

TOWARDS AN INFERENTIALIST ACCOUNT OF INFORMATION THROUGH PROOF-THEORETIC SEMANTICS

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ABSTRACT. Information is one of the most widely-discussed concepts of the current era. However, a great deal of insightful work notwithstanding, it is yet to be given wholly convincing logical or mathematical foundations. Without them, we lack adequate reasoning tools for understanding the complex ecosystems of systems upon which the society depends. We seek to rectify this by taking a first step towards developing an inferentialist semantic theory of information. There are three key interacting components. First, conceptual analysis: the metaphysics of information. Dretske expressed the key concepts of information in terms of intentionality, truth, and transmissibility. We replace truth with inferability, and trace the consequences of this replacement. Second, logic: proof-theoretic semantics (P-tS) provides a mathematical-logical realization of inferentialist reasoning. Using P-tS, we develop the first steps towards a mathematical-logical theory of an inferentialist primitive unit of information, the ‘inferon’. This proof-theoretic approach counterpoints the model-theoretic view of information articulated in situation theory. Furthermore, we argue that it facilitates addressing all three components of van Benthem and Martinez’s categorization of the understandings of information, as range, as correlation, and as code. Our focus is on information-as-correlation. Third, systems: the P-tS tools we develop provide the basis for a mathematical account of distributed systems modelling — a key tool from informatics for understanding the organization of information processing systems. This yields a reasoning-based theory of information flow in models of distributed systems. Overall, we seek to give a conceptually rigorous mathematical-logical account of information and its role within informatics, grounded in inference and reasoning.

1. INTRODUCTION

Information is one of the most widely-discussed concepts of the current era. However, a great deal of insightful work notwithstanding, it is yet to be given wholly convincing logical or mathematical foundations. Without them, we lack adequate reasoning tools for understanding the complex ecosystems of systems upon which the society depends. We seek to rectify this by taking a first step towards developing an inferentialist semantic theory of information. There are three key interacting components. First, conceptual analysis: the metaphysics of information. Dretske expressed the key concepts of information in terms of intentionality, truth, and transmissibility. We replace truth with inferability, and trace the consequences of this replacement. Second, logic: proof-theoretic semantics (P-tS) provides a mathematical-logical realization of inferentialist reasoning. Using P-tS, we develop the first steps towards a mathematical-logical theory of an inferentialist primitive unit of information, the ‘inferon’. This proof-theoretic approach counterpoints the model-theoretic view of information articulated in situation theory. Furthermore, we argue that it facilitates addressing all three components of van Benthem and Martinez’s categorization of the understandings of information, as range, as correlation, and as code. Our focus is on information-as-correlation. Third, systems: the P-tS tools we develop provide the basis for a mathematical account of distributed systems modelling — a key tool from informatics for understanding the organization of information processing systems. This yields a reasoning-based theory of information flow in models of distributed systems. Overall, we seek to give a conceptually rigorous mathematical-logical account of information and its role within informatics, grounded in inference and reasoning.

Barwise and Seligman, *Information Flow: The Logic of Distributed Systems*, Cambridge University Press, 1997, page 22:

Inference has something crucial to do with information. On the one hand, inference is often characterized as the extraction of implicit information from explicit information. This seems reasonable, because it is clearly possible to obtain information by deduction from premisses. In carrying out these inferences in real life, we take

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as premisses some background theory of how the world works. On the other hand, getting information typically requires inference.

Our starting point is the position of inferentialism — that the meaning of linguistic entities, including propositions and proofs, is determined by their use. Inferentialism has been articulated with clarity and elegance by Brandom (and also Hlobil) in a sequence of major publications, most notably [18, 19, 55]. We adopt an inferentialist perspective on, and reinterpretation of Dretske’s metaphysics of information and employ in our formal development a mathematical-logical realization of inferentialism known as proof-theoretic semantics.

We begin the story in Section 2 where we summarize Dretske’s metaphysics of information [30], explaining how we seek to replace his requirement of truth with a requirement of inferability. We also discuss van Benthem and Martinez’s characterization of formal theories of information into the categories information-as-range, information-as-correlation, and information-as-code [89] and how our approach, while largely within information-as-correlation, accounts for significant parts of all three categories.

Next, we have two important background sections. The first, Section 3, recalls the basic ideas of situation theory that are relevant to our development. The second, Section 4, introduces the key ideas of proof-theoretic semantics, stressing what is widely known as base-extension semantics, which is an essential tool in our later development. In Section 3, we summarize the key ideas of situation theory, as developed logically by Barwise, Etechemendy, Devlin, Perry, Seligman, and others [13, 14, 8, 10, 28, 15, 81]. The core idea of situation theory in our context is that of the (atomic) *infor* — a basic unit of information that is very closely related to a first-order atomic predicate, which derives its meaning from a first-order model.

In Section 4, we introduce the key ideas of proof-theoretic semantics [67, 79, 80, 33], stressing the role of base-extension semantics [76, 65, 77, 75, 44, 50], in particular as formulated for intuitionistic logic, which provides our logical point of departure. The key idea here is that of a *base* of atomic rules, such as

$$\frac{}{p} \quad \frac{p_1 \dots p_k}{p} \quad \text{and} \quad \frac{[P_1] \quad \dots \quad [P_k]}{p}$$

where each p and p_i is an atom, not a formula or a variable ranging over formulæ. Conceptually, the key point here is that base rules are pre-logical — they involve no use of logical operators. We discuss how bases work in Section 4.

In Section 5, we get to the core of our ideas about information, giving a range of simple examples in Section 6. Central to this development is the concept of the inferentialist *infor*, or *inferon* — this too is a basic unit of information and also resembles a first-order atomic predicate, but its meaning is derived not from a first-order model, but rather from a global base of atomic rules, describing the context for the *inferon*, together with the *inferon*’s own local base of atomic rules, describing the inferential capacity associated with the *inferon*.

Then, in Section 7, we develop an account of information flow in the inferentialist setting that is analogous to the theory presented by Barwise and Seligman [15]. Section 8 relates the theory of information flow introduced in Section 7 to a quite generic, yet practically motivated, approach to modelling distributed systems. Finally, in Section 9, we consider a range of further developments of the programme initiated by the work presented herein.

2. THE METAPHYSICS OF INFORMATION: DRETSKE AND VAN BENTHEM

Building on ‘Knowledge and the Flow of Information’ [31], Dretske gives a basic account of the metaphysics of information in [30].

Dretske [30] identifies that information is an important concept in many aspects of science and everyday life that, despite that importance, lacks a general theory. This is an especially important and difficult problem as the use of the word has become quite different and specialized as it has been integrated into different disciplines. To address this Dretske aims to identify the concept that stands behind these specific uses and constitutes our shared understanding of information. A statistician, a computing scientist, a philosopher, and a layman, for example, might all talk about information differently, but with each them we understand that what they are talking about is, in fact, information.

Dretske identifies three properties of information:

- Intentionality
- Truth
- Transmissibility.

Intentionality requires information to be about something. With any information we have to be able to ask the question ‘information about what?’. It follows that pieces of information must be semantic objects. A letter is not information even if the message conveyed by it might be. Furthermore, this means that the same signal might carry different information depending on the sender, the receiver, and the general circumstances, with obvious examples being words like ‘you’ and ‘I’.

Truth simply requires the information to be correct. This is a much discussed property called the *Veridicality Thesis* (see, for example, [40]). Dretske argues that misinformation or false information are not actually information, but rather are mistakenly represented as information, much like a fake diamond is not actually a diamond. The truth of information is important because of its epistemological role: Information is a necessary condition for knowledge. A false proposition works just as well to check the validity of an argument as a true one. However, as Dretske puts it ‘What good is flawless reasoning if everything you conclude is false?’ [30]. Information is sought after exactly because of its truth and that truth, together with correct reasoning, allows us to draw true conclusions. Israel and Perry [57], in their treatment of ‘information reports’, assume that information conveyed within such reports must be ‘factive’; that is, true.

Transmissibility requires information to have the potential to be sent and received. Meaning can, for example, be transmitted by simply writing it down. For information, however, more is required. As information is only real when it is true, it is not enough to transmit the truth, but you have to do so in a way that the receiver can know the truth you transmit. For example, looking at a thermometer conveys information about the temperature because the receiver already considers the thermometer to be a reasonably accurate tool for measuring temperature. Similarly a trusted expert can convey information on a topic in a way a random person cannot. A message from an unknown source can, of course, still be information if it includes the necessary evidence to generate knowledge in the receiver.

We aim to give an inferentialist account of information and so differ from Dretske in various ways. The most significant such difference is with the notion of truth. Instead of being given in terms of truth and falsity, information is given in terms of inferability. In our specific case, this means the derivability of basic units of information from a set of atomic rules called a base. This has a significant impact on the discussion of veridicality. We conjecture that an inferentialist version of the veridicality thesis holds. Our approach requires inferences to be valid in order to count as information and so the idea that information has to be correct remains, fundamentally, preserved. As this is supposed to be a first step, a more in-depth discussion of the details and intricacies is outside the scope of this paper and left for future work (see Section 9).

Less obvious is how the inferentialist view impacts the idea of transmissibility. Information still has to be sent and received, of course, but we can give a precise account of the idea that information has to be transmitted so that the receiver can know it. We take a piece of information to include a base, as mentioned above, so that the statement the information is about can be derived from the rules in that base together with the rules the receiver already had access to. This gives an easy way of checking whether a message fails the transmissibility criterion: whenever this inference is not possible. Only intentionality remains largely unchanged. However, derivability, more so than truth, asks the question ‘for whom?’ and so puts further importance on the question of the rôle of the agents.

For the purposes of this paper, we have taken Dretske’s metaphysics of information [31, 30] as our point of departure. There is a significant literature on the philosophy and logic of information — for example, Bimbó [16], Carnap [23, 24], Dunn [34] (and ‘A “Reply” to my “Critics” ’ in [17]), Floridi [38, 40, 41] to name just a few contributors of very many.

In particular, Johan van Benthem has written a great deal about the relationship between logic and information. In [89], ‘The Stories of Logic and Information’, van Benthem and Martínez characterize three categories of formal theories of information:

- (1) *Information-as-range*. In this view, information is modelled by the range of possibilities that are consistent with it — the more options one has to consider, the less information one has. From a logical point view, epistemic logics [62, 90] beginning with [92, 54] and including Public Announcement Logic (PAL) [6] provide the basic logical tools.
- (2) *Information-as-correlation*. This view takes what information is *about* as the focus of study. Information is treated as existing in different situations with *dependencies* between such situations — situation theory, and ‘infor logic’ [14, 28, 29, 15, 2] are the primary tools for this view.
- (3) *Information-as-code*. Here information is thought of as *encoded* in sentences and messages. This allows for a syntactical investigation of information which relies on proof theory rather than semantics. The inferences possible given a certain message tell us its informational content — Shannon’s work [82] and Abramsky’s process-oriented approach [1] provide good examples of this view.

These ideas are also conveyed by Adriaans and van Benthem in [3], ‘Information is What Information Does’.

While the work discussed above, and much more besides,¹ has considered many aspects of the philosophy, logic, and algorithmics of information, and while this work exposes many challenging issues and has attracted much comment too various to discuss here, it has not addressed the challenge of establishing a unified logical theory. By contrast, through our approach, we seek to be able to address at least some of the concerns of each of, as indicated below.

Our first point of departure, as discussed above, is Dretske’s metaphysics of information and we propose an inferentialist variation of Dretske’s analysis in which the requirement of truth is replaced by a requirement of inferability. Our second point of departure, also as discussed above, is van Benthem and Martinez’s categorization of formal theories of information. We have argued that van Benthem and Martinez’s categorization can be seen as being underpinned by Dretske’s metaphysics. Our inferentialist view of van Benthem and Martinez’s categorization of formal theories of information — in which information-as-correlation is based not on model-theoretic inforons and situations, but rather on proof-theoretic inforons and situations, grounded in proof-theoretic semantics — can evidently be seen as being underpinned by our inferentialist variation of Dretske’s analysis.

While the primary purpose of the present paper is to develop an inferentialist theory of information-as-correlation, as described in terms of inforons and situations, we conjecture that our approach based on proof-theoretic semantics is also able to account for at least significant parts of information-as-range and information-as-code:

- (1) *Information-as-range*. Rather than considering a range of models or states that model facts, this understanding of information can be seen as employing a range of bases that support inforons (see Section 5 for the meaning of this).

We can expect initially to work with epistemic logics given by B-eS. We have established background work for intuitionistic modal logics (K through to S4) [21], classical modal logics (K through to S4 and S5) [35, 37], and Public Announcement Logic (PAL) [36]. However, we will seek more general accounts of epistemic logics and agency — see Section 5 for some first thoughts on agency — in the setting of the account of information given by proof-theoretic semantics. We can expect to find substantive connections with van Benthem’s work on logical dynamics and information [86, 87] and perhaps Floridi’s [39], though this beyond our present scope.

- (2) *Information-as-correlation*. The inferentialist formulation of this through proof-theoretic semantics is the main topic of this paper.
- (3) *Information-as-code*. Our proof-theoretic approach to information is able to address some aspects of the information-as-code understanding, in which the inferences that are possible from a given message tell us its informational content. In our setting, such inferences are part of the inferential semantics — characterized as support for an inforon with respect to a base (again, see Section 5 for the meaning of this) — rather than being accessible only through meta-theoretical reasoning.

¹For example, see [60] for a discussion of Shannon’s theory, some derivatives of it, and some other approaches [7, 26, 27, 83, 63, 22, 58, 91]. More detailed discussion of these is beyond our present scope.

Three worthwhile observations are the following:

- For the first-order inferonic logic, we can work with the theory of Peano Arithmetic (PA) [44, 45] and so represent numerical measure of information within our semantics.
- We can develop second-order inferonic logic (based on [51]), with the theory of second-order arithmetic (building on [44]), so allowing aspects of set theory and analysis to be deployed.
- We can explore a proof-relevant inferon logic, where proof-objects are associated with the clauses of the support relation (cf. BHK semantics, as discussed in connexion with proof-theoretic validity in [79]; see also [4]).

We discuss our intended approach to these further in Section 9. This is somewhat related to Prawitz’s original conception of proof-theoretic validity, which we discuss briefly in Section 4.

Finally, a remark on agency: in each of the three categories, but especially in ‘range’ and ‘correlation’, notions of agency are relevant. In epistemic logics (e.g., [62, 90, 88]), this is commonplace notion and in our setting we should seek to explore, echoing Dretske, ‘knowledge and the flow of information’. In our present development, we introduce, in Section 5, a notion of agency through ‘action’ or ‘agent’ modalities. Further development of these ideas, to which we return briefly in Section 9, is deferred to future work.

Before we proceed to develop our inferentialist theory of inferons and situations, we give, in Section 3, a brief overview of situation theory as presented by Barwise, Devlin, Etchemendy, Perry, Seligman, and others.

3. SITUATIONS

Situation theory is a mathematical theory of information and information flow: see, among other sources, [13, 14, 8, 9, 10, 28, 15, 81]. The focus of situation theory is on making an applicable theory of how pieces of information are constructed and interpreted, including how they combine, how one piece of information may lead to another piece of information, and how they flow between situations.

The techniques employed in the mathematical presentation of situation theory are given in the general style of model-theoretic semantics, using interpretation functions to give meaning to propositions. Before proceeding to develop our inferentialist account of information and information flow, grounded in proof-theoretic semantics, we give a brief sketch of the most basic concepts of situation theory. We do this in a simplified way, eliding much of the substantial complexity of the full set-up, an extensive account of which is provided by Devlin [28]. In reality, there are several different situation theories, and we here try to give a flavour of some key concepts.

In situation theory, pieces of information are formalized using syntactic entities called *infons*. The form of an (atomic) infon is, essentially, $\langle\langle R, a_1, \dots, a_n, \mathbf{b} \rangle\rangle$, where R is an n -ary multi-sorted relation symbol, and a_1, \dots, a_n is a sequence of constants of the appropriate sorts to instantiate R , and $\mathbf{b} \in \{0, 1\}$ is a polarity. The idea is that this piece of syntax stands for ‘ a_1, \dots, a_n do (do not) stand in relation R ’. Compound infons are generated using connectives for conjunction, implication, quantifiers, and so on.

Infons themselves are taken to be neither true nor false: the relevant question is whether any given situation *supports* the infon (or not). To this end, a semantic judgement relation is used, of the form $s \models \sigma$, for s a situation and σ an infon. In applications, s is often an *abstract situation*, which is a set of infons. In this case, the support relation, \models , is just the ‘element-of’ relation, \in . We remark that Mares [61] and Restall [73] consider how Routley–Meyer ternary-relation semantics for relevance logic can be used to give a model-theoretic logic of infons.

The general set-up (e.g., [28]) uses a system of types. In particular, situations can be typed using a typing judgement. This takes the form $s : T$, where s is a situation of type T . In Barwise and Perry’s early set-up [14], a type is (essentially) a set of infons. Situations then consist of a type together with a location. In Devlin’s set-up [28], one can abstract to form the types of situations supporting particular infons. For example, for a 0-ary predicate (proposition) *fire*, a corresponding type of situations (in which there is fire) can be defined in which $\langle\langle \text{fire}, 1 \rangle\rangle$. The notion of type employed in some parts of situation theory is rich and complicated and requires considerable mathematical machinery from classical model theory to realize set-theoretically. By contrast, the approach we develop in Section 5 is grounded in proof theory.

A *constraint* is a binary relation on situation types, written $S \Rightarrow S'$, where S and S' are situation types. The idea of the constraint is, essentially, that any situation s that has $s : S$ can be related to some situation $s' : S'$. In particular, a constraint $S \Rightarrow S'$ gives us that if S is the type of situations that support σ , and S' is the type of situations that support σ' , then if there is $s \models \sigma$, then there is some $s' \models \sigma'$. For example, S may be the type of all smokey situations and S' may be the type of all firey situations, and a constraint might express ‘there is no smoke without fire’. More technical parts of the set-up allow closer control of how the resulting witness s' is linked to the triggering s . Agents are said to be *attuned* to constraints when they are aware of them, and (consciously or unconsciously) accept them.

It’s clear then that even in the most basic conception, as well as the commitments to ontology, situation theory also commits methodologically to model-theoretic semantics and, indeed, bivalent truth. Moreover, a constraint is a semantic construct designed to capture a possible inference rule, while attunement means accepting the validity of that inference rule.

Situation theory requires the rich notion of type as described in order to support all the goals of situation semantics. Consequently, the realization of it in terms of sets (models) is somewhat complex. Later, simplified theories were produced to give a simpler treatment of information flow. One such theory, involving sites and channels in information networks, and using graph structures and substructural logic, was given by Barwise, Gabbay, and Hartonas [11]. We discuss a development inspired by this approach in Section 8.

In a second simplified theory, Barwise and Seligman developed the theory of classifications and infomorphisms, under the name ‘The Logic of Distributed Systems’. The standard introductory reference for this is the monograph [15]. We return to the technical development of these ideas when we introduce our corresponding inferentialist notion in Section 7.

Our development in the sequel is compatible with the four Principles of Information Flow [15]. Information flow requires some things to flow between, and this requires some conception of system that is distributed over various locations, and that has ‘regularities’ describing the informational linkage of the distributed components. Flow requires us to be able to identify the components of a system, say which are connected, and to reason about the (related, by regularity) informational properties of the various components. The level of detail of the model/analysis of the distributed system determines what are the regularities that can be identified.

4. LOGICAL BACKGROUND: PROOF-THEORETIC SEMANTICS

Recall that our purpose is to construct an inferentialist foundation for information, drawing upon situation theory, but grounded in proof-theoretic semantics rather than model-theoretic semantics. Accordingly, we now give a summary of the aspects of the theory of proof-theoretic semantics that we require.

Proof-theoretic semantics is a counterpart for model-theoretic semantics in which the meanings of proofs and formulæ are grounded in inference rather than truth (in the sense of Tarski). As we have remarked, can be seen as a logical realization of inferentialism [18, 19, 55]. See Schroeder-Heister’s [79] for a brief explanation.

We adopt closely the introduction to proof-theoretic semantics as presented in [71]. As in [71], our discussion, in Section 4.1, of proof-theoretic validity, which is concerned with the validity of proofs, is not strictly necessary for this paper. However, it provides valuable background for our discussion, in Section 4.2, of base-extension semantics, which is concerned with the validity of formulæ. Also, the final paragraph of Section 4.1 provides a brief discussion of the necessity for considering ‘base extensions’.

4.1. Proof-theoretic Validity. Our technical starting point is the logical realization of an inferentialist theory of meaning known as *proof-theoretic semantics* (P-tS). P-tS has its origins in the work of Prawitz [67, 68, 69, 70] and Dummett [32, 33]. Proof-theoretic semantics stands in contrast to model-theoretic semantics, in which the meaning of formulæ and of proofs is given denotationally [33].

We can usefully consider P-tS to have two main logical lines of development. The first, the earlier approach, is directly in the spirit of Prawitz and Dummett and is concerned with the validity of proofs. We refer to this as proof-theoretic validity (P-tV). The second line of development, *base-extension semantics* (B-eS), is concerned with the validity of formulæ.

This paper will focus on the second approach. To better motivate and understand B-eS, however, we find it helpful to place this in the context of P-tV. Readers familiar with P-tV may want to skip to Section 4.2 on B-eS.

Proof-theoretic validity is an attempt to answer the following seemingly naive question: how can the validity of a proof (or derivation) be judged? Before looking at the answer provided by P-tV, there is one answer already given by model-theoretic semantics. Following Schroeder-Heister [79], we remark that this view lies in the tradition of Tarskian semantics, which works through the transmission of truth, as follows: given a model (i.e., some semantic structure) \mathfrak{M} , $\Gamma \models_{\mathfrak{M}} \phi$ if, and only if, whenever $\models_{\mathfrak{M}} \psi$ holds for all $\psi \in \Gamma$, $\models_{\mathfrak{M}} \phi$ holds as well. The validity of a consequence is then given as follows:

$$\Gamma \models \phi \quad \text{iff} \quad \text{for all models } \mathfrak{M}, \Gamma \models_{\mathfrak{M}} \phi$$

In this setting, proofs can be understood denotationally; that is, as operations on the interpretation of formulæ in models, by which consequences are yielded; here, for example, an interpretation of a proof of ϕ from proofs of those ψ in Γ . However, such a model-theoretic definition of validity is unsatisfactory for answering the specific question one asks here about justification of validity: M-tS provides little insight into the structure of the formulæ involved in a valid sequence; it also presumes an epistemic priority of models over proofs which is itself ungrounded.

In proof-theoretic semantics (P-tS), one seeks to give meaning to propositions and to proofs purely in terms of inference. More specifically, meaning is given in terms of inferences in systems of pre-logical inference rules between atoms, called *atomic rules* [67]. A set of such rules is called a *base*. An atomic rule is one in which an atomic proposition is inferred from atomic propositions and which makes no reference to logical constants (connectives or other operators). For example, we have that if, say, Amber is a vixen, we can infer that it is a fox, and that it is female. In the standard format of proof rules, we have:

$$\frac{\text{Amber is a vixen}}{\text{Amber is a fox}} \quad \text{and} \quad \frac{\text{Amber is a vixen}}{\text{Amber is female}}$$

Similarly, if we have that Amber is a fox and that Amber is female, then we can infer that Amber is a vixen:

$$\frac{\text{Amber is female} \quad \text{Amber is a fox}}{\text{Amber is a vixen}}$$

In general, atomic rules have forms such as

$$\frac{}{p} \quad \frac{p_1 \dots p_k}{p} \quad \text{and} \quad \frac{[P_1] \quad \dots \quad [P_k]}{p}$$

and more (e.g., [65, 77, 75]). The latter allows each of the p_i s to be proved from dischargeable hypotheses P_i s [67, 76, 66].² Again, we emphasize that such atomic rules are *pre-logical* — they contain only atoms and do not refer to any logical constant. The use of the discharging of hypotheses in atomic rules is discussed in [76].

The propositional bases we have described can be generalized to bases of rules for atomic predicates [44, 51]. Such predicate bases remain pre-logical in that they make no reference to logical connectives. Predicate bases provide a key tool for our project of providing an inferential account of information using the logical tools of proof-theoretic semantics.

As we have seen, situation theory's account of information is grounded in the idea of infon of the form $\langle\langle R, a_1, \dots, a_n, \mathfrak{b} \rangle\rangle$ in which R is relation, the intended interpretation of a corresponding predicate symbol. This set-up is realized in terms of model-theoretic semantics, which provides the foundation for the development of the logic of infons: infons and situations are interpreted truth-functionally.

In the manner of proof-theoretic semantics, predicate bases provide an inferential alternative to this model-theoretic interpretation: we develop, in Section 5, an inferential counterpart to the concept of the infon, called 'inferon', in which predicate bases give meaning to relational assertions.

The starting point for this development will be bases defined over atoms of the form $\langle p, \mathfrak{b} \rangle$, where p is a proposition or predicate and \mathfrak{b} is a polarity. This will be explained in Section 5

²The choice of the form of atomic rules in bases has a profound effect on the strength of the semantics that is obtained [65, 77, 75].

P-tS in its modern form can be considered to begin with Appendix A of [67], ‘Validity of Derivations’, in which Prawitz discusses the validity of derivations in terms of bases. We refer to this topic as proof-theoretic validity (P-tV). In the remainder of this section, we give a very brief summary of P-tV in the sense of Prawitz [67, 78]. While this topic is an important part of the background to the present paper, it may be omitted by readers who wish to progress quickly to our core concerns. Note, however, that our final paragraph stresses an important point.

While there are a few variations of P-tV, we consider here a slightly generalized expression of the original presentation by Prawitz [67], and elegantly presented and discussed by Schroeder-Heister [78, 79]. The validity of proofs is reduced to so-called S -validity, where S is a base (i.e., a set of atomic rules), using the following auxiliary concepts:

- a set of *canonical derivations*, \mathbf{C} , typically derivations whose last step uses an introduction rule,
- a justification \mathbf{J} that maps proof-structures to a proof-structures.

Then, \mathcal{B} -validity can be defined as follows:

- All derivations formed using atomic rules from \mathcal{B} are \mathcal{B} -valid;
- A closed canonical derivation — that is, a canonical derivation that has no open assumptions — is \mathcal{B} -valid;
- A closed non-canonical derivation is \mathcal{B} -valid if \mathbf{J} reduces it to a canonical one;
- An open derivation $\mathcal{D}: \psi_1, \dots, \psi_k \triangleright \varphi$ (i.e., \mathcal{D} has open assumptions among ψ_1, \dots, ψ_k and conclusion φ) is \mathcal{B} -valid if, for any $\mathcal{B}' \supseteq \mathcal{B}$ and \mathcal{B}' -valid closed derivations \mathcal{D}_i s, when we close each open assumption ψ_i with \mathcal{D}_i , the resulting derivation is \mathcal{B}' -valid.

Validity relative to a base \mathcal{B} and a justification \mathbf{J} , $\langle \mathbf{J}, \mathcal{B} \rangle$ -validity is now defined as follows: a proof-structure Φ is $\langle \mathbf{J}, \mathcal{B} \rangle$ -valid — that is, represents a proof — if either $\Phi \in \mathbf{C}$ or if \mathbf{J} can be applied to Φ to yield an element of \mathbf{C} . This definition gives rise to a consequence relation as follows [79]:

Let $\langle \mathcal{J}, \mathcal{B} \rangle \Vdash \phi$ denote that \mathcal{J} generates a valid closed proof of ϕ relative to \mathcal{B} .

Then we obtain the following proof-theoretic notion of logical consequence:

$$\phi_1, \dots, \phi_k \Vdash \phi \quad \text{iff} \quad \text{there is a } \mathcal{J} \text{ s.t., for every } \mathcal{B} \text{ and all } \mathcal{J}_1, \dots, \mathcal{J}_k, \\ \text{if } \langle \mathcal{J}_1, \mathcal{B} \rangle \Vdash \phi_1, \dots, \langle \mathcal{J}_k, \mathcal{B} \rangle \Vdash \phi_k, \text{ then } \langle \mathcal{J}, \mathcal{B} \rangle \Vdash \phi$$

If \mathbf{J} specifies a reduction system — as with normalization in natural deduction or BHK-semantics, as described in [67, 79] — $\langle \mathbf{J}, \mathcal{B} \rangle$ -valid proofs can be defined inductively on their component structure.

Other choices of justification are, of course, possible and P-tV remains an active area of research.

Finally, we consider the role of arbitrary extensions \mathcal{B}' of the current atomic system \mathcal{B} in the \mathcal{B} -validity condition for open derivations. This is to prevent open derivations from being \mathcal{B} -valid simply due to that some of its open assumptions be *undervivable* in \mathcal{B} . It is helpful to read a base \mathcal{B} as some ‘epistemic stage’ (cf. Kripke’s semantics of implication): an open derivation is \mathcal{B} -valid if at any future point one can close the open assumption with closed derivations, then the resulting closed derivation is also valid. Such a potential completion at a future point justifies the need for extensions of the current base \mathcal{B} .³

4.2. Base-extension Semantics. While P-tV deals with the validity of proofs, it does so without considering directly the validity of formulæ (though, of course, validity of formulæ could be indirectly derived by saying $\Gamma \triangleright \varphi$ is valid precisely when there exists some valid derivation $\mathcal{D}: \Gamma \triangleright \varphi$). The second line of development, which we call *base-extension semantics* (B-eS), is concerned with the validity of formulæ (and sequents). Sandqvist [76] has given an elegant B-eS for intuitionistic propositional logic (IPL). This analysis demonstrates very clearly the basic principles of B-eS. Strong relationships between P-tV and B-eS in the setting of IPL have been discussed in [50].

The basic idea of such inductive definition of validity of formulæ can be articulated conveniently by comparison with the model-theoretic definition of the validity of formulæ. In the Kripke (and related) semantics of intuitionistic propositional logic based on ordered worlds, meaning is defined inductively as follows: the truth of an atomic formula at a world is determined by its interpretation in a model; that is, the atom is true at the world just in case the world is an element of the

³In fact, Prawitz [67] points out that the extensions considered can be restricted to those required by \mathbf{J} for the construction of ψ .

$$\frac{}{p} \quad \frac{p_1 \cdots p_n}{p} \quad \text{and} \quad \frac{\frac{[P_1]}{q_1} \cdots \frac{[P_n]}{q_n}}{r}$$

FIGURE 1. Base rules: atomic, level 1, and level 2, respectively

valuation of the atom. The meaning of the remaining connectives is then defined inductively, with the meaning of implication formulæ requiring, analogously to the requirement for base-extensions described above, judgements relative to worlds higher in the ordering. In fact, looking at the current world instead will incur the vacuous satisfaction problem similar to that in P-tV as mentioned at the end of Section 4.1: $\varphi \supset \psi$ could be true at a world w simply because φ is false at w (but not necessarily false at all worlds greater than w).

By contrast, base-extension semantics gives the meaning ($\Vdash_{\mathcal{B}}$) of atomic formulæ relative to a base \mathcal{B} in terms of provability ($\vdash_{\mathcal{B}}$) in \mathcal{B} , where the underlying set of atoms is assumed to be denumerably infinite:

$$\Vdash_{\mathcal{B}} p \quad \text{iff} \quad \vdash_{\mathcal{B}} p$$

The meanings of the connectives are then given inductively, as in Kripke's semantics.

Atomic derivability in a base \mathcal{B} of rules of the form given in Figure 1 is defined using the combinators (Ref) and (App), as follows:

$$\begin{aligned} (\text{Ref}) \quad & P, p \vdash_{\mathcal{B}} p \\ (\text{App}_{\mathcal{R}}) \quad & \text{if } \mathcal{R} \in \mathcal{B} \text{ and, for all } i \in [1, n], \text{ where } n \geq 0, Q, P_i \vdash_{\mathcal{B}} q_i, \text{ then } Q \vdash_{\mathcal{B}} r \end{aligned}$$

Note that the rule \mathcal{R} encompasses the axiom and hypothesis-free rules ($n = 0$ and P_i s empty, respectively). For brevity, the rule \mathcal{R} in Figure 1 is also written inline as

$$((P_1 \Rightarrow q_1), \dots, (P_n \Rightarrow q_n)) \Rightarrow r$$

There is a substitution (cut) operation on bases that maps derivations $P \vdash_{\mathcal{B}} p$ and $Q, p \vdash_{\mathcal{B}} q$ to a derivation $P, Q \vdash_{\mathcal{B}} q$ [76].

Before proceeding to the remainder of the discussion that we have sketched, we first recall and fix some notational conventions that we shall need throughout this section. These are compatible with Sandqvist [76], Gheorghiu [44], and Gheorghiu and Pym [50] and are summarized in Figure 2.

-
- p, q, r, \dots denote propositional letters
 - $\mathbf{b}, \mathbf{c}, \mathbf{d}, \dots$ denote polarities
 - x, y, z, \dots denote variables
 - a, b, c, \dots denote constants
 - f, g, h, \dots denote function symbols
 - t, s, r, \dots denote terms
 - P, Q, R, \dots denote either propositional letters or atomic predicates
 - ϕ, ψ, χ, \dots denote formulæ
 - $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \dots$ denote (possibly empty, possibly countably infinite) sets of atoms — propositional letters and atomic predicates (note this is a minor though wholly compatible variation on Sandqvist's notational scheme, as used in Section 4.2)
 - $\Gamma, \Delta, \Theta, \dots$ denote (possibly empty, possibly infinite) sets of formulæ
 - $\mathcal{B}, \mathcal{C}, \mathcal{D}, \dots$ denote atomic systems
 - $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \dots$ denote (possibly empty, possibly infinite) sets of inferonic atoms (see Definition 2)

The sets of all variables, constants, terms, propositional atoms, and formulæ are V, C, T, A , and F , respectively; the subsets containing only closed elements (where appropriate) are denoted $\text{Cl}(-)$. We write FV to denote the function that takes a term, atom, or formula and returns the set of its free variables. We write quantified formulæ as $\forall x.\phi$ and $\exists x.\phi$, etc., and let $[t/x]$ to denote a substitution function that replaces all free occurrences of x by t .

FIGURE 2. Notational conventions

We use \perp , \wedge , \vee , \supset , \forall , \exists as the logical signs. We do not regard \perp as an atomic formula; intuitively, following Dummett [33], it has logical structure that renders it logically/semantically complex even though it is indeed syntactically atomic. In this intuitionistic setting, we elide \top .

We write ‘Atoms’ for the (denumerably infinite) set of propositional and predicate atomic formulae in a specified context. Then the propositional and first-order formulae are given by the usual grammar:

$$\phi ::= P \mid \perp \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \supset \phi \mid \forall x.\phi \mid \exists x.\phi$$

where atomic predicates defined over terms $f(t_1, \dots, t_k)$ in the usual way. Terms and predicates are *closed* if they have no free variables.

The B-eS clause for disjunction can be given in the form that is familiar from Kripke semantics,

$$\Vdash_{\mathcal{B}} \phi_1 \vee \phi_2 \text{ iff } \Vdash_{\mathcal{B}} \phi_1 \text{ or } \Vdash_{\mathcal{B}} \phi_2$$

but such a choice leads to a failure of completeness [64, 66, 76].

(At)	$\Vdash_{\mathcal{B}} P$	iff	$\vdash_{\mathcal{B}} P$ for closed P
(\wedge)	$\Vdash_{\mathcal{B}} \phi_1 \wedge \phi_2$	iff	$\Vdash_{\mathcal{B}} \phi_1$ and $\Vdash_{\mathcal{B}} \phi_2$
(\vee)	$\Vdash_{\mathcal{B}} \phi_1 \vee \phi_2$	iff	for every closed P and every $\mathcal{C} \supseteq \mathcal{B}$, if $\phi_1 \Vdash_{\mathcal{C}} P$ and $\phi_2 \Vdash_{\mathcal{C}} P$, then $\Vdash_{\mathcal{C}} P$
(\supset)	$\Vdash_{\mathcal{B}} \phi_1 \supset \phi_2$	iff	$\phi_1 \Vdash_{\mathcal{B}} \phi_2$
(Inf)	for $\Theta \neq \emptyset$, $\Theta \Vdash_{\mathcal{B}} \phi$	iff	for every $\mathcal{C} \supseteq \mathcal{B}$, if $\Vdash_{\mathcal{C}} \psi$, for every $\psi \in \Theta$, then $\Vdash_{\mathcal{C}} \phi$
(\perp)	$\Vdash_{\mathcal{B}} \perp$	iff	for all closed P , $\Vdash_{\mathcal{B}} P$

(\forall)	$\Vdash_{\mathcal{B}} \forall x.\phi$	iff	$\Vdash_{\mathcal{B}} \phi[t/x]$ for all $t \in \text{Cl}(T)$
(\exists)	$\Vdash_{\mathcal{B}} \exists x.\phi$	iff	for any $\mathcal{C} \supseteq \mathcal{B}$ and any closed P , if $\phi[t/x] \Vdash_{\mathcal{C}} P$, for any $t \in \text{Cl}(T)$, then $\Vdash_{\mathcal{C}} P$

(Validity)	$\Gamma \Vdash \phi$	iff	for all \mathcal{B} , $\Gamma \Vdash_{\mathcal{B}} \phi$
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FIGURE 3. Sandqvist’s B-eS for intuitionistic propositional logic and Gheorghiu’s extension of it to first-order

While Figure 3, which gives the inductively defined semantics, looks similar to the Kripke semantics of IPL, there are some key points to note:

- The use of base-extension in the (Inf) clause reflects Prawitz’s argument for its necessity in giving meaning to implication, the same reasons underlying the consideration of extensions in P-tV (see the last paragraph of Section 4.1).
- Base-extension is transmitted to the implication connective \supset through its clause (\supset).
- The form of the clause for \perp may seem a little odd at first sight, but it gives the usual intuitionistic introduction and elimination rules for negation [69], defined as $\neg\phi = \phi \supset \perp$, as well as Ex Falso Quodlibet [33, 76, 75]. Note that the set of atoms (ps) is assumed to be denumerably infinite.
- The form of the clause for disjunction is critical: it is this form that allows the completeness theorem (q.v., below) — see, for example, [76, 71, 50] for explanations of this, which are beyond the scope of this brief introduction. In fact, the typical way of defining that a disjunction is true iff at least one of the disjuncts is true results in incompleteness [66].

Given all this, *validity* (denoted as \Vdash) is defined as support in all bases: $\Gamma \Vdash \phi$ iff, for all \mathcal{B} , $\Gamma \Vdash_{\mathcal{B}} \phi$. Note that we have transitivity for \Vdash as given in Figure 3: if $\Gamma \Vdash \phi$ and $\Gamma, \phi \Vdash \psi$, then $\Gamma \Vdash \psi$.

We write $\Gamma \vdash \phi$ to denote that ϕ is provable from Γ in the natural deduction systems NJ (cf. [76]). For reference, NJ, in sequential form, is given in Figure 4.

Sandqvist [76] establishes the soundness and completeness of the natural deduction calculus NJ [43, 69] with respect to this semantics. Gheorghiu [44] extends soundness and completeness to first-order (classical and) intuitionistic logic. In the notation of Figure 4, we obtain soundness and completeness for validity .

Theorem 1 (Sandqvist [76] and Gheorghiu [44]). $\Gamma \vdash \phi$ iff $\Gamma \Vdash \phi$. □

$$\begin{array}{c}
\frac{}{\Gamma, \phi \vdash \phi} (Ax) \qquad \frac{\Gamma \vdash \perp}{\Gamma \vdash \phi} \perp E \\
\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge I \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} \wedge E \quad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi} \wedge E \\
\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \vee I \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi} \vee I \quad \frac{\Gamma \vdash \phi \vee \psi \quad \Gamma, \phi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi} \vee E \\
\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi} \supset I \qquad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi \supset \psi}{\Gamma \vdash \psi} \supset E \\
\hline
\frac{\Gamma \vdash \phi(x)}{\Gamma \vdash \forall x. \phi(x)} \quad (x \text{ not free in } \Gamma) \quad \forall I \qquad \frac{\Gamma \vdash \forall x. \phi(x)}{\Gamma \vdash \phi[t/x]} \quad (t \text{ free for } x \text{ in } \phi) \quad \forall E \\
\frac{\Gamma \vdash \phi[t/x]}{\Gamma \vdash \exists x. \phi(x)} \quad (t \text{ free for } x \text{ in } \phi) \quad \exists I \qquad \frac{\Gamma \vdash \exists x. \phi(x) \quad \Gamma, \phi \vdash \psi}{\Gamma \vdash \psi} \quad (x \text{ not free in } \Gamma \text{ or } \psi) \quad \exists E
\end{array}$$

FIGURE 4. The calculus NJ in sequential form (eliding \top) [69, 84, 52]

It is base-extension semantics that provides the logical basis for the use of bases to provide the inferential alternative to infons that we call *inferons*. In this setting, developed in Section 5, we need to define base rules of atoms of the form $\langle p, \mathbf{b} \rangle$, where p is a propositional or predicate atom and \mathbf{b} is a boolean polarity.⁴

5. AN INFERENTIALIST LOGICAL BASIS FOR INFORMATION

With the background ideas and techniques in place, we now employ base-extension semantics to establish an inferentialist logic of information. Our set-up follows the base-extension semantics for intuitionistic predicate logic as described in Section 4.2. However, the key distinction is that atomic predicates are replaced by *inferons*, our inferentialist version of infons.

It should be noted that our choice to ground our account of information on an intuitionistic logic of inferons, building on the B-eS for intuitionistic logic [76, 44] as described above, is a wholly pragmatic one. It does not represent a commitment or suggestion that intuitionistic logic as presented. Rather, it is the simplest viable point of departure.

Borrowing some notation from situation theory, we take *inferons* to be of the form

$$\langle\langle R(t_1, \dots, t_n), \mathcal{B}, \mathbf{b} \rangle\rangle$$

This stands in contrast to form of *infons* taken by Barwise, Devlin et al. (see [28]), namely

$$\langle\langle R(t_1, \dots, t_n), \mathbf{b} \rangle\rangle$$

The latter gains meaning through the assumption of the availability of a notion of truth for predicates $R(t_1, \dots, t_n)$, which can be obtained only from a presumed model within which predicates can be interpreted.⁵

Our position is that we seek to replace such a presumption with an account of meaning that is inferential. In the present setting, we seek to achieve this through proof-theoretic semantics and, in particular, through base-extension semantics. In this view, as we have explained in Section 4.2, the meaning of propositions — specifically, their validity — is given by the idea of derivability in a base of atomic rules. So, an atomic predicate $R(t_1, \dots, t_n)$ is valid in a base \mathcal{B} iff it is derivable in \mathcal{B} :

$$\Vdash_{\mathcal{B}} R(t_1, \dots, t_n) \quad \text{iff} \quad \vdash_{\mathcal{B}} R(t_1, \dots, t_n)$$

⁴For now, we take polarities to be Boolean, but we conjecture that other choices are of substantive interest.

⁵Note that while, following situation theory, we retain the presence of a polarity \mathbf{b} within the structure of an inferon, it is not clearly that its presence is strictly essential, at least theoretically. However, their presences seems to allow us to avoid, to begin with, some delicate logical issues around negation. Furthermore, as we shall see in Section 8, for example, polarities are useful in the context of modelling systems inferentially. Moreover, both theoretically and for modelling purposes, the question of the possible value of richer types of polarity is intriguing.

So, the ambient, extensional notion of truth presumed in the infon [28] $\langle\langle R(t_1, \dots, t_n), \mathbf{b} \rangle\rangle$ is replaced by derivability in a base.

At this point, we have two options:

- we can presume a global base to be used to give meaning to all atomic predicates, which is analogous to the use of an implicit model to give meaning to infons, or
- we can associate a local base with an atomic predicate and so obtain our notion of inferon.

We choose the latter. Why? The former is of course possible, but, we would argue, simply not as useful or as convenient as an account of information.

It's really a design choice that, from the inferentialist perspective, it is natural for an assertion to be accompanied by an account — albeit a context-sensitive one — of from where it is intended to derive its meaning. Indeed, it might be argued that the absence of such an account is a weakness of the set-up of Barwise et al. [28]. While it is in principle possible to address this by associating with each assertion a model from which it acquires its meaning, we would argue that that would represent a large, perhaps unmanageable, overhead in modelling information-theoretic scenarios. By contrast, the inferentialist set-up can fix this weakness simply by associating a base, which, for practical modelling purposes can be considered essentially finite.

Furthermore, with this set-up, we can now naturally ask how the meaning of a predicate in this setting is affected by its context, as represented by a base \mathcal{C} ; that is,

$$\Vdash_{\mathcal{C}} \langle\langle P, \mathcal{B}, \mathbf{b} \rangle\rangle \quad \text{iff} \quad \vdash_{\mathcal{C} \cup \mathcal{B}} \langle\langle P, \mathbf{b} \rangle\rangle$$

in which the local and contextual bases are combined. Indeed, we would stress that inferons have no inherent meaning. In Dretske's terms, they are 'objective commodities' and meaning is associated with them only through their use in inferential contexts. The force of this is particularly clear if $\langle\langle P, \mathbf{b} \rangle\rangle$ is not derivable in \mathcal{B} , but is derivable with the addition of \mathcal{C} .

Moreover, this set-up is naturally aligned with a key concept in the metaphysics of information as we see it: that of agency. Agents — the literature is vast, we make no attempt here to review it, though [88] (see also [86]) provides a comprehensive, elegant, and insightful account of the state of the art — can naturally be taken to have inferential capacities. In our setting, we associate with an agent a set of bases $\mathbf{Bases}(a)$ representing its inferential capacities in various contexts — an agent may have different abilities in different contexts, such as different language environments — and capture this in the associated inferential semantics. Here ι denotes an inferon. So,

$$\Vdash_{\mathcal{C}} [a]\iota \quad \text{iff} \quad \text{for all } \mathcal{A} \in \mathbf{Bases}(a), \Vdash_{\mathcal{C} \cup \mathcal{A}} \iota$$

and

$$\Vdash_{\mathcal{C}} \langle a \rangle \iota \quad \text{iff} \quad \text{for some } \mathcal{A} \in \mathbf{Bases}(a), \Vdash_{\mathcal{C} \cup \mathcal{A}} \iota$$

and so on.

So, while it is possible to reduce the set-up to employ fixed global bases, it seems conceptually unnatural and technically inconvenient to do so — especially from a modelling perspective.

Definition 2 (Inferonic atoms). An *inferonic atom* is a pair $\langle\langle P, \mathbf{b} \rangle\rangle$, where P denotes a closed proposition (i.e., a propositional letter or a closed atomic predicate) and \mathbf{b} denotes a *polarity*, taking the values 0 or 1. \square

The polarities associated with propositions correspond to the 'assertions' and 'denials', in the sense of Rumfitt [74], as used by Gheorghiu and Buzoku [46] in giving a Sandqvist-style base-extension semantics for classical propositional logic.

In the base-extension semantics for intuitionistic propositional logic [76],⁶ bases are defined over propositional letters, p . In establishing a logic of inferons, we need to define bases over inferonic atoms, $\langle\langle P, \mathbf{b} \rangle\rangle$. We use \mathfrak{P} to denote sets of inferonic atoms. Figure 5 gives the form of inferonic base rules that we consider.

Definition 3 (Inferonic base). An *inferonic base* \mathcal{B} is a base of rules over a denumerable set of *inferonic atoms*, as given in Figure 5. \square

In using the logic we develop below — q.v. Figure 6 — as a tooling for modelling an information-theoretic phenomenon, we can think of inferonic bases as the *basic situation* describing the inherent inferential properties of the phenomenon. Then supported inferons $\langle P, \mathcal{P}, \mathbf{b} \rangle$ provide additional

⁶Indeed, also in the base-extension semantics for many other, though by no means all, logics.

inferential content. Later, we shall see how we can incorporate the local reasoning capabilities of agents through support relations for agent-modalities.

$$\begin{array}{c}
 \text{(1)} \quad \frac{}{\langle\langle P, \mathbf{b} \rangle\rangle} \qquad \text{(2)} \quad \frac{\langle\langle P_1, \mathbf{b}_1 \rangle\rangle \dots \langle\langle P_k, \mathbf{b}_k \rangle\rangle}{\langle\langle P, \mathbf{b} \rangle\rangle} \qquad \text{(3)} \quad \frac{\begin{array}{c} [\mathfrak{P}_1] \qquad \dots \qquad [\mathfrak{P}_k] \\ \langle\langle P_1, \mathbf{b}_1 \rangle\rangle \qquad \dots \qquad \langle\langle P_k, \mathbf{b}_k \rangle\rangle \end{array}}{\langle\langle P, \mathbf{b} \rangle\rangle}
 \end{array}$$

FIGURE 5. Inferonic base rules

Other forms of bases rules are also possible (e.g., [77]). It could be suggested that the presence of polarities in inferonic atoms constitutes a degree of ‘semantic pollution’ [72]. We would argue that our set-up lies within the scope of Avron’s criterion [5] for acceptability, that

‘... the framework should be independent of any particular semantics. One should not be able to guess, just from the form of the structures which are used, the intended semantics of a given proof-system’

Definition 4 (Inferonic basis). An *inferonic basis* \mathbf{B} is a denumerable set of inferonic bases. \square

Note that atoms may be propositional or predicates, following Gheorghiu’s set-up [44]. Note that we follow Sandqvist’s notational conventions [76].

Definition 5 (Inferon). An *inferon* is a triple $\iota = \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle$, where P is an atom — that is, a propositional letter or atomic predicate — \mathcal{P} is an inferonic base, and $\langle\langle P_1, \mathbf{b}_1 \rangle\rangle$ is an inferonic atom. We let ι range over inferons. $\langle\langle \phi, \mathcal{P}, \mathbf{b} \rangle\rangle$ is closed if P is closed. \square

An inferonic base, \mathcal{B} , of rules of the form given in Figure 5, generates a basic derivability relation $\vdash_{\mathcal{B}}$. This relates a finite set of inferonic atoms, \mathfrak{P} to a single inferonic atom:

$$\mathfrak{P} \vdash_{\mathcal{B}} \langle\langle P, \mathbf{b} \rangle\rangle$$

Definition 6 (A logic of inferons). For a given inferonic base \mathcal{B} , a *logic of inferons* is given by a support relation for validity. Figure 3 gives the support relation for logics or inferons corresponding, when the polarities are all 1 to intuitionistic propositional and predicate logics). \square

The logical connectives defined by the support relation in Figure 6 construct compound inferons — henceforth called just inferons — from basic inferons.

Definition 7 (Inferential situation). An *inferential situation* is then a set of inferons and support for an inferon by an inferential situation is given by the support relation given in Figure 6. \square

In the sequel, where no confusion is likely, we shall frequently contract ‘inferential situation’ to just ‘situation’.

Definition 8 (Derivations in a base). For inferonic base rules of the form given in Figure 5, here for convenience written as

$$\frac{\begin{array}{c} [\mathfrak{P}_1] \qquad \dots \qquad [\mathfrak{P}_k] \\ \langle\langle Q_1, \mathbf{b}_1 \rangle\rangle \qquad \dots \qquad \langle\langle Q_k, \mathbf{b}_k \rangle\rangle \end{array}}{\langle\langle R, \mathbf{c} \rangle\rangle} \mathcal{R}$$

where \mathfrak{P} s denote sets of inferonic atoms, derivations are constructed using the following combinators:

$$\begin{array}{l}
 \text{(Ref)} \quad \mathfrak{P}, \langle\langle P, \mathbf{b} \rangle\rangle \vdash_{\mathcal{B}} \langle\langle P, \mathbf{b} \rangle\rangle \\
 \text{(App}_{\mathcal{R}}) \quad \text{if } \mathcal{R} \in \mathcal{B} \text{ and, for all } i \in [1, n], \mathcal{Q}, \mathfrak{P}_i \vdash_{\mathcal{B}} \langle\langle Q_i, \mathbf{b}_i \rangle\rangle, \text{ then } \mathcal{Q} \vdash_{\mathcal{B}} \langle\langle R, \mathbf{c} \rangle\rangle
 \end{array}$$

\square

As in the purely propositional setting, there is a substitution (cut) operation on base derivations.

Definition 9 (Consistent base). A base \mathcal{B} is *consistent* if, for all P in the given language, it is not the case that both $\vdash_{\mathcal{B}} \langle\langle P, \mathbf{o} \rangle\rangle$ and $\vdash_{\mathcal{B}} \langle\langle P, \mathbf{1} \rangle\rangle$. \square

Many different logics of inferons can be defined using support relations that build upon inferonic As we have remarked, inferonic bases over sets/algebras of polarities other than just \mathbf{o} and $\mathbf{1}$ can of course also be considered. We begin with an intuitionistic logic of inferons — similar to Sandqvist’s treatment of IPL [76], with the support relation as given in Figure 3.⁷

Later, we extend this to first-order following Gheorghiu [44].

(At)	$\Vdash_{\mathcal{B}} \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle$	iff	$\vdash_{\mathcal{B} \cup \mathcal{P}} \langle\langle P, \mathbf{b} \rangle\rangle$, for atomic propositions P
(\wedge)	$\Vdash_{\mathcal{B}} \phi_1 \wedge \phi_2$	iff	$\Vdash_{\mathcal{B}} \phi_1$ and $\Vdash_{\mathcal{B}} \phi_2$
(\vee)	$\Vdash_{\mathcal{B}} \phi_1 \vee \phi_2$	iff	for every inferon ι and every $\mathcal{C} \supseteq \mathcal{B}$, if $\phi_1 \Vdash_{\mathcal{C}} \iota$ and $\phi_2 \Vdash_{\mathcal{C}} \iota$, then $\Vdash_{\mathcal{C}} \iota$
(\supset)	$\Vdash_{\mathcal{B}} \phi_1 \supset \phi_2$	iff	$\phi_1 \Vdash_{\mathcal{B}} \phi_2$
(Inf)	for $\Theta \neq \emptyset$, $\Theta \Vdash_{\mathcal{B}} \phi$	iff	for every $\mathcal{C} \supseteq \mathcal{B}$, if $\Vdash_{\mathcal{C}} \psi$, for every $\psi \in \Theta$, then $\Vdash_{\mathcal{C}} \phi$
(\perp)	$\Vdash_{\mathcal{B}} \perp$	iff	for all closed ι , $\Vdash_{\mathcal{B}} \iota$

(Validity) $\Gamma \Vdash \phi$ iff for all \mathcal{B} , $\Gamma \Vdash_{\mathcal{B}} \phi$

FIGURE 6. Support relation for inferons: propositional case

The base-extension semantics presented here very closely corresponds that for intuitionistic logic, the difference being the inferonic structure of propositions. For the soundness and completeness of this theory of inferons, we consider the derivability relation given by NJ with inferons as atomic formulæ and with the (Inferon) axiom, as given in Figure 7. Note that \Vdash is transitive: that is, if $\Gamma \Vdash \phi$ and $\Gamma, \phi \Vdash \psi$, then $\Gamma \Vdash \psi$.

(Inferon)	$\frac{}{\Gamma \Vdash \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle}$	iff	for all $\mathcal{P}' \supseteq \mathcal{P}$ s.t., for all $\phi \in \Gamma$, if $\Vdash_{\mathcal{P}'} \phi$, then $\vdash_{\mathcal{P}'} \langle\langle P, \mathbf{b} \rangle\rangle$
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FIGURE 7. The axiom (Inferon) for the theory of inferons

The (Inferon) axiom, as given in Figure 7, is a generalization, in the atomic case of the principal formula, of the (Ax) in NJ. Its purpose is to axiomatize the theory of inferons. In the case in which Γ is empty, it reduces, as one would expect, to

$$\frac{}{\vdash \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle} \text{ iff } \vdash_{\mathcal{P}} \langle\langle P, \mathbf{b} \rangle\rangle$$

which simply enforces the coherence of NJ and base proof for inferons.

The (Inferon) axiom also provides a clear illustration of how — not only in B-eS but also in P-tV — bases replace models in the giving of meaning to formulæ and proofs. Specifically, an appeal to the truth of an atomic formula in a model is replaced by the requirement of provability in a base. We remark that it would be interesting to compare and contrast Mares’s model-theoretic semantics, expressed in terms of ‘info models’ [61], and Restall’s analysis [73] with our B-eS for our logic of inferons.

The need for the richer form of (Inferon) arises technically from the presence of local bases within inferons — it handles the effect on the support relation that is induced via the axiom rule.

Remark 10 (Veridicality). It might be asked whether our choice to follow situation theory in employing both the assertion and denial of inferons, through the parameter \mathbf{b} , entails a violation of veridicality [40, 42] — that is, the principle that information must be true. We would argue that it does not. In inferon logic, veridicality is respected at the level of the support relation — the support of a denial is determined to hold. That said, we observe that it would seem to be valuable to consider in more detail the relationship between validity and assertion/denial.

⁷The same support relation gives both intuitionistic and classical logics of inferons. Classical logic is obtained if bases are restricted to rules of types (1) and (2) and intuitionistic logic is obtained if rules of type (3) are permitted.

For the propositional logic of inferons, as given by the propositional support relation and the (Inferon) axiom with propositional atoms, we can adapt Sandqvist's proofs of soundness and completeness [76].

Theorem 11 (Soundness for the propositional logic of inferons). *If $\Gamma \vdash \phi$, then $\Gamma \Vdash \phi$.*

Proof. We extend the proof in [76] by the case for the (Inferon) rule. The proof goes by showing that a translation of the rules of \vdash into the support relation still hold. For example, here are the translations of the rules for implication:

$$\begin{aligned} (\supset I)' & \text{ If } \Gamma, \phi \Vdash \psi, \text{ then } \Gamma \Vdash \phi \supset \psi \\ (\supset E)' & \text{ If } \Gamma \Vdash \phi \supset \psi \text{ and } \Gamma \Vdash \phi, \text{ then } \Gamma \Vdash \psi \end{aligned}$$

For $(\supset I)'$, assume that $\Gamma, \phi \Vdash \psi$. Take some \mathcal{B} s.t. $\Vdash_{\mathcal{B}} \chi$ for every $\chi \in \Gamma$. For every $\mathcal{C} \supseteq \mathcal{B}$ s.t. $\Vdash_{\mathcal{C}} \phi$ we have $\Vdash_{\mathcal{C}} \psi$. So, we have $\phi \Vdash_{\mathcal{B}} \psi$ and $\Vdash_{\mathcal{B}} \phi \supset \psi$. From this we can conclude $\Gamma \Vdash \phi \supset \psi$.

For $(\supset E)'$, note, by (Inf), that if $\phi \Vdash_{\mathcal{B}} \psi$ and $\Vdash_{\mathcal{B}} \phi$, then $\Vdash_{\mathcal{B}} \psi$. By (\supset) (in Figure 6), $\phi \Vdash_{\mathcal{B}} \psi$ iff $\Vdash_{\mathcal{B}} \phi \supset \psi$ and so, if $\Vdash_{\mathcal{B}} \phi \supset \psi$ and $\Vdash_{\mathcal{B}} \phi$, then $\Vdash_{\mathcal{B}} \psi$. So, $\phi \supset \psi, \phi \Vdash \psi$. By transitivity, for every Γ s.t. $\Gamma \Vdash \phi \supset \psi$ and $\Gamma \Vdash \phi$, we have $\Gamma \Vdash \psi$.

For our proof here, we have added only the (Inferon) rule and so it suffices to show that

$$(\text{Inferon})' \quad \Gamma \Vdash \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle \quad \text{iff} \quad \text{for all } \mathcal{P}' \supseteq \mathcal{P} \text{ s.t., for all } \phi \in \Gamma, \text{ if } \Vdash_{\mathcal{P}'} \phi, \text{ then } \vdash_{\mathcal{P}'} \langle\langle P, \mathbf{b} \rangle\rangle$$

where $(\text{Inferon})'$ is, following Sandqvist's presentation [76], the translation of (Inferon) into the support relation.

By validity and (Inf), $\Gamma \Vdash \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle$ iff for all \mathcal{B} and $\mathcal{B}' \supseteq \mathcal{B}$ s.t., for all $\phi \in \Gamma$, if $\Vdash_{\mathcal{B}'} \phi$, $\Vdash_{\mathcal{B}} \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle$. By (At), $\Vdash_{\mathcal{B}} \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle$ iff $\Vdash_{\mathcal{B} \cup \mathcal{P}} \langle\langle P, \mathbf{b} \rangle\rangle$. It suffices to point out that for every \mathcal{P}' , there is a \mathcal{B} s.t. $\mathcal{B} \cup \mathcal{P} = \mathcal{P}'$ and vice-versa. \square

Theorem 12 (Completeness for the propositional logic of inferons). *If $\Gamma \Vdash \phi$, then $\Gamma \vdash \phi$.*

Proof. Again, we adapt the proof of [76], with some important differences.

The basic strategy is to create a base \mathcal{N} so that $\vdash_{\mathcal{N}}$ behaves like \vdash . This is done by encoding the rules of \vdash into base rules in \mathcal{N} with the help of a flattening function that assigns an atomic formula to each formula. We then show two properties of \mathcal{N} ((\dagger) and (\ddagger) below) that, together with the property $\mathbb{T} \vdash_{\mathcal{B}} p$ iff $\mathbb{T} \Vdash_{\mathcal{B}} p$ (Theorem 3.1 in [76]), suffice to prove the completeness result. Our proof here follows that strategy but differs in some key aspects. First, we extend the base-support relation to formulae of the form $\langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}}$ such that $\langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}}$ is supported at a base \mathcal{B} if and only if $\langle\langle P, \mathbf{b} \rangle\rangle$ is supported at $\mathcal{B} \cup \mathcal{P}$. We require two flattening functions: one that assigns every formula an inferon and another that assigns them a formula of this new type. We then prove (3) below which takes the role of Theorem 3.1 in the proof. Given that, (\dagger) and (\ddagger) follow as before and we obtain the wanted result.

Let Γ' be the set containing all members of $\Gamma \cup \phi$ and their subsentences. We largely retain the flattening function ϕ^b so that for any $\phi \in \Gamma$ we associate a $\phi^b \notin \Gamma$ s.t. for every $\phi, \psi \in \Gamma$ $\phi^b \neq \psi^b$ whenever $\phi \neq \psi$ with $\langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle$ taking the role of the atom and so, $\langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle^b = \langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle$. Our one change to ϕ^b is that we take $\perp^b = \perp$. As before for a set of formulae Γ , let Γ^b be the sequence obtained by replacing any $\phi \in \Gamma$ with ϕ^b .

For technical convenience in adapting Sandqvist's proof, we extend the base-support relation to a new type of inferonic atom of the form $\langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}}$ as follows:

$$\mathfrak{P} \vdash_{\mathcal{B}} \langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}} \quad \text{iff} \quad \mathfrak{P} \vdash_{\mathcal{B} \cup \mathcal{P}} \langle\langle P, \mathbf{b} \rangle\rangle$$

With this device we can define a further flattening function ϕ^{b-} just like ϕ^b expect to inferonic atoms so that $\langle\langle P, \mathcal{P}, \mathbf{b} \rangle\rangle^{b-} = \langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}}$. Again, let Γ^{b-} be Γ with all ϕ replaced by ϕ^{b-} . We also define the functions ϕ^{\ddagger} as in [76] and $\phi^{\ddagger-}$ accordingly, that is so that for any $\phi \in \Gamma'$, $(\phi^b)^{\ddagger} = \phi$ and $(\phi^{b-})^{\ddagger-} = \phi$.

We prove the following property of $\langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}}$ formulae as it will be required in our proof. It is analogous to Lemma 2.2 in [76]: $\mathbb{T} \vdash_{\mathcal{B}} p$ iff for every $\mathcal{B}' \supseteq \mathcal{B}$, if $\vdash_{\mathcal{B}'} p$ for every $p \in \mathbb{T}$, then $\vdash_{\mathcal{B}'} p$.

$$(1) \quad \mathfrak{P} \vdash_{\mathcal{B}} \langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}} \quad \text{iff} \quad \text{for every } \mathcal{B}' \supseteq \mathcal{B}, \text{ if } \vdash_{\mathcal{B}'} \phi \text{ for every } \phi \in \mathfrak{P}, \text{ then } \vdash_{\mathcal{B}'} \langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}}$$

Note, by definition, $\vdash_{\mathcal{B}'} \langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}}$ iff $\vdash_{\mathcal{B}' \cup \mathcal{P}} \langle\langle P, \mathbf{b} \rangle\rangle$ and $\mathfrak{P} \vdash_{\mathcal{B}} \langle\langle P, \mathbf{b} \rangle\rangle_{\mathcal{P}}$ iff $\mathfrak{P} \vdash_{\mathcal{B} \cup \mathcal{P}} \langle\langle P, \mathbf{b} \rangle\rangle$. By a straightforward generalization of Lemma 2.2 above, $\mathfrak{P} \vdash_{\mathcal{B} \cup \mathcal{P}} \langle\langle P, \mathbf{b} \rangle\rangle$ iff for every $\mathcal{C} \supseteq \mathcal{B} \cup \mathcal{P}$, if $\vdash_{\mathcal{C}} \phi$ for every $\phi \in \mathfrak{P}$, then $\vdash_{\mathcal{C}} \langle\langle P, \mathbf{b} \rangle\rangle$. So it suffices to show that for every $\mathcal{B}' \supseteq \mathcal{B}$, if $\vdash_{\mathcal{B}'} \phi$ for every

$\phi \in \mathfrak{P}$, then $\vdash_{\mathcal{B}' \cup \mathcal{P}} \langle\langle P, \mathfrak{b} \rangle\rangle$ iff for every $\mathcal{C} \supseteq \mathcal{B} \cup \mathcal{P}$, if $\vdash_{\mathcal{C}} \phi$ for every $\phi \in \mathfrak{P}$, then $\vdash_{\mathcal{C}} \langle\langle P, \mathfrak{b} \rangle\rangle$. For the left-to-right direction, it suffices to point out that $\mathcal{C} \supseteq \mathcal{B}'$ and that $\mathcal{C} \cup \mathcal{P} = \mathcal{C}$. Right-to-left follows because if $\vdash_{\mathcal{B}'} \phi$, then also $\vdash_{\mathcal{B}' \cup \mathcal{P}} \phi$ and $\mathcal{B}' \cup \mathcal{P} \supseteq \mathcal{B} \cup \mathcal{P}$.

In [76, 44], this completeness proof is reliant on the fact that the following holds for atomic \mathfrak{p} in their semantics: $\top \vdash_{\mathcal{B}} \mathfrak{p}$ iff $\top \Vdash_{\mathcal{B}} \mathfrak{p}$. Note that such a result cannot hold for this semantics as $\vdash_{\mathcal{B}}$ deals with inferonic atoms and $\Vdash_{\mathcal{B}}$ with inferons.

Our extension of the support relation and the new flattening function ϕ^{b-} allow us to prove an analogous result sufficient to give completeness:

$$(2) \quad \Gamma^{b-} \vdash_{\mathcal{B}} \langle\langle P, \mathfrak{b} \rangle\rangle_{\mathcal{P}} \quad \text{iff} \quad \Gamma^b \Vdash_{\mathcal{B}} \langle\langle P, \mathcal{P}, \mathfrak{b} \rangle\rangle$$

For empty Γ , this follows directly from the definition of $\langle\langle P, \mathfrak{b} \rangle\rangle_{\mathcal{P}}$ and (At). Otherwise this follows from those together with (Inf) and (1). Since for any $\phi^b = \langle\langle P, \mathcal{P}, \mathfrak{b} \rangle\rangle$ we have $\phi^{b-} = \langle\langle P, \mathfrak{b} \rangle\rangle_{\mathcal{P}}$, we also get the following:

$$(3) \quad \Gamma^{b-} \vdash_{\mathcal{B}} \phi^{b-} \quad \text{iff} \quad \Gamma^b \Vdash_{\mathcal{B}} \phi^b$$

The rest of the proof remains unchanged except uses of (3) are substituted for any uses of $\mathfrak{P} \vdash_{\mathcal{B}} \mathfrak{p}$ iff $\mathfrak{P} \Vdash_{\mathcal{B}} \mathfrak{p}$ and ϕ^{b-} replaces ϕ^b on the level of base-support.

The strategy involves defining a base \mathcal{N} that mimics the rules of natural deduction. \mathcal{N} is defined as the base containing all and only the rules of the following form:

1. $(\phi^{b-} \Rightarrow \psi^{b-}) \Rightarrow (\phi \supset \psi)^{b-}$
2. $(\Rightarrow (\phi \supset \psi)^{b-}), (\Rightarrow \phi^{b-}) \Rightarrow \psi^{b-}$
3. $(\Rightarrow \phi^{b-}), (\Rightarrow \psi^{b-}) \Rightarrow (\phi \wedge \psi)^{b-}$
4. $(\Rightarrow (\phi \wedge \psi)^{b-}) \Rightarrow \phi^{b-}$
5. $(\Rightarrow (\phi \wedge \psi)^{b-}) \Rightarrow \psi^{b-}$
6. $(\Rightarrow \phi^{b-}) \Rightarrow (\phi \vee \psi)^{b-}$
7. $(\Rightarrow \psi^{b-}) \Rightarrow (\phi \vee \psi)^{b-}$
8. $(\Rightarrow (\phi \vee \psi)^{b-}), (\phi^{b-} \Rightarrow \langle\langle P, \mathfrak{b} \rangle\rangle), (\psi^{b-} \Rightarrow \langle\langle P, \mathfrak{b} \rangle\rangle) \Rightarrow \langle\langle P, \mathfrak{b} \rangle\rangle$
9. $(\Rightarrow \perp^{b-}) \Rightarrow \langle\langle P, \mathfrak{b} \rangle\rangle$

With these changes, the following are proved as in [76]:

- (†) for every $\phi \in \Gamma'$ and every $\mathcal{B} \supseteq \mathcal{N}$, $\Vdash_{\mathcal{B}} \phi^b$ iff $\Vdash_{\mathcal{B}} \phi$
- (‡) for any \mathfrak{P} and $\langle\langle P, \mathfrak{b} \rangle\rangle_{\mathcal{P}}$ if $\mathfrak{P} \vdash_{\mathcal{N}} \langle\langle P, \mathfrak{b} \rangle\rangle_{\mathcal{P}}$ then $\mathfrak{P}^{\sharp} \vdash (\langle\langle P, \mathfrak{b} \rangle\rangle_{\mathcal{P}})^{\sharp}$

The final result follows from (†), (‡), and (3): Assume $\Gamma \Vdash \phi$. For every $\mathcal{B} \supseteq \mathcal{N}$ such that $\Vdash_{\mathcal{B}} \psi^b$ for every $\psi^b \in \Gamma^b$, it follows by (†) that $\Vdash_{\mathcal{B}} \psi$ for every $\psi \in \Gamma$. So, since $\Gamma \Vdash \phi$, $\Vdash_{\mathcal{B}} \phi$ and, again by (†), $\Vdash_{\mathcal{B}} \phi^b$. It follows that $\Gamma^b \Vdash \phi^b$. By (3), we get $\Gamma^{b-} \vdash_{\mathcal{N}} \phi^{b-}$ and so, by (‡), we can conclude $\Gamma \vdash \phi$. \square

For the first-order logic of inferons, we extend the support relation as in Figure 8 (building on [44]), and of course consider the (Inferon) axiom with closed atomic predicates.

(At) $\Vdash_{\mathcal{B}} \langle\langle P, \mathfrak{b} \rangle\rangle$ iff $\vdash_{\mathcal{B} \cup \mathcal{P}} \langle\langle P, \mathfrak{b} \rangle\rangle$ for closed P
.....
(∀) $\Vdash_{\mathcal{B}} \forall x. \phi$ iff $\Vdash_{\mathcal{B}} \phi[t/x]$ for all $t \in \text{Cl}(T)$
(∃) $\Vdash_{\mathcal{B}} \exists x. \phi$ iff for any $\mathcal{C} \supseteq \mathcal{B}$ and all closed ι , if $\phi[t/x] \Vdash_{\mathcal{C}} \iota$, for all $t \in \text{Cl}(T)$, then $\Vdash_{\mathcal{C}} \iota$

FIGURE 8. Support relation for inferons: first-order case

Theorem 11 (soundness) and Theorem 12 (completeness) can be extended to the first-order case (see [44]), with the same adaptations to handle the atomic cases.

As [44] uses a Hilbert-type axiomatic proof-system, this extension is not completely trivial. However, our treatment of quantifiers in the semantics is equivalent to that in [44] and so the differences between the two semantics reside only in the treatment of the atomic cases, which we have treated and shown to be unproblematic in Theorem 11 and Theorem 12.

Our basic notion of inferon — what is intended when we refer to inferons — is atomic in that the propositional component is taken to be a closed atom. As presented, the support relation supports logical combinations of *inferons*. For example,

$$\langle\langle P, \mathcal{P}, \mathfrak{b} \rangle\rangle \wedge \langle\langle Q, \mathcal{Q}, \mathfrak{c} \rangle\rangle$$

However, we can allow connectives to occur within inferons provided the support relation preserves the associated base and polarity. For example, we can meaningfully write

$$\Vdash_{\mathcal{B}} \langle\langle P_1 \wedge P_2, \mathcal{P}, \mathfrak{b} \rangle\rangle \quad \text{iff} \quad \Vdash_{\mathcal{B}} \langle\langle P_1, \mathcal{P}, \mathfrak{b} \rangle\rangle \quad \text{and} \quad \Vdash_{\mathcal{B}} \langle\langle P_2, \mathcal{P}, \mathfrak{b} \rangle\rangle$$

and so construct *compound inferons*. Thus the support relation then allows us to extend the inferon notation to incorporate the connectives, as in Figure 9.

$$\begin{array}{ll} \Vdash_{\mathcal{B}} \langle\langle \phi_1 \wedge \phi_2, \mathcal{P}, \mathfrak{1} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \langle\langle \phi_1, \mathcal{P}, \mathfrak{1} \rangle\rangle \wedge \langle\langle \phi_2, \mathcal{P}, \mathfrak{1} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \phi_1 \wedge \phi_2, \mathcal{P}, \mathfrak{o} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \langle\langle \phi_1, \mathcal{P}, \mathfrak{o} \rangle\rangle \vee \langle\langle \phi_2, \mathcal{P}, \mathfrak{o} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \phi_1 \vee \phi_2, \mathcal{P}, \mathfrak{1} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \langle\langle \phi_1, \mathcal{P}, \mathfrak{1} \rangle\rangle \vee \langle\langle \phi_2, \mathcal{P}, \mathfrak{1} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \phi_1 \vee \phi_2, \mathcal{P}, \mathfrak{o} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \langle\langle \phi_1, \mathcal{P}, \mathfrak{o} \rangle\rangle \wedge \langle\langle \phi_2, \mathcal{P}, \mathfrak{o} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \phi_1 \supset \phi_2, \mathcal{P}, \mathfrak{1} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \langle\langle \phi_1, \mathcal{P}, \mathfrak{1} \rangle\rangle \supset \langle\langle \phi_2, \mathcal{P}, \mathfrak{1} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \phi_1 \supset \phi_2, \mathcal{P}, \mathfrak{o} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \langle\langle \phi_1, \mathcal{P}, \mathfrak{1} \rangle\rangle \wedge \langle\langle \phi_2, \mathcal{P}, \mathfrak{o} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \forall x.\phi, \mathcal{P}, \mathfrak{1} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \forall x.\langle\langle \phi, \mathcal{P}, \mathfrak{1} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \forall x.\phi, \mathcal{P}, \mathfrak{o} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \exists x.\langle\langle \phi, \mathcal{P}, \mathfrak{o} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \exists x.\phi, \mathcal{P}, \mathfrak{1} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \exists x.\langle\langle \phi, \mathcal{P}, \mathfrak{1} \rangle\rangle \\ \Vdash_{\mathcal{B}} \langle\langle \exists x.\phi, \mathcal{P}, \mathfrak{o} \rangle\rangle & \text{iff} \quad \Vdash_{\mathcal{B}} \forall x.\langle\langle \phi, \mathcal{P}, \mathfrak{o} \rangle\rangle \end{array}$$

FIGURE 9. Compound inferons, with internal logical connectives

Lemma 13. For any compound inferon $\langle\langle \phi, \mathcal{P}, \mathfrak{b} \rangle\rangle$, there exists a non-compound inferon ψ such that, for any \mathcal{B} ,

$$\Vdash_{\mathcal{B}} \langle\langle \phi, \mathcal{P}, \mathfrak{b} \rangle\rangle \quad \text{iff} \quad \Vdash_{\mathcal{B}} \psi$$

Proof. This follows from a simple induction on the complexity of ϕ using the extension of the support relation in Figure 9. \square

So far, we have considered just Boolean polarities in inferons, just as in classical situation theory. This is closely related to Rumfitt's conception of assertion and denial [74], which is also employed by Gheorghiu and Buzoku [46] in their treatment of base-extension semantics for classical propositional logic. Here, $\Vdash_{\mathcal{B}} \langle\langle \phi, \mathcal{P}, \mathfrak{1} \rangle\rangle$ expresses that the property denoted by ϕ is, given its associated \mathcal{B} , supported by \mathcal{P} , whereas $\Vdash_{\mathcal{B}} \langle\langle \phi, \mathcal{P}, \mathfrak{o} \rangle\rangle$ expresses that, given \mathcal{B} , the property denoted by ϕ is not supported by \mathcal{P} . It seems clear that richer notions of polarity will be of interest, especially in the context of richer logics.

Finally, we remark that presentation of the given base-extension semantics for first-order readily generalizes to second-order logic [51], offering the possibility of a more expressive logic of inferons.

6. FIRST EXAMPLES

Below are some examples that indicate the use of inferons and situations in modelling informational scenarios.

Example 14 shows how attunement to a constraint (aligning with the meaning of those words from situation theory) can, in some cases, be formalized as an inference rule linking inferonic atoms. When immersed in an inferential context supplying the premise of the rule, the rule can be used to infer the conclusion of the rule, as the antecedent of the constraint. This takes (sets of) inferonic atoms to characterize situation types. Moreover, the example shows the usefulness of the formulation of the At rule of Figure 6 in decomposing the inferential bases into a contextual part and a part associated to a particular inferon. Example 15 shows how attunement can then be captured by the $\langle a \rangle$ modality.

Example 16 shows how agents can resolve epistemic puzzles using fixed inferential capability. Specifically, they do not need unbounded capability to consider possible states of the world, given public announcements. Justification of the base rules used by agents does require knowledge of

the bases used by other agents. Alternative, modal-relational approaches to epistemic logic are discussed elsewhere by Eckhardt and Pym [37, 36].

Example 14 (No smoke without fire). We adapt a classic example discussed in situation theory [28]. Smoke is observed arising from a mountaintop, and an observer some distance away concludes that there is fire on the mountaintop. The observer cannot observe the fire directly, but is ‘attuned’ to a constraint $\text{Smoke} \Rightarrow \text{Fire}$ that says that smokey-type situations entail firey-type situations.

In our set-up, we shall be concerned with — in addition to an ambient inferonic base \mathcal{B} containing the background description of the scenario, including the constraint — two other inferonic bases. The first base, \mathcal{P}_1 , describes immediate perceptions about the local frame of the observer. The second base, \mathcal{P}_2 , contains facts about the remote frame (the mountaintop). We consider atomic sentence P_1 to be an abbreviation for ‘smoke is observed in the local frame’ and P_2 to mean ‘fire is present on the mountaintop’. Here, the base \mathcal{B} contains a single rule

$$(4) \quad \frac{\langle\langle P_1, \mathbf{1} \rangle\rangle}{\langle\langle P_2, \mathbf{1} \rangle\rangle}$$

Suppose, moreover, that

$$\overline{\langle\langle P_1, \mathbf{1} \rangle\rangle}$$

is the only rule contained in base \mathcal{P}_1 .

Suppose that the observer uses base $\mathcal{B} \cup \mathcal{P}_1$, because they are attuned to the constraint and have observed smoke. It is then the case that we can derive $\langle\langle P_2, \mathbf{1} \rangle\rangle$ in the base $\mathcal{B} \cup \mathcal{P}_1$, and therefore $\Vdash_{\mathcal{B}} \langle\langle P_2, \mathcal{P}_1, \mathbf{1} \rangle\rangle$.

The key point here is that a reasoner with access to only \mathcal{P}_1 and \mathcal{B} has $\Vdash_{\mathcal{B}} \langle\langle P_2, \mathcal{P}_1, \mathbf{1} \rangle\rangle$, even without direct access to the reasoning resources, including basic facts (coded as atoms), of \mathcal{P}_2 . The At rule allows for the decomposition between rules specific to the context \mathcal{P}_1 , and rules formalizing context-independent facts, such as the rule for the constraint (4), within \mathcal{B} . \square

Example 15 (Inferential capability via modality). In Example 14, the availability of the rule (4), the constraint linking inferonic bases, was central to the observer drawing the conclusion of interest. Different agents may have different capabilities, say, because they are attuned to different constraints. The agent modalities, $\langle a \rangle$ and $[a]$, introduced in Section 5 are supported using decomposition of a base into the part used by a specific, named, agent, and an ambient part. This allows for logical expressions that capture different agent capabilities.

We continue with the set-up from Example 14. Suppose that we have an agent a that can reason with the inferonic base \mathcal{B} , so that $\mathcal{B} \in \mathbf{Bases}(a)$. In that case, $\Vdash_{\emptyset} \langle a \rangle \langle\langle P_2, \mathcal{P}_1, \mathbf{1} \rangle\rangle$ is an instance of the support relation.

Suppose another agent a' with, say, $\mathbf{Bases}(a') = \{\mathcal{B}'\}$, where \mathcal{B}' includes no rules with a conclusion $\langle\langle P_2, \mathbf{1} \rangle\rangle$. We do not have $\langle a' \rangle \langle\langle P_2, \mathcal{P}_1, \mathbf{1} \rangle\rangle$ supported over the empty basis. \square

Example 16 (The wise men). The following example is a variant of a well-known puzzle:

‘A certain king wishes to test his three wise men a_1, a_2, a_3 . a_1 can see both a_2 and a_3 . Man a_2 can see only a_3 . Man a_3 is blindfolded. They all know this to be the case. The king tells them that he will put a white or black spot on each of their foreheads but that at least one spot will be white. In fact, all three spots are white. He then repeatedly asks them, in turn (first man, second man, third man), “Do you know the color of your spot?” What do they answer?’

With agents a_1, a_2, a_3 who make sufficient use of information, the sequence should go ‘no’, ‘no’, ‘yes’. Critical to this are the inferences that are drawn from announcements that are themselves tied to inferences that cannot be made by agents earlier in the sequence. It assumes that the second man can draw an inference from an inference not made by the first (a_2 must be able to infer reasons why a_1 was unable to infer his own colour). It assumes that the second man can combine the result of this inference with an observation to draw an inference that he may not infer his own colour (a_2 can see that a_3 ’s spot is white, and so a_2 cannot infer his own colour). a_3 draws the same inference as a_2 from a_1 ’s announcement. Man a_3 infers from a_2 ’s announcement that his own spot must be white, for a_2 did not infer his own colour.

Our focus below is on agents reasoning about the use of available inferences of agents earlier in the chain, rather than on state-based reasoning as in many classical state-based accounts of these types of problems epistemic logic.

We suppose a basic situation $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ for each wise man, relating to inferences that he can make using only the observations visually made of colour (for example, ‘if the other two have black spots then I have a white spot’ and ‘he has a black spot’). Specifically, each \mathcal{P}_i does not capture the inferences from inferences-or-not flowing from the announcements. The first man has access only to \mathcal{P}_1 . The second man has access to \mathcal{P}_2 . The third man has access only to \mathcal{P}_3 .

Let $W(a_n)$ be the proposition ‘Man n has a white dot’ for each $n = 1, 2, 3$. An inferonic atom $\langle\langle W(a_n), \mathbf{1} \rangle\rangle$ is an assertion of this fact, and an inferonic atom $\langle\langle W(a_n), \mathbf{o} \rangle\rangle$ is a denial of it. Let P_n be the proposition ‘Man n knows the colour of his own dot’. An inferonic atom $\langle\langle P_n, \mathbf{1} \rangle\rangle$ is therefore an affirmation of the content of this announcement (that a_n knows his own colour), and an inferonic atom $\langle\langle P_n, \mathbf{o} \rangle\rangle$ is a denial of it.

(The first man). The first man, a_1 , has his visual observations, which we formalize as inferences

$$\overline{\langle\langle W(a_2), \mathbf{1} \rangle\rangle} \quad \overline{\langle\langle W(a_3), \mathbf{1} \rangle\rangle}$$

in a base \mathcal{P}_1^O . The first man also has ‘static’ rules

$$(5) \quad \frac{\langle\langle W(a_2), \mathbf{1} \rangle\rangle}{\langle\langle P_1, \mathbf{o} \rangle\rangle} \quad \frac{\langle\langle W(a_3), \mathbf{1} \rangle\rangle}{\langle\langle P_1, \mathbf{o} \rangle\rangle}$$

and

$$(6) \quad \frac{\langle\langle W(a_2), \mathbf{o} \rangle\rangle \quad \langle\langle W(a_3), \mathbf{o} \rangle\rangle}{\langle\langle P_1, \mathbf{1} \rangle\rangle}$$

in a base \mathcal{P}_1^S . Define $\mathcal{P}_1 = \mathcal{P}_1^O \cup \mathcal{P}_1^S$. We therefore have a derivation of $\langle\langle P_1, \mathbf{o} \rangle\rangle$ over \mathcal{P}_1 . We then have $\Vdash_{\mathcal{P}_1^O} \langle\langle P_1, \mathcal{P}_1^S, \mathbf{o} \rangle\rangle$, while $\langle\langle P_1, \mathcal{P}_1^S, \mathbf{1} \rangle\rangle$ is not supported over \mathcal{P}_1^O . The rule (6) which is not, in fact, applied when all agents are dotted white, is justified directly by consideration of the King’s announcement.

(The second man). The second man has rules

$$\overline{\langle\langle W(a_3), \mathbf{1} \rangle\rangle} \quad \overline{\langle\langle P_1, \mathbf{o} \rangle\rangle}$$

relating to observations and announcements. Let these be elements of base \mathcal{P}_2^O . The second man has static rules

$$(7) \quad \frac{\langle\langle P_1, \mathbf{o} \rangle\rangle \quad \langle\langle W(a_3), \mathbf{1} \rangle\rangle}{\langle\langle P_2, \mathbf{o} \rangle\rangle}$$

and

$$(8) \quad \frac{\langle\langle P_1, \mathbf{o} \rangle\rangle \quad \langle\langle W(a_3), \mathbf{o} \rangle\rangle}{\langle\langle P_2, \mathbf{1} \rangle\rangle}$$

and

$$(9) \quad \frac{\langle\langle P_1, \mathbf{1} \rangle\rangle}{\langle\langle P_2, \mathbf{1} \rangle\rangle}.$$

Let these static rules be in a base \mathcal{P}_2^S . Take $\mathcal{P}_2 = \mathcal{P}_2^O \cup \mathcal{P}_2^S$. There is a derivation of $\langle\langle P_2, \mathbf{o} \rangle\rangle$ over the base \mathcal{P}_2 using the first of the two rules from \mathcal{P}_2^S . There is the following instance of the support relation $\Vdash_{\mathcal{P}_2^O} \langle\langle P_2, \mathcal{P}_2^S, \mathbf{o} \rangle\rangle$, but $\langle\langle P_2, \mathcal{P}_2^S, \mathbf{1} \rangle\rangle$ is not supported over \mathcal{P}_2^O .

Notice that support for a disjunction $\langle\langle W(a_2), \mathcal{P}_1^S, \mathbf{1} \rangle\rangle \vee \langle\langle W(a_3), \mathcal{P}_1^S, \mathbf{1} \rangle\rangle$ over \mathcal{P}_2^O is not required.

The rule (7) can be justified by considering \mathcal{P}_1 with \mathcal{P}_1^S fixed as above, but an unknown \mathcal{P}_1^O that is a subset of $\{\Rightarrow \langle\langle W(a_2), \mathbf{o} \rangle\rangle, \Rightarrow \langle\langle W(a_2), \mathbf{1} \rangle\rangle, \Rightarrow \langle\langle W(a_3), \mathbf{o} \rangle\rangle, \Rightarrow \langle\langle W(a_3), \mathbf{1} \rangle\rangle\}$. Suppose $\Vdash_{\mathcal{P}_1^O} \langle\langle P_1, \mathcal{P}_1^S, \mathbf{o} \rangle\rangle$ and $\Vdash_{\mathcal{P}_1^O} \langle\langle W(a_3), \mathcal{P}_1^S, \mathbf{1} \rangle\rangle$. However, the rules (6) allow these support relation instances to hold for both $\mathcal{P}_1^O = \{\Rightarrow \langle\langle W(a_2), \mathbf{o} \rangle\rangle, \Rightarrow \langle\langle W(a_3), \mathbf{1} \rangle\rangle\}$ and $\mathcal{P}_1^O = \{\Rightarrow \langle\langle W(a_2), \mathbf{1} \rangle\rangle, \Rightarrow \langle\langle W(a_3), \mathbf{1} \rangle\rangle\}$. That is, one cannot infer either of $\langle\langle W(a_2), \mathbf{1} \rangle\rangle$ or $\langle\langle W(a_2), \mathbf{o} \rangle\rangle$. By contrast, for rule (8), if we have both $\Vdash_{\mathcal{P}_1^O} \langle\langle P_1, \mathcal{P}_1^S, \mathbf{o} \rangle\rangle$ and $\Vdash_{\mathcal{P}_1^O} \langle\langle W(a_3), \mathcal{P}_1^S, \mathbf{o} \rangle\rangle$, then it must be the case that $\vdash_{\mathcal{P}_1} \langle\langle W(a_2), \mathbf{1} \rangle\rangle$. For rule (9), if $\Vdash_{\mathcal{P}_1^O} \langle\langle P_1, \mathcal{P}_1^S, \mathbf{1} \rangle\rangle$ for unknown \mathcal{P}_1^O , then it must be the case that $\vdash_{\mathcal{P}_1} \langle\langle W(a_2), \mathbf{1} \rangle\rangle$.

(The third man). The third man has observations from announcements

$$\overline{\langle\langle P_1, \mathbf{o} \rangle\rangle} \quad \overline{\langle\langle P_2, \mathbf{o} \rangle\rangle}$$

in a basis \mathcal{P}_2^O . He has static rules

$$(10) \quad \frac{\langle\langle P_1, \mathbf{o} \rangle\rangle \quad \langle\langle P_2, \mathbf{o} \rangle\rangle}{\langle\langle P_3, \mathbf{1} \rangle\rangle}$$

and

$$(11) \quad \frac{\langle\langle P_1, \mathbf{1} \rangle\rangle}{\langle\langle P_3, \mathbf{o} \rangle\rangle}$$

and

$$(12) \quad \frac{\langle\langle P_2, \mathbf{1} \rangle\rangle}{\langle\langle P_3, \mathbf{o} \rangle\rangle}$$

in base \mathcal{P}_3^S . Taking $\mathcal{P}_3 = \mathcal{P}_3^O \cup \mathcal{P}_3^S$, there is a derivation of $\langle\langle p_3, \mathbf{1} \rangle\rangle$ over \mathcal{P}_3 . We have that

$$\Vdash_{\mathcal{P}_3^O} \langle\langle P_3, \mathcal{P}_3^S, \mathbf{o} \rangle\rangle$$

but $\langle\langle P_3, \mathcal{P}_3^S, \mathbf{1} \rangle\rangle$ is not supported over \mathcal{P}_3^O .

In particular, rule (10) can be justified by reasoning about the second man's available inferences. If $\Vdash_{\mathcal{P}_2^O} \langle\langle P_2, \mathcal{P}_2^S, \mathbf{o} \rangle\rangle$ where \mathcal{P}_2^S is as above, but \mathcal{P}_2^O is an unknown subset of $\{\Rightarrow \langle\langle P_1, \mathbf{1} \rangle\rangle, \Rightarrow \langle\langle P_1, \mathbf{o} \rangle\rangle, \Rightarrow \langle\langle W(a_3), \mathbf{1} \rangle\rangle, \Rightarrow \langle\langle W(a_3), \mathbf{o} \rangle\rangle\}$, then it must be that $\vdash_{\mathcal{P}_2} \langle\langle W(a_3), \mathbf{1} \rangle\rangle$. The rule (12) is also justified by considering $\Vdash_{\mathcal{P}_2}$ for unknown \mathcal{P}_2^O , while the rule (11) can be justified by considering $\Vdash_{\mathcal{P}_1}$ for unknown \mathcal{P}_1^O but known \mathcal{P}_1^S . \square

7. INFEROMORPHISMS AND INFORMATION FLOW

As we have previously mentioned, in Section 3, in order to produce a simpler treatment of information flow, Barwise, Seligman and others developed the theory of classifications and infomorphisms, under the name ‘The Logic of Distributed Systems’ which we abbreviate as LDS. The standard introductory reference for this is the monograph [15]. Before proceeding to develop our inferentialist counterpart to these ideas, we summarize some background on LDS from [15].

A *classification* is a structure $\mathbf{A} = \langle \Sigma_{\mathbf{A}}, A, \vDash_{\mathbf{A}} \rangle$, where A is a set (of ‘tokens’), $\Sigma_{\mathbf{A}}$ is a set of ‘types’, and $\vDash \subseteq A \times \Sigma_{\mathbf{A}}$ is a binary relation. We think of a relational instance $a \vDash_{\mathbf{A}} \alpha$ as saying that the token a is classified as being of type α . Classification abstracts from the support relation of situation theory, where there are situations classified $s \vDash T$ by types T , including the type T of situations that satisfy a particular infon: $T = [\dot{s} \mid \dot{s} \vDash \sigma]$.

An *infomorphism* $f : \mathbf{A} \rightarrow \mathbf{C}$ from classification \mathbf{A} to classification \mathbf{C} consists of a pair of functions $f^\wedge : \Sigma_{\mathbf{A}} \rightarrow \Sigma_{\mathbf{C}}$, $f^\vee : C \rightarrow A$ such that the Chu condition (see [15] for the background) holds:

$$(13) \quad f^\vee(c) \vDash_{\mathbf{A}} \alpha \text{ iff } c \vDash_{\mathbf{C}} f^\wedge(\alpha)$$

for every $c \in C$ and $\alpha \in \Sigma_{\mathbf{A}}$.

Just as in situation theory, constraints are still viewed as the primary concept to analyse information flow, but general situated propositions are replaced by assertions about tokens being assigned to classes, while (logical) constraints themselves are understood as arising from certain configurations of the concept of infomorphism known as ‘channels’. We remark that Mares [61] and Restall [73] discuss how constraints can be captured model-theoretically in the Routley–Meyer ternary-relation semantics for relevance logic.

A *channel* is an indexed family $(f_i : A_i \rightarrow C \mid i \in I)$ of infomorphisms, where each member of the family has the same target, C , called the *core* of the channel. Channels provide a means to identify distributed systems. A system, in a particular configuration/state, is identified by a single $c \in C$. The distributed components are the $a_i = f_i^\vee(c) \in A_i$. Components a_i and a_j are *connected* if there is some c such that $a_i = f_i^\vee(c)$ and $a_j = f_j^\vee(c)$.

Later, in Section 8, we shall apply the tools we develop herein to a quite generic notion of distributed system that, while consistent with the ideas of [15], derives from quite practical modelling concerns (see, for example, [25, 56] and the references therein).

For an example of our present set-up, we can imagine a flashlight as a system consisting of a bulb and a switch (eliding other relevant components). We can imagine a domain S of switches and a domain B of bulbs. Taking F to be the domain of flashlights, each flashlight connects a unique switch and a unique bulb. This provides mappings $f_S^\vee : F \rightarrow S$ and $f_B^\vee : F \rightarrow B$. An important type for bulbs is LIT, those that are showing light, and we can imagine this to be recognizable also for flashlights, $f_B^\wedge(\text{LIT})$. Similarly, for switches, we have a type ON and for flashlights $f_S^\vee(\text{ON})$.

For any classification A , a *sequent* of A is a pair $\langle \Gamma, \Delta \rangle$ of sets of types of A . A token $a \in A$ *satisfies* a sequent $\langle \Gamma, \Delta \rangle$ if $a \vDash_A \alpha$ for every $\alpha \in \Gamma$ implies that $a \vDash_A \alpha'$ for some $\alpha' \in \Delta$. The theory $\text{Th}(A)$ is the set of all sequents that are satisfied by all $a \in A$.

The central proposal for the notion of information flow now goes quite naturally. Suppose a channel that is made from infomorphisms including $f : A \rightarrow C$ and $g : B \rightarrow C$ over the core C . Suppose that a is a token of type A , α is a type of A , b is a token of type B , and β is a type of B . Then, relative to C , we say $a \vDash_A \alpha$ *carries the information* that $b \vDash_B \beta$ if a and b are connected and $\langle f^\wedge(\alpha), g^\wedge(\beta) \rangle$ is in $\text{Th}(C)$.

7.1. Inferomorphisms. We now develop our inferentialist analysis, taking inspiration from the methodology of LDS. Our target remains to produce an analysis of information flow in distributed systems. To that end, we need to be able to identify and reason about definite components of the system. We use a first-order language, and instead of taking individual components to be elements from domains that are sets, individuals must simply be identified as equivalence classes of closed terms under the equality relation.

We use the notational conventions from Figure 2. In particular, terms t are generated from: individual constants a, b, c, \dots ; individual variables x, y, z, \dots ; fixed arity relation/predicate symbols P, Q, R, \dots for various arities. Recall that \mathcal{T} is the set of terms, and $\text{Cl}(\mathcal{T})$ is the set of closed terms of this language. For each natural number n , let Pred_n be the set of n -ary relation symbols of the language.

Below we write expressions for terms that are potentially undefined. We use the Kleene-equality symbol below: for two such expressions t_1 and t_2 , we write $t_1 \simeq t_2$ if $t_1 = t_2$ whenever t_1 and t_2 are both defined, and t_1 is defined if and only if t_2 is defined.

Definition 17. A *pre-inferomorphism* $f : \mathcal{P} \rightarrow \mathcal{P}'$ over \mathcal{B} consists of a partial function

$$f^\vee : \text{Cl}(\mathcal{T}) \rightarrow \text{Cl}(\mathcal{T})$$

and a dependent function

$$f^\wedge : \prod_{n \in \mathbb{N}} (\text{Pred}_n \rightarrow \text{Pred}_n).$$

The partial function f^\vee is required to have a substitution property: if $t_1 = t_2$, then $f^\vee(t_1) \simeq f^\vee(t_2)$. The pair of functions are required to satisfy the condition below:

$$(14) \quad \vdash_{\mathcal{C} \cup \mathcal{P}'} \langle R(f^\vee(t_1), \dots, f^\vee(t_n)), \mathfrak{b} \rangle \quad \text{iff} \quad \vdash_{\mathcal{C} \cup \mathcal{P}} \langle f^\wedge(n)(R)(t_1, \dots, t_n), \mathfrak{b} \rangle$$

for every $\mathcal{C} \supseteq \mathcal{B}$, every $n \geq 0$, every relation symbol R of arity n , and every sequence of closed terms t_1, \dots, t_n of length n . \square

Given a pre-inferomorphism $f : \mathcal{P} \rightarrow \mathcal{P}'$ over \mathcal{B} , we define:

$$\begin{aligned} \langle R(t_1, \dots, t_n), \mathcal{P}, \mathfrak{b} \rangle^{f, \vee} &\simeq \langle R(f^\vee(t_1), \dots, f^\vee(t_n)), \mathcal{P}, \mathfrak{b} \rangle \\ \langle R(t_1, \dots, t_n), \mathcal{P}'', \mathfrak{b} \rangle^{f, \vee} &= \langle R(t_1, \dots, t_n), \mathcal{P}'', \mathfrak{b} \rangle && \text{for } \mathcal{P}'' \not\equiv \mathcal{P} \\ \langle R(t_1, \dots, t_n), \mathcal{P}, \mathfrak{b} \rangle^{f, \wedge} &= \langle (f^\wedge(R))(t_1, \dots, t_n), \mathcal{P}', \mathfrak{b} \rangle \\ \langle R(t_1, \dots, t_n), \mathcal{P}'', \mathfrak{b} \rangle^{f, \wedge} &= \langle R(t_1, \dots, t_n), \mathcal{P}'', \mathfrak{b} \rangle && \text{for } \mathcal{P}'' \not\equiv \mathcal{P} \end{aligned}$$

and then for $\dagger \in \{\wedge, \vee\}$, indicating the components of the morphism, we take

$$\begin{aligned} (\phi_1 \circ \phi_2)^{f, \dagger} &= (\phi_1^{f, \dagger}) \circ (\phi_2^{f, \dagger}) && \text{for } \circ \in \{\wedge, \vee, \supset\} \\ \perp^{f, \dagger} &= \perp \\ \Theta^{f, \dagger} &= \{\psi^{f, \dagger} \mid \psi \in \Theta\} \\ (\Theta \Vdash_{\mathcal{B}} \phi)^{f, \dagger} &= (\Theta^{f, \dagger}) \Vdash_{\mathcal{B}} (\phi^{f, \dagger}). \end{aligned}$$

In particular, this defines a function $\iota \mapsto \iota^{f, \wedge}$ and a partial function $\iota \mapsto \iota^{f, \vee}$ on inferons.

Definition 18. A pre-inferomorphism is a *quasi-inferomorphism* if $\iota \mapsto \iota^{f, \vee}$ and $\iota \mapsto \iota^{f, \wedge}$ are surjective. \square

For every \mathcal{B} , there is a category with objects the elements of the basis \mathcal{M} and morphisms the inferomorphisms over \mathcal{B} . Note that $(fg)^\vee = g^\vee \circ f^\vee$ and $(fg)^\wedge = f^\wedge \circ g^\wedge$ for $f : \mathcal{P} \rightarrow \mathcal{P}'$ and $g : \mathcal{P}' \rightarrow \mathcal{P}''$. The following substitution result can then be shown.

Proposition 19. Suppose the first-order language of inferons and base-extension semantics as set out in Figure 6. Let $f : \mathcal{P} \rightarrow \mathcal{P}'$ be a quasi-inferomorphism over \mathcal{B} . Let ϕ be a formula and Θ be a finite set of formulæ. Then

$$(15) \quad (\Theta \Vdash_{\mathcal{B}} \phi)^{f,\vee} \text{ iff } (\Theta \Vdash_{\mathcal{B}} \phi)^{f,\wedge}$$

Proof. We show the stronger result that $(\Theta \Vdash_{\mathcal{C}} \phi)^{f,\vee}$ if, and only if $(\Theta \Vdash_{\mathcal{C}} \phi)^{f,\wedge}$, for all $\mathcal{C} \supseteq \mathcal{B}$.

We use a function γ that defines complexity $\gamma(\Theta, \phi)$ of pairs (Θ, ϕ) . First, define (overloading the symbol γ) complexity of a formula:

$$\begin{aligned} \gamma(\iota) &= 1 \\ \gamma(\perp) &= 1 \\ \gamma(\phi \circ \phi') &= 1 + \gamma(\phi) + \gamma(\phi') \text{ for } \circ \in \{\wedge, \supset, \vee\} \\ \gamma(\forall x.\phi) &= 1 + \gamma(\phi) \\ \gamma(\exists x.\phi) &= 2 + \gamma(\phi). \end{aligned}$$

From there, extend this to complexity of pairs (Θ, ι) :

$$\gamma(\Theta, \iota) = 1 + \max(\{\gamma(\iota)\} \cup \{\gamma(\kappa) \mid \kappa \in \Theta\}).$$

The proof works by induction on complexity of pairs (Θ, ϕ) that appear in support relation instances for some base, with Θ on the left of \Vdash and ϕ on the right. The induction hypothesis is that for all (Θ', ϕ') less complex than (Θ, ϕ) , and all bases $\mathcal{C} \supseteq \mathcal{B}$, that

$$(\Theta' \Vdash_{\mathcal{C}} \phi')^{f,\vee} \text{ iff } (\Theta' \Vdash_{\mathcal{C}} \phi')^{f,\wedge}$$

Given the induction hypothesis, we show that (15) holds.

(\perp) Immediate, since $\perp^{f,\vee} = \perp = \perp^{f,\wedge}$ and $\Theta = \emptyset$

(At) We have $\Theta = \emptyset$. We take $\phi' = \iota = \langle R(t_1, \dots, t_n), \mathcal{P}'', \mathbf{b} \rangle$. In the case where $\mathcal{P}'' \neq \mathcal{P}$ the result is trivial because the two operations, $()^\vee$ and $()^\wedge$ make no change to the inferon.

In the case where $\mathcal{P}'' = \mathcal{P}$, we have:

$$\begin{aligned} \Vdash_{\mathcal{C}} \langle R(t_1, \dots, t_n), \mathcal{P}, \mathbf{b} \rangle^{f,\vee} &\text{ iff } \Vdash_{\mathcal{C}} \langle R(f^\vee(t_1), \dots, f^\vee(t_n)), \mathcal{P}, \mathbf{b} \rangle \\ &\text{ iff } \vdash_{\mathcal{C} \cup \mathcal{P}} \langle R(f^\vee(t_1), \dots, f^\vee(t_n)), \mathbf{b} \rangle \\ &\text{ iff } \vdash_{\mathcal{C} \cup \mathcal{P}'} \langle (f^\wedge(n)(R))(t_1, \dots, t_n), \mathbf{b} \rangle \\ &\text{ iff } \Vdash_{\mathcal{C}} \langle (f^\wedge(n)(R))(t_1, \dots, t_n), \mathcal{P}', \mathbf{b} \rangle \\ &\text{ iff } \Vdash_{\mathcal{C}} \langle R(t_1, \dots, t_n), \mathcal{P}, \mathbf{b} \rangle^{f,\wedge} \end{aligned}$$

($\wedge, \supset, \forall x$) Straightforward use of the induction hypothesis, noting that the ‘only if’ and ‘if’ directions of the proof rely, respectively, only on the same direction (‘only if’, ‘if’) in the induction hypothesis. For \supset , the fact that $\gamma(\phi_1, \phi_2) < \gamma(\phi_1 \supset \phi_2)$ is used.

(Inf and \vee) Straightforward application of the induction hypothesis, noting that both the ‘if’ and ‘only if’ directions in the hypothesis are required to prove each of the ‘if’ and ‘only if’ directions. For the case of (\vee), surjectivity of $\iota \mapsto \iota^{f,\wedge}$ is used in showing the ‘only if’ direction, while surjectivity of $\iota \mapsto \iota^{f,\vee}$ is used in showing the ‘if’ direction.

($\exists x$) The ‘only if’ direction uses the surjectivity of $\iota \mapsto \iota^{f,\wedge}$, and the ‘if’ direction uses the surjectivity of $\iota \mapsto \iota^{f,\vee}$. Both directions use both of the ‘if’ and ‘only if’ directions of the induction hypothesis. \square

A *stock* (M, \mathcal{P}) consists of a base \mathcal{P} and a set of closed terms M that is closed under equality. Recall that the *domain* of a partial function is the set of elements where it is defined.

Definition 20. Let (M, \mathcal{P}) and (N, \mathcal{P}') be stocks. An *inferomorphism* $f : (N, \mathcal{P}') \rightarrow (M, \mathcal{P})$ over \mathcal{B} is a quasi-inferomorphism $f : \mathcal{P} \rightarrow \mathcal{P}'$ over \mathcal{B} such that

- the domain of the partial function f^\vee is equal to M
- f^\vee defines a function from M to N . \square

Stocks and inferomorphisms over \mathcal{B} form a category.

Definition 21. Let (L, \mathcal{P}) be a stock. A *channel over \mathcal{B} with core (L, \mathcal{P})* is a family

$$(f_i : (M_i, \mathcal{P}_i) \longrightarrow (L, \mathcal{P}) \mid i \in I)$$

of inferomorphisms over \mathcal{B} . Two terms $t_i \in M_i$ and $t_j \in M_j$ are *connected* in the channel if there is $t \in L$ such that $t_i = f^\vee(t)$ and $t_j = g^\vee(t)$. \square

For a channel as above, we can think of an element $t \in L$ as identifying a particular system that is made up of connected components from the family $(f_i^\vee(t) \in M_i \mid i \in I)$. The base \mathcal{P}_i provides supports some facts about elements of M_i . The base \mathcal{P} supports facts about the entire system. In particular, it will allow us to compare and combine sentences of the forms $f_i^\wedge(\phi_i)$ and $f_j^\wedge(\phi_j)$ where ϕ_i is a sentence supported by \mathcal{P}_i and ϕ_j is a sentence supported by \mathcal{P}_j .

Definition 22. Suppose a channel with core (L, \mathcal{P}'') and with inferomorphisms $f : (M, \mathcal{P}) \longrightarrow (L, \mathcal{P}'')$ and $g : (N, \mathcal{P}') \longrightarrow (L, \mathcal{P}'')$ over \mathcal{B} . Let R and S be unary predicates. Let $t \in M$ and $t' \in N$ be terms. Relative to the channel, we say $\Vdash_{\mathcal{B}} \langle\langle R(t), \mathcal{P}, \mathfrak{b} \rangle\rangle$ carries the information that $\Vdash_{\mathcal{B}} \langle\langle S(t'), \mathcal{P}', \mathfrak{b} \rangle\rangle$ if t and t' are connected by the channel for all terms $t'' \in L$, and

$$\langle\langle f^\wedge(R)(t''), \mathcal{P}'', \mathfrak{b} \rangle\rangle \Vdash_{\mathcal{B}} \langle\langle g^\wedge(S)(t''), \mathcal{P}'', \mathfrak{b} \rangle\rangle \quad \square$$

Suppose that we have a channel as in Definition 22 and that $t = f^\vee(t'') \in M$ and $t' = g^\vee(t'') \in N$ for some $t'' \in L$. Considering Proposition 19, if $\Vdash_{\mathcal{B}} \langle\langle R(t), \mathcal{P}, \mathfrak{b} \rangle\rangle$, then $\Vdash_{\mathcal{B}} \langle\langle f^\wedge(R)(t''), \mathcal{P}'', \mathfrak{b} \rangle\rangle$. If $\Vdash_{\mathcal{B}} \langle\langle R(t), \mathcal{P}, \mathfrak{b} \rangle\rangle$ carries the information that $\Vdash_{\mathcal{B}} \langle\langle S(t'), \mathcal{P}', \mathfrak{b} \rangle\rangle$, then by Inf-clause in Figure 3 of the support relation it is the case that $\Vdash_{\mathcal{B}} \langle\langle g^\wedge(S)(t''), \mathcal{P}'', \mathfrak{b} \rangle\rangle$. Applying Proposition 19 again, we find $\Vdash_{\mathcal{B}} \langle\langle S(t'), \mathcal{P}', \mathfrak{b} \rangle\rangle$. \square

Example 23. This example is adapted from an example regarding inferomorphisms and classifications [15]. We can think of a flashlight as a system consisting of a switch and a bulb. Regard stock (L, \mathcal{P}'') , modelling flashlights (by providing a set of terms identifying flashlights and a base of facts about them), as a way of describing the interface between (M, \mathcal{P}) , modelling switches, and (N, \mathcal{P}') , modelling bulbs. In particular, \mathcal{P}'' will support constraints that link the state of the bulb to the state of the switch. A closed term $t'' \in L$ determines both $t = f^\vee(t'') \in M$, a particular switch, and $t' = g^\vee(t'') \in N$, a particular bulb. The term t'' , as the whole flashlight, represents the particular connection, so which switch is connected to which bulb and how they are connected.

We consider the relationship between inferons $\langle\langle \text{ON}(x), \mathcal{P}, \mathbf{1} \rangle\rangle$ saying that switch x is on, and inferons $\langle\langle \text{LIT}(y), \mathcal{P}', \mathbf{1} \rangle\rangle$ saying that bulb y is lit. We can think of ON as an abbreviation for the predicate ‘is a switch and is on’, and LIT as meaning ‘is a bulb and is lit’. In particular, then, we must have $\{t \in \text{Cl(Terms)} \mid \Vdash_{\mathcal{B}} \langle\langle \text{ON}(t), \mathbf{1} \rangle\rangle\} \subseteq M$ and $\{t \in \text{Cl(Terms)} \mid \Vdash_{\mathcal{B}} \langle\langle \text{LIT}(t), \mathbf{1} \rangle\rangle\} \subseteq N$.

We are supposing a particular flashlight named by $t'' \in L$, and that its component switch, $t = f^\vee(t'') \in M$, is in the on state, as derivable in \mathcal{P} , so that we have both $\vdash_{\mathcal{P}} \langle\langle \text{ON}(t), \mathbf{1} \rangle\rangle$ and

$$\Vdash_{\emptyset} \langle\langle \text{ON}(t), \mathcal{P}, \mathbf{1} \rangle\rangle$$

for all bases \mathcal{P} .

Suppose also predicates fON and fLIT such that $\{t \in \text{Cl(Terms)} \mid \Vdash_{\mathcal{B}} \langle\langle \text{fON}(t), \mathbf{1} \rangle\rangle\} \subseteq L$ and $\{t \in \text{Cl(Terms)} \mid \Vdash_{\mathcal{B}} \langle\langle \text{fLIT}(t), \mathbf{1} \rangle\rangle\} \subseteq L$ for all bases \mathcal{B} . These predicates say ‘is a flashlight with a switch that is on’, and ‘is a flashlight with a bulb that is lit’, respectively. For the purposes of this example, abbreviate $f^\wedge(\mathbf{1})$ as f^\wedge , as we are only interested in unary predicates. Define $f^\wedge(\text{ON}) = \text{fON}$ and $g^\wedge(\text{LIT}) = \text{fLIT}$. We are supposing the condition (14), which guarantees that $\vdash_{\mathcal{P}''} \langle\langle (f^\wedge(\text{ON}))(t''), \mathbf{1} \rangle\rangle$ since $\vdash_{\mathcal{P}} \langle\langle \text{ON}(f^\vee(t'')), \mathbf{1} \rangle\rangle$.

Suppose stock (L, \mathcal{P}'') represents only ‘normal’ (properly functioning [15]), flashlights, x : that is, suppose that

$$\vdash_{\mathcal{P}''} \forall x. (\langle\langle \text{fON}(x), \mathcal{P}'', \mathbf{1} \rangle\rangle \supset \langle\langle \text{fLIT}(x), \mathcal{P}'', \mathbf{1} \rangle\rangle)$$

Then the constraint obtains that if the switch is on, then the bulb is lit holds at the flashlight t'' :

$$\langle\langle (f^\wedge(\text{ON}))(t''), \mathcal{P}'', \mathbf{1} \rangle\rangle \vdash_{\mathcal{P}''} \langle\langle (g^\wedge(\text{LIT}))(t''), \mathcal{P}'', \mathbf{1} \rangle\rangle$$

In other words, $\Vdash_{\mathcal{P}''} \langle\langle \text{ON}(t), \mathcal{P}, \mathbf{1} \rangle\rangle$ carries the information that $\Vdash_{\mathcal{P}''} \langle\langle \text{LIT}(t'), \mathcal{P}', \mathbf{1} \rangle\rangle$. From this it follows by the Inf clause of the support relation that $\Vdash_{\mathcal{P}''} \langle\langle (g^\wedge(\text{LIT}))(t''), \mathcal{P}'', \mathbf{1} \rangle\rangle$ and therefore $\vdash_{\mathcal{P}''} \langle\langle (g^\wedge(\text{LIT}))(t''), \mathbf{1} \rangle\rangle$. Applying condition (14) again we get $\vdash_{\mathcal{P}'} \langle\langle \text{LIT}(g^\vee(t'')), \mathbf{1} \rangle\rangle$ and so

$$\Vdash_{\emptyset} \langle\langle \text{LIT}(g^\vee(t'')), \mathcal{P}', \mathbf{1} \rangle\rangle$$

saying that the component bulb $t = g^\vee(t'')$ is lit according to base \mathcal{P}' . \square

8. INFORMATION FLOW AND SYSTEM STRUCTURE

8.1. Information Flow from Inferomorphisms and Inferonic Bases. Recall the intuition behind the notion of ‘constraint’ from situation theory in which logical properties’ holding (that is, infons’ being ‘supported’) in one situation necessarily entail logical properties in another situation. That is, constraints describe a form of information flow in the information-as-correlation sense [85, 89]. The existence of a constraint is often presumed to be tied to some structural factor relating situations, known as a ‘channel’. In the ‘Logic of Distributed Systems’ approach [15] using classifications, infomorphisms, when combined into channels, give a mathematically precise notion of correlation of logical properties (instances of classification relations) across linked system components.

In the approach of Section 5, the logical language and the base-extension support relation semantics are designed to allow for reasoning, relative to an inferonic base, about properties of other inferonic bases, via the inferon construct. Flow of information can then be handled, as in Example 14, by ensuring constraints connecting inferons regarding inferonic bases are supported in the inferonic base. An example of this is the rule (4), where support is ensured by requiring the constraint as a rule within a certain inferonic base.

In Section 7, an inferomorphism, because it leads to Proposition 19, provides an alternative view, in which the (now, bidirectional) constraint is not formalized within the inferonic base, but rather holds because of the properties of the inferomorphism mapping.

In this section, we study another notion of situation, namely ‘site’ that comes along with a natural notion of channel. Our sites are built from inferonic atoms (Definition 2). The logical languages and base-extension semantics will be generalizations of that introduced in Section 5. This approach combines the view of information developed above with the structural approach of Barwise, Gabbay, and Hartonas [12].

8.2. A Logic of Inferons with Sites. Building on ideas given by Barwise and Gabbay in [12], we define a *site*, \mathfrak{P} to be a finite set of inferonic atoms and recall that sites form the contexts used in the basic derivability relation for any base.

Given this notion of site, we can extend the logic of inferons by giving a *contextual support relation* $\Vdash^{\mathfrak{P}}$ defined as in Figure 10. The soundness and completeness theorems of Section 5 can be extended to this logic.

(At)	$\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \langle\langle P, \mathcal{C}, \mathfrak{b} \rangle\rangle$	iff	$\mathfrak{P} \vdash_{\mathcal{B} \cup \mathcal{C}} \langle\langle P, \mathfrak{b} \rangle\rangle$ for closed P
(\wedge)	$\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi_1 \wedge \phi_2$	iff	$\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi_1$ and $\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi_2$
(\vee)	$\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi_1 \vee \phi_2$	iff	for every closed ι and every $\mathcal{C} \supseteq \mathfrak{B}$, if $\phi_1 \Vdash_{\mathcal{C}}^{\mathfrak{P}} \iota$ and $\phi_2 \Vdash_{\mathcal{C}}^{\mathfrak{P}} \iota$, then $\Vdash_{\mathcal{C}} \iota$
(\supset)	$\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi_1 \supset \phi_2$	iff	$\phi_1 \Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi_2$
(Inf)	for $\Theta \neq \emptyset$, $\Theta \Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi$	iff	for every $\mathcal{C} \supseteq \mathfrak{B}$, and every \mathfrak{P}' , if $\Vdash_{\mathcal{C}}^{\mathfrak{P}'} \psi$, for every $\psi \in \Theta$, then $\Vdash_{\mathcal{C}}^{\mathfrak{P} \cup \mathfrak{P}'} \phi$
(\perp)	$\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \perp$	iff	for all closed ι , $\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \iota$
.....			
	(\forall) $\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \forall x. \phi$	iff	$\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi[t/x]$ for all $t \in \text{Cl}(\mathcal{T})$
	(\exists) $\Vdash_{\mathfrak{B}}^{\mathfrak{P}} \exists x. \phi$	iff	for all $\mathcal{C} \supseteq \mathfrak{B}$ and any closed ι , if $\phi[t/x] \Vdash_{\mathcal{C}}^{\mathfrak{P}} \iota$, for any $t \in \text{Cl}(\mathcal{T})$, then $\Vdash_{\mathcal{C}}^{\mathfrak{P}} \iota$
.....			
	(Validity) $\Gamma \Vdash \phi$ iff for all \mathfrak{B} , $\Gamma \Vdash_{\mathfrak{B}}^{\emptyset} \phi$		

FIGURE 10. Contextual support relation

For any site \mathfrak{P} , let $\mathcal{B}(\mathfrak{P})$ be the base consisting of the rule set $\{\Rightarrow \iota \mid \iota \in \mathfrak{P}\}$. The following is easily shown:

$$\mathfrak{P} \vdash_{\mathcal{B}} \iota \text{ iff } \vdash_{\mathcal{B} \cup \mathcal{B}(\mathfrak{P})} \iota$$

for all ι , \mathfrak{P} and \mathcal{B} . From this it follows that

$$\Theta \Vdash_{\mathfrak{B}}^{\mathfrak{P}} \phi \text{ iff } \Theta \Vdash_{\mathcal{B} \cup \mathcal{B}(\mathfrak{P})} \phi$$

for all Θ , ϕ , \mathfrak{P} and \mathcal{B} .

When site \mathfrak{P} is a subseq of \mathfrak{P}' , we say that the inclusion is a *channel*. Such a channels is labelled by the relative complement $\lambda := \mathfrak{P}' \setminus \mathfrak{P}$. We write the channel in the notation $\mathfrak{P} \xrightarrow{\lambda} \mathfrak{P}'$. Channels can be composed by function composition, with labels combining by union. As a result, site, channels and composition give a simple example of an *information network* in the sense of Barwise, Gabbay and Hartonas [11].

The support relation $\Vdash_{\mathcal{B}}^{\mathfrak{P}}$ provides us with instances of the form $\Vdash_{\mathcal{B}}^{\mathfrak{P}} \phi$ at sites \mathfrak{P} . We can take this base-extension support relation instance to be a rendering of the situation-theoretic ‘ \mathfrak{P} supports ϕ ’ relation instance, relativized to \mathcal{B} .

The base-extension support relation instances of the form $\Gamma \Vdash_{\mathcal{B}}^{\mathfrak{P}} \phi$ for Γ non-empty can be viewed as formulæ that are situated at channels, relative to bases. For a fixed base \mathcal{B} , examining again the Inf-rule of Figure 10, we see that this concerns the flow of information along the channel (inclusion) from \mathfrak{P}' to $\mathfrak{P} \cup \mathfrak{P}'$. The channel underpins the logical constraints according to Inf.

The base-extension support relation semantics ties together, in a systematic way, support at sites and support at channels. For example, this is evident in the support relation clauses

$$\begin{aligned} \Vdash_{\mathcal{B}}^{\mathfrak{P}} \phi \supset \psi & \text{ iff } \phi \Vdash_{\mathcal{B}}^{\mathfrak{P}} \psi \\ & \text{ iff for all } \mathcal{C} \supseteq \mathcal{B}, \text{ and all } \mathfrak{P}', \text{ if } \Vdash_{\mathcal{C}}^{\mathfrak{P}'} \phi, \text{ then } \Vdash_{\mathcal{C}}^{\mathfrak{P} \cup \mathfrak{P}'} \psi \end{aligned}$$

This can be seen as a way of formalizing constraints, via flow along channels, and also of internalizing this in the logic using implication.

Example 24 (Access control). We consider an access-control system that consumes a password and a token, and then grants access. The system consumes an inferon $\langle\langle p, \mathcal{B}_1, \mathbf{1} \rangle\rangle$, representing (information carried by) supply of the password. It becomes a sub-process that consumes a token, carrying $\langle\langle q, \mathcal{B}_2, \mathbf{1} \rangle\rangle$, and makes a decision to grant access, corresponding to atom $\langle\langle r, \mathcal{B}_3, \mathbf{1} \rangle\rangle$. We want a constraint, informally expressed as ‘if p is presented, then q is presented, then access, r , is granted.’ We specify the system, without the token p , by a formula $\langle\langle p, \mathcal{B}_1, \mathbf{1} \rangle\rangle \supset (\langle\langle q, \mathcal{B}_2, \mathbf{1} \rangle\rangle \supset \langle\langle r, \mathcal{B}_3, \mathbf{1} \rangle\rangle)$. In this example, we do not want anything other than the context to lead to support for the three inferons, and so we take $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \emptyset$.

A model of the system consists of a base and a site, such that this formula is supported. We take the site to be empty, since we consider it in isolation. The complete system, without the token, is modelled by some \mathcal{B} with $\Vdash_{\mathcal{B}}^{\emptyset} \langle\langle p, \emptyset, \mathbf{1} \rangle\rangle \supset (\langle\langle q, \emptyset, \mathbf{1} \rangle\rangle \supset \langle\langle r, \emptyset, \mathbf{1} \rangle\rangle)$. The significance of the use of \emptyset will be seen in capturing the desired constraints:

$$(16) \quad \text{if } \Vdash_{\mathcal{B}}^{\mathfrak{P}} \langle\langle p, \emptyset, \mathbf{1} \rangle\rangle \text{ and } \Vdash_{\mathcal{B}}^{\mathfrak{P}'} \langle\langle q, \emptyset, \mathbf{1} \rangle\rangle, \text{ then } \Vdash_{\mathcal{B}}^{\mathfrak{P} \cup \mathfrak{P}'} \langle\langle r, \emptyset, \mathbf{1} \rangle\rangle$$

parametrized on sites $\mathfrak{P}, \mathfrak{P}'$.

Assume that we do have a base \mathcal{B} providing a model. This yields $\langle\langle p, \emptyset, \mathbf{1} \rangle\rangle \Vdash_{\mathcal{B}}^{\emptyset} \langle\langle q, \emptyset, \mathbf{1} \rangle\rangle \supset \langle\langle r, \emptyset, \mathbf{1} \rangle\rangle$ using the support relation clause for \supset . For any site \mathfrak{P} , there is a channel from site \mathfrak{P} to $\mathfrak{P} \cup \emptyset = \mathfrak{P}$. Now suppose $\Vdash_{\mathcal{B}}^{\mathfrak{P}} \langle\langle p, \emptyset, \mathbf{1} \rangle\rangle$ as in the premiss of the desired constraint (16). Then Inf transfers this along the channel to give $\Vdash_{\mathcal{B}}^{\mathfrak{P}} \langle\langle q, \emptyset, \mathbf{1} \rangle\rangle \supset \langle\langle r, \emptyset, \mathbf{1} \rangle\rangle$. The support relation clause at \supset then tells us that $\langle\langle q, \emptyset, \mathbf{1} \rangle\rangle \Vdash_{\mathcal{B}}^{\mathfrak{P}} \langle\langle r, \emptyset, \mathbf{1} \rangle\rangle$. Assuming $\Vdash_{\mathcal{B}}^{\mathfrak{P}'} \langle\langle q, \emptyset, \mathbf{1} \rangle\rangle$, the channel from \mathfrak{P} to $\mathfrak{P} \cup \mathfrak{P}'$ along provides the constraint instance of Inf ensuring that $\Vdash_{\mathcal{B}}^{\mathfrak{P} \cup \mathfrak{P}'} \langle\langle r, \emptyset, \mathbf{1} \rangle\rangle$. \square

Example 25 (Airport security). A model of the departure-security process at an airport is depicted in Figure 11 (taken from [47]). Passengers enter the airport and proceed to the check-in desk. Their baggage enters the hold-baggage security process and they enter the cabin and passenger security process, and so on. Critically, the two processes are synchronized at the aircraft, where the baggage in the hold is reconciled with the passengers in the cabin to whom it belongs.

A particular passenger, o , passes through security controls prior to a flight. The passenger is travelling with hold-baggage h , and is using passport p . We describe how information is processed throughout a simplified and idealized version of process. The focus in our description here is solely on securing the flight itself. The architecture of the system is indicated in Figure 11.

For each $j = 1, \dots, 6$, at each (physical) location l_j we define a site S_j and a base \mathcal{B}_j . For $j = 1, \dots, 5$, the base is the base underpinning the (relevant for process component) background reasoning capacity of the agent at that location, prior to execution of the process. The site component is the atomic inferonic information that becomes available for use with the base at the location as the process executes. A slightly different set-up applies at location l_6 .

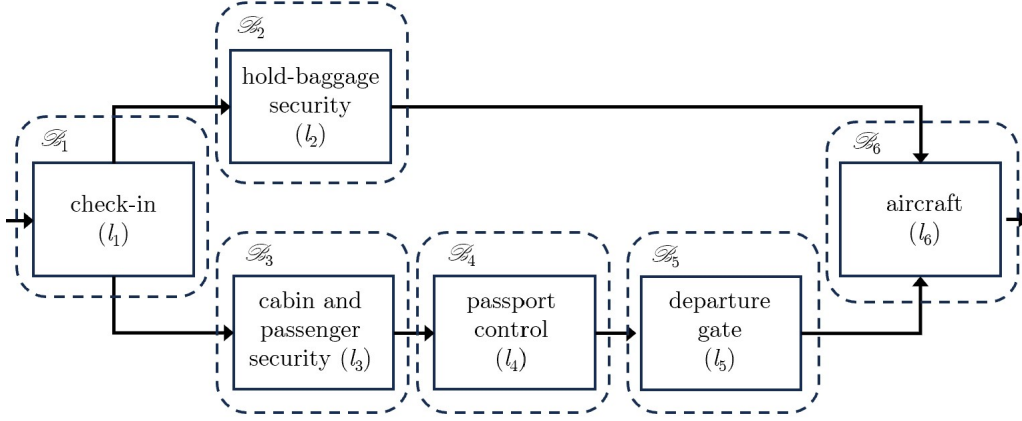


FIGURE 11. The airport security system (taken from [47])

l_1 (check-in): The passport p is checked for validity. This involves using some of the atomic inferonic information $\langle\langle I^W(p), \mathbf{1} \rangle\rangle$ carried by the passport. A ticket t is issued. This functions as an authentication token that is used in subsequent stages. The ticket carries this authentication information as an inferonic atom $\langle\langle T(t), \mathbf{1} \rangle\rangle$. Not all of the information carried by the passport can be used by the check-in agent, but will be used at a later stage: we represent this by the atom $\langle\langle I^S(p), \mathbf{1} \rangle\rangle$. The site $S_1 = \{\langle\langle I^W(p), \mathbf{1} \rangle\rangle, \langle\langle I^S(p), \mathbf{1} \rangle\rangle\}$ represents the passport information carried by the passport.

Base \mathcal{B}_1 is used by the agent at l_1 . This includes rules

$$\frac{\langle\langle I^W(p), \mathbf{1} \rangle\rangle}{\langle\langle P^W(p), \mathbf{1} \rangle\rangle} \quad \frac{\langle\langle I^W(p), \mathbf{1} \rangle\rangle}{\langle\langle A(h, t, p), \mathbf{1} \rangle\rangle} \quad \frac{\langle\langle I^W(p), \mathbf{1} \rangle\rangle}{\langle\langle T(t), \mathbf{1} \rangle\rangle}.$$

The first rule is used to generate an inferonic atom $\langle\langle P^W(p), \mathbf{1} \rangle\rangle$ meaning that the passenger with passport p has passed the (weak) check of the passport conducted by the check-in desk agent. The second rule is used to generate the inferonic atom $\langle\langle A(h, t, p), \mathbf{1} \rangle\rangle$, used to support information carried by the luggage label, and associating o 's baggage, passport, and ticket. The third rule is used to generate an inferonic atom $\langle\langle T(t), \mathbf{1} \rangle\rangle$ used to support information carried by the ticket.

We now have $S_1 \vdash_{\mathcal{B}_1} \langle\langle P^W(p), \mathbf{1} \rangle\rangle$ and $S_1 \vdash_{\mathcal{B}_1} \langle\langle A(h, t, p), \mathbf{1} \rangle\rangle$ and $S_1 \vdash_{\mathcal{B}_1} \langle\langle T(t), \mathbf{1} \rangle\rangle$, and so

$$\Vdash_{\emptyset}^{S_1} \langle\langle P^W(p), \mathcal{B}_1, \mathbf{1} \rangle\rangle \quad \text{and} \quad \Vdash_{\emptyset}^{S_1} \langle\langle A(h, t, p), \mathcal{B}_1, \mathbf{1} \rangle\rangle \quad \text{and} \quad \Vdash_{\emptyset}^{S_1} \langle\langle T(t), \mathcal{B}_1, \mathbf{1} \rangle\rangle$$

representing the information that is generated by the check-in agent.

l_2 (hold-baggage security): The tag on the baggage carries with it information to this location. The 3-way association of h , t , and p , flows from the site at l_1 to the site at l_2 is explicit in the tag. Implicitly, the information that p has been checked at check-in is inferrable, and thus also carried along this channel. The baggage scanner at l_2 provides an inferonic atom $\langle\langle H(h), \mathbf{1} \rangle\rangle$, meaning that the bag is clear, to the baggage-handler agent at this location. The agent at l_2 makes use of site $S_2 = S_1 \cup \{\langle\langle P^W(p), \mathbf{1} \rangle\rangle, \langle\langle A(h, t, p), \mathbf{1} \rangle\rangle, \langle\langle H(h), \mathbf{1} \rangle\rangle\}$, and we have a channel $S_1 \rightsquigarrow S_2$. We have that $\langle\langle T(t), \mathbf{1} \rangle\rangle$ is not in S_2 , representing the fact that this information does not flow to l_2 .

The agent uses the three atoms to generate information that the bag should be, and is, loaded. This is supported by an inferonic atom $\langle\langle L(h), \mathbf{1} \rangle\rangle$. Base \mathcal{B}_2 , used by the agent, contains the rule

$$\frac{\langle\langle P^W(p), \mathbf{1} \rangle\rangle \quad \langle\langle A(h, t, p), \mathbf{1} \rangle\rangle \quad \langle\langle H(h), \mathbf{1} \rangle\rangle}{\langle\langle L(h), \mathbf{1} \rangle\rangle}$$

Then $S_2 \vdash_{\mathcal{B}_2} \langle\langle L(h), \mathbf{1} \rangle\rangle$ and $\Vdash_{\emptyset}^{S_2} \langle\langle L(h), \mathcal{B}_2, \mathbf{1} \rangle\rangle$.

l_3 (cabin and passenger security): At this location, the ticