truth*, since logical structure is defined in terms of interpretations.
- For example, if T is defined as the least set satisfying certain properties, including '$\phi \wedge \psi \in T$ iff $\phi \in T$ and $\psi \in T$', then no information is gained about $\wedge$ by saying that it satisfies this relationship.
- Moreover, M-tS fails to provide a genuinely consequential relationship between Γ and ϕ.

Logical Consequence

- Tennant, in *Entailment and Proofs*, observes that a consequential reading of a consequence judgment $\Gamma \vdash \phi$ implies that ϕ follows from Γ by some valid reasoning. This requires a notion of a *valid argument* that encapsulates the forms of valid reasoning.
- We must, therefore, explicate the semantic conditions required for an argument that demonstrates

$$\psi_1, \ldots, \psi_n, \text{ therefore, } \phi$$

to be valid.

Logical Consequence

- So, that's how we think about model-theoretic semantics.
- However, there's always been an alternative view, in which *inference* is taken as the foundation for semantics.
- Following Prawitz, as above, these semantic conditions ought to be based on the logical structure of $\psi_1, \ldots, \psi_n$ and some fixed laws of thought.
- 'Meaning-as-Use' is a rather established idea:
  - Wittgenstein, Brandom, etc., as discussed.
  - The Brouwer-Heyting-Kolmogorov semantics (weakly, I guess).
  - Structural Operational Semantics (strongly).
- Also connections to simulation modelling (Kuorikoski).

Logical Consequence

- Consequently, we abandon the denotationalist perspective on logic, on which M-tS rests, where meaning is given relative to interpretation.
- Instead, we adopt the *inferentialist* perspective, in which meaning is given in terms of inferential relationships, that we have discussed.

Proof-theoretic Semantics (P-tS)

- Here are going to explore *Proof-theoretic Semantics* (P-tS), paying attention to two logical points of view:
  - Proof-theoretic Validity (P-tV) — the tradition of Prawitz (starting from *Ideas and Results in Proof Theory*) and Dummett (*The Logical Basis of Metaphysics*) — in which we ask how can the validity of proofs be defined in terms of inference
  - Base-extension Semantics (B-eS) — dispenses with logical proofs as such and is concerned with defining the validity of formulae in terms of inference.
- B-eS will be the primary focus of these lectures, but we will begin — for context and for historical completeness — with a summary of P-tV.

Proof-theoretic Semantics (P-tS)

Dummett (see *The Logical Basis of Metaphysics*) on circularity in semantics:

- Fairly common philosophical position: no effective justification of a logical law is possible.
- Why? Because all justifications require reasoning/argument, necessarily employing the target logical law.
- Arises because logicians tend to regard soundness and completeness theorems as being of value that is independent of justifications of meaning: mathematically, we are typically concerned with the *relationship* between model-theoretic and proof-theoretic consequence, without unpacking how models confer meaning beyond their having independent mathematical existence.
- Prawitz delivers this attack against model-theoretic semantics in *On the idea of a general proof theory*.
- And P-tS gives us a *degree* of control.

Proof-theoretic Semantics (P-tS

Proof-theoretic Validity (P-tV)

- How can the validity of a proof be judged inferentially?
- Key background, at least in historical terms, builds on the idea of natural deduction rules, which 'introduce' and 'eliminate' the logical constants.
- The relationship between introduction and elimination rules leads to notions of 'reduction' on proofs.
- Key references her are Gentzen's *Untersuchungen uber das logische Schließen* [Investigations into Logical Deduction] (1934) and, especially, Prawitz's *Natural Deduction: A Proof-Theoretical Study* (1971).
- Let's have a very brief reminder of what a natural deduction system looks like.

Proof-theoretic Validity (P-tV)

The calculus NJ:

$$\frac{\phi_1 \qquad \phi_2}{\phi_1 \wedge \phi_2} \quad \wedge I \qquad\qquad \frac{\phi_1 \wedge \phi_2}{\phi_i}\ (i = 1, 2) \quad \wedge E$$

$$\frac{\phi_i}{\phi_1 \vee \phi_2}\ (i = 1, 2) \quad \vee I \qquad\qquad \frac{\phi_1 \vee \phi_2 \qquad \begin{matrix}[\phi_1] \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\phi_2] \\ \vdots \\ \chi\end{matrix}}{\chi} \quad \vee E$$

$$\frac{\begin{matrix}[\phi] \\ \vdots \\ \psi\end{matrix}}{\phi \supset \psi} \quad \supset I \qquad\qquad \frac{\phi \qquad \phi \supset \psi}{\psi} \quad \supset E$$

Proof-theoretic Validity (P-tV)

Normalization:

- Intro-Elim reductions:

$$
\dfrac{\dfrac{\begin{array}{c}\not\phi \\ \vdots\ \Psi \\ \psi\end{array}}{\phi \to \psi}\ \to\text{I} \qquad \begin{array}{c}\vdots\ \Phi \\ \phi\end{array}}{\psi}\ \to\text{E}
\qquad \text{normalizes to} \qquad
\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}
$$

- A point to note is that normalization steps in natural deduction correspond to (β) reduction steps in the λ-calculus.
- Correspondingly, $(\lambda x : [\phi].[\Psi])[\Phi] \to_\beta [\Psi][\,[\Phi]/x\,]$ in the λ-calculus.
- The required set of reductions is quite large and quite delicate to identify: see Girard, Lafont, and Taylor's *Proofs and Types* for a clear and efficient presentation.

Proof-theoretic Validity (P-tV)

- A key motivation lies in the following remarks by Gentzen (see Szabo's *Collected Papers of Gerhard Gentzen*):

  The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'.

- Prawitz used his normalization theory for NJ to develop a semantic concept reflecting this intuition. Dummett later developed the philosophical underpinnings of the idea.
- Later, we shall see that this is very far from being the only way to think about things.

Proof-theoretic Validity (P-tV)

- The basic idea of P-tV in the Dummett-Prawitz tradition is that arguments are valid by virtue of their form.
- One begins with some class of canonical proofs relative to which validity is computed. Arguments are valid if they *represent* a canonical proof, with basic premisses:
  - priority of canonical proofs
  - reduction of closed non-canonical arguments to canonical ones
  - the substitutional view of open arguments — open arguments are justified by their closed instances.

  See Schroeder-Heister's *Validity Concepts in Proof-theoretic Semantics* for a formal account of this version of P-tS.
- Closely related to the *Brouwer-Heyting-Kolmogorov* (BHK) interpretation of intuitionism (see, e.g., Schroeder-Heister's *Model-Theoretic vs. Proof-Theoretic Semantics*.
- P-tV is fundamentally a constructional semantics.

Bases (Atomic Rules)

Bases (Atomic Rules)

- Axiom case and simple case (Level 1):

$$\frac{}{p}\ \text{Axiom} \qquad \frac{p_1 \ldots p_k}{p}$$

For example, with apologies to Tor and Tammy,

$$\frac{\text{Luna is a fox} \qquad \text{Luna is female}}{\text{Luna is a vixen}}$$

$$\frac{\text{Luna is a vixen}}{\text{Luna is female}} \qquad \frac{\text{Luna is a vixen}}{\text{Luna is a fox}}$$

- Interesting case (Level 2):

$$\frac{\begin{array}{ccc}[P_1] & & [P_n] \\ q_1 & \ldots & q_n\end{array}}{r}$$

with dischargeable hypotheses — we'll return to this later

– Lots of other cases, with subtle choices

Dischargaeable Hypotheses

- Does discharging arise in atomic systems?
- Examples may readily be found in natural language.
- Conceptually related in such a way as to call for hypothesis-discharging modes of inference: if I accept that Sandy is a sibling of Mary, then I am committed to accepting anything that follows both from the hypothesis that Sandy is a brother of Mary and from the hypothesis that Sandy is a sister of Mary.
- So we have, for any sentence p and using $\Rightarrow$ for consequences, the rule

$$(\Rightarrow \text{'Sandy is a sibling of Mary'}),\\ (\text{'Sandy is a brother of Mary'} \Rightarrow p),\\ (\text{'Sandy is a sister of Mary'} \Rightarrow p) \Rightarrow p$$

- More discussion and pointers in Sandqvist's *Base-extension semantics for intuitionistic sentential logic*.

Bases (Atomic Rules)

- The choice of the form of atomic rules in bases has a profound effect on the strength of the semantics that is obtained — see, for example, Tor Sandqvist's presentation at the World Logic Day event at UCL in 2022,

  `https://sites.google.com/view/wdl-ucl2022/home`
- Note that base rules are *pre-logical* — they do not refer to the logical constants.
- Note that base rules such as the Level 2 rules mentioned above permit discharge of assumptions (cf. NJ, discussed above).
- This class is used in the B-eS for intuitionistic propositional logic Lecture 2).

Proof-theoretic Validity in the Dummett-Prawitz Tradition

$\langle \mathbf{J}, S \rangle$-validity — A General View

Prawitz, in *Ideas and Results in proof Theory*, introduces the idea of S-validity of proofs for a base S. To set up such a semantics of proofs relative to a base S, we need a few auxiliary ideas:

- Let S be a base of atomic rules: bases are essential to give ground validity — they assert that atoms have a grounding in inferential validity. Prawitz has been discussing this even recently.
- Let $\mathcal{D}$ be a system of proof rules.
- Suppose that the 'canonical' proofs of $\mathcal{D}$ are those Φ that are elements of the set $\mathbf{C}(\mathcal{D})$.
- Let $\mathcal{S}$ denote the class of proof-structures, regulated by the rules of $\mathcal{D}$.
- Let $\mathbf{J}$ be a procedure on proof-structures that yields proof-structures.

$\langle \mathbf{J}, S \rangle$-validity — A General View

$\langle \mathbf{J}, S \rangle$-validity — A General View

We can define validity of proofs, relative to a base and a justification.

A *closed* proof does *not* depend on assumptions; *open*, otherwise.

$\langle \mathbf{J}, S \rangle$-validity: *closed proofs*

1. Every closed proof in S is $\langle \mathbf{J}, S \rangle$-valid.
2. A closed canonical proof is $\langle \mathbf{J}, S \rangle$-valid if its immediate subproofs are $\langle \mathbf{J}, S \rangle$-valid.
3. A closed non-canonical proof is $\langle \mathbf{J}, S \rangle$-valid if it reduces via $\mathbf{J}$ to a $\langle \mathbf{J}, S \rangle$-valid canonical proof.

$\langle \mathbf{J}, S \rangle$-validity — A General View

Case 3 requires the role of **J**: a proof-structure Φ is $\langle \mathbf{J}, S \rangle$-valid — that is, represents a proof — if either $\Phi \in \mathbf{C}(\mathcal{D})$ or if **J** can be applied to Φ to yield an element of $\mathbf{C}(\mathcal{D})$.

$\langle \mathbf{J}, S \rangle$-validity — A General View

An *open proof*

$$\begin{array}{c} \phi_1, \ldots, \phi_k \\ \mathbf{\Phi} \\ \psi \end{array}$$

is $\langle \mathbf{J}, S \rangle$-valid if, for every list of closed $\langle \mathbf{J}, S \rangle$-valid proofs,

$$\begin{array}{ccc} \mathbf{\Phi}_1 & & \mathbf{\Phi}_k \\ & \ldots & \\ \phi_1 & & \phi_k \end{array}$$

the proof

$$\begin{array}{ccc} \mathbf{\Phi}_1 & & \mathbf{\Phi}_k \\ & \ldots & \\ & \mathbf{\Phi} & \\ & \psi & \end{array}$$

is $\langle \mathbf{J}, S \rangle$-valid

$\langle \mathbf{J}, S \rangle$-validity

A notion of consequence w.r.t $\mathbf{J}$ and S is now given as follows:

$$\phi_1, \ldots, \phi_k \models_{\langle \mathbf{J}, S \rangle} \psi$$

holds if there is an open proof

$$\begin{array}{c} \phi_1, \ldots, \phi_k \\ \Phi \\ \psi \end{array}$$

s.t. for all S and every list of $\langle \mathbf{J}, S \rangle$-valid closed proofs

$$\begin{array}{ccc} \Phi_1 & \ldots & \Phi_k \\ \phi_1 & & \phi_k \end{array}$$

the proof

$$\begin{array}{ccc} \Phi_1 & \ldots & \Phi_k \\ \phi_1 & & \phi_k \\ & \Phi & \\ & \psi & \end{array}$$

is $\langle \mathbf{J}, S \rangle$-valid

$\langle \mathbf{J}, S \rangle$-validity

Examples of justifications:

- Normalization of NJ proofs, in the sense of Prawitz — see *Ideas and Results in Proof Theory*, and Peter S-H's *Validity Concepts in Proof-theoretic Semantics* — choices around notion of canonical, intro- and elim-based versions of definitions.
- And so, the reduction theory of the corresponding (according to 'Curry-Howard') typed λ-terms.
- The BHK-type interpretations — see Troelstra and van Dalen, *Constructivism in Mathematics*. These essentially trivialize the set-up.
- The reduction theory of the $\lambda\mu$-calculus and its extensions.
- With a bit of effort/imagination: cut-reduction in proof-nets ... .

These examples all live within 'deductive logic'. In fact, 'reductive logic', which we'll encounter in Lecture 3, is perhaps a more natural place to look.

$\langle \mathbf{J}, S \rangle$-validity

- Consider the setting of NJ with justification given by its usual normalization procedure.
- Can consider that the 'canonical' proofs of $\mathcal{D}$ — i.e., those Φ that are elements of the set of $\mathbf{C}(\mathcal{D})$ — are those with an introduction rule as the last step.
- But other choices.

$\langle \mathbf{J}, S\rangle$-validity

- Recall

$$\phi_1, \ldots, \phi_k \models_{\langle \mathbf{J}, S\rangle} \psi$$

  defined in terms of the existence of proofs.
- We can enrich this idea — see Lecture 3 — with a notion of *realizer*, allowing us to consider the space of witnesses for such consequences.

$\langle \mathbf{J}, S \rangle$-validity

- If **J** specifies a reduction system — as with normalization in natural deduction, as described in, for example *Ideas and Results in proof Theory* or *Model-theoretic vs. Proof-theoretic Semantics* — $\langle \mathbf{J}, S \rangle$-valid proofs can be defined inductively on their component structure.
- In this case, the semantics of proofs of implications presents a particular issue.

$\langle \mathbf{J}, S \rangle$-validity

- We expect a construction of an implicational formula $\phi \supset \psi$ to be a construction that given a construction of ϕ yields a construction of ψ.
- However, such a condition on a construction of $\phi \supset \psi$, as formulated above, would be satisfied vacuously if there be no construction of ϕ relative to the system S in question.
- It follows (cf. Kripke's semantics of implication) that it's better to give the semantics of proofs of implications relative to all possible extensions $S \subseteq S'$.
- In fact, Prawitz (Ideas and Results in Proof Theory) points out that the extensions considered can be restricted to those required by **J** for the construction of ψ.

$\langle \mathbf{J}, S \rangle$-validity

In summary, all this set-up yields a notion of logical consequence with respect to $\mathbf{J}$ and S as follows:

- Denote by $\langle \mathbf{J}, S \rangle \models \phi$ that $\mathbf{J}$ generates with respect to S a closed canonical proof of ϕ.
- Then

$$\phi_1, \ldots, \phi_k \models_{\langle \mathbf{J}, S \rangle} \psi \quad \text{iff} \quad \begin{array}{l} \text{there is a } \mathbf{J} \text{ such that, for every } S \\ \text{and all } \mathbf{J}_1, \ldots, \mathbf{J}_k\text{, if} \\ \langle \mathbf{J}_1, S \rangle \models \phi_1, \ldots, \langle \mathbf{J}_k, S \rangle \models \phi_k, \\ \text{then } \langle \mathbf{J}, S \rangle \models \psi \end{array}$$

- In the case of the BHK interpretation, the structure of proofs is trivialized.
- The set-up remains truly more general, however: we'll later, in Lecture 3, that P-tV provides a conceptual framework for reductive logic, where many notions of 'canonical' and 'justification' are of interest.

Prawitz's Conjecture

- Recall our remark early on that P-tV is fundamentally a constructional semantics.
- In fact, Prawitz conjectured that his notion of P-tV corresponds to intuitionistic propositional logic (IPL) — remains open.
- Piecha et al. have shown that, for a slightly simplified P-tV, IPL is incomplete.
- Stafford has shown that Piecha et al's semantics amounts to 'general inquisitve logic', which is the intermediate logic that extends IPL with (the admissible, but not derivable)

$$\frac{H \supset (\phi \vee \psi)}{(H \supset \phi) \vee (H \supset \psi)}$$

where H is hereditary Harrop (we'll meet hHfs again in Lecture 3).

Prawitz's Conjecture

In a recent paper — *From Proof-theoretic Validity to Base-extension Semanics for Intuitionistic Propositional Logic*, *Studia Logica, 2025* — Gheorghiu and Pym have show that IPL encapsulates the declarative content of a P-tV based on elimination rules.

The details of this are beyond our scope here.

The Dogma of Standard Semantics (following Peter S-H)

- As we have seen, the model-theoretic and proof-theoretic accounts of logical consequence are fundamentally different.
- Two key, interrelated ideas are common to them, however:
  (1) the assumption that a *categorical* concept — that is, unqualified, unconditional, absolute — is prior to the *hypothetical* —- that is, contingent upon a theory — concept of *consequence*; for classical logic,
    - model-theoretically, this is the notion of truth in structure the
    - proof-theoretically, this is validity with respect to a construction or justification
  (2) the *transformational* view of consequence; for classical logic,
    - the transmission of truth between interpretations of formulae in a model
    - the transmission of validity from the premisses to the conclusion of a model.

The Dogma of Standard Semantics (following Peter S-H)

- The two assumptions have been called *the dogma of standard semantics* (Schroeder-Heister and Contu, *Logik in der Philosophie*, 2005).
- Denote constructions of of proof-theoretic structures by $\mathfrak{C}$s (simplifying out earlier more detailed notation) and define validity with respect to a structure by $\mathfrak{C} \models \phi$,

$$\phi_1, \ldots, \phi_k \models \phi \quad \text{iff} \quad \text{for all } \mathfrak{C}_1 \ldots \mathfrak{C}_k,\ \mathfrak{C}_1 \models \phi_1, \ldots, \mathfrak{C}_k \models \phi_k \text{ implies } f(\mathfrak{C}_1 \ldots \mathfrak{C}_k) \models \phi$$

  where f is a constructive transformation that generates a structure that validates the conclusion from structures that validate the premisses.
- This notion of the transmission of (the categorical concept of) validity is at the core of P-tV, but is it really inferential?

The Dogma of Standard Semantics (following Peter S-H)

- It can be argued that on consequence as transmission of categorical concepts blocks the way towards a concept that is really based on inference and deserves the name 'inferential semantics'.
- It can, therefore, be argued that need to avoid forms of definition in which assumptions are interpreted as placeholders for proofs.
- This problem is particularly acute in P-tV that is based on ideas of introduction and elimination'.

Definitional Reflection (following Peter S-H)

- So, the dogma of standard semantics effectively imposes the primacy of closed proofs over open proofs.
- How can we give up the dogma of standard semantics?
- Sequent calculi, in the sense of Gentzen's LJ and LK, provide a way forward.
- This view will also provide a technical connection to the idea of reductive logic that we'll consider in Lecture 3.

Aside: Sequent Calculus

We don't need to get into a lot of detail, but the basic idea is that the rules somehow directly manipulate consequences by introducing logical operators either on the left or the right. For example,

$$\frac{\Gamma, \phi_1, \phi_2, \Gamma' \vdash \Delta}{\Gamma, \phi_1 \wedge \phi_2, \Gamma' \vdash \Delta} \wedge L \qquad \frac{\Gamma \vdash \Delta, \phi_1, \Delta' \qquad \Gamma \vdash \Delta, \phi_2, \Delta'}{\Gamma \vdash \Delta, \phi_1 \wedge \phi_2, \Delta'} \wedge R$$

$$\frac{\Gamma, \phi_1, \Gamma' \vdash \Delta \qquad \Gamma, \phi_1, \Gamma' \vdash \Delta}{\Gamma, \phi_1 \vee \phi_2, \Gamma' \vdash \Delta} \vee L \qquad \frac{\Gamma \vdash \Delta, \phi_1, \phi_2, \Delta'}{\Gamma \vdash \Delta, \phi_1 \vee \phi_2, \Delta'} \vee R$$

$$\frac{\Gamma, \Gamma' \vdash \Delta, \phi, \Delta' \quad \Gamma, \psi, \Gamma' \vdash \Delta, \Delta'}{\Gamma, \phi \supset \psi, \Gamma' \vdash \Delta, \Delta'} \supset L \qquad \frac{\Gamma, \phi, \Gamma' \vdash \Delta, \psi, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \phi \supset \psi, \Delta'} \supset R$$

$$\frac{}{\Gamma, \phi, \Gamma' \vdash \Delta, \phi, \Delta'} Ax \qquad \frac{\Gamma, \Gamma' \vdash \Delta, \phi, \Delta' \qquad \Gamma, \phi, \Gamma' \vdash \Delta, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} Cut$$

Definitional Reflection (following Peter S-H)

- Simplifying a bit, let's consider

$$\frac{}{\phi \vdash \phi}\,Ax \qquad \frac{\Gamma \vdash \phi \qquad \phi, \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \psi}\,Cut$$

- The view that assumptions are placeholders for proofs now corresponds to cuts

$$\frac{\vdash \phi_1 \ldots \vdash \phi_k \qquad \phi_1, \ldots, \phi_k \vdash \psi}{\vdash \psi}$$

- So we restore an inferential foundation.

Definitional Reflection (following Peter S-H)

- How can sequent-style rules be justified?
- We can use a theory of *external definitions* that is inspired by *logic programming*.
- This is the first of a few connections between P-tS and logic programming.

Definitional Reflection (following Peter S-H)

A *definition* is a system of clauses of the form

$$\mathfrak{D} \begin{cases} a_1 \Leftarrow A_{11} \\ \quad\vdots \vee \\ a_1 \Leftarrow A_{1k_1} \\ \quad\vdots \wedge \\ a_n \Leftarrow A_{n1} \\ \quad\vdots \vee \\ a_n \Leftarrow A_{nk_n} \end{cases}$$

where the a_is are atoms and the A_{ij}s are sets of atoms (so we have an inductive definition). Generalizing this view leads us to the characterization of logic programs through hereditary Harrop formulae.

Definitional Reflection (following Peter S-H)

- How are such definitions used?
- Through the left- and right-rules (in the sense of the sequent calculus), *definitional closure* and *definitional reflection*, respectively:

$$\frac{\Gamma \vdash A_{ij}}{\Gamma \vdash a_i} \vdash a_i \quad \text{and} \quad \frac{\Gamma, A_{i1} \vdash \phi \ \ldots \ \Gamma, A_{ik_i} \vdash \phi}{\Gamma, a_i \vdash \phi} a_i \vdash$$

  respectively — everything that follows from all the defining conditions of a_i follows from a_i.
- This leads us to reductive logic, which we shall consider in Lecture 3.

Definitional Reflection (following Peter S-H)

- A last word on Definitional Reflection (DR).
- As Peter S-H and Lars Hallnäs have shown, in natural deduction, DR shows how to derive elimination rules from introduction rules. Since we have

$$\frac{\phi}{\phi \vee \psi} \qquad \frac{\psi}{\phi \vee \psi}$$

  the defining conditions of $\phi \vee \psi$ are ϕ and ψ.
- So DR warrants the elimination rule

$$\frac{\phi \vee \psi \quad \begin{array}{c}[\phi]\\ \vdots\\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \vdots\\ \chi\end{array}}{\chi}$$

Summary: Paradigms of Meaning

denotationalism	–	inferenatialism
meaning based on truth	–	meaning based on inference
model-theoretic semantics	–	proof-theoretic semantics
e.g., Tarski/Kripke semantics	–	e.g., base-extension semantics
e.g., interpretation of proofs in models	–	e.g., proof-theoretic validity

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Schedule

1. What is Proof-theoretic Semantics (P-tS)?
   - Inferentialism.
   - Consequence.
   - Proof-theoretic Validity (P-tV).
2. Base-extension Semantics (B-eS):
   - B-eS for Intuitionistic Propositional Logic.
   - Naturality, categorically speaking.
   - B-eS and P-tV.

Schedule

3. Reductive Logic, Tactical Proof, and Logic Programming:
   - Reductive Logic and P-tV.
   - Tactical Proof.
   - Remarks on Logic Programming, B-eS, and Coalgebra.
4. Modal and Substructural Logics, Resource Semantics, and Modelling:
   - B-eS for Modal Logics.
   - B-eS for Substructural Logics.
   - Resource Semantics and Modelling with B-eS.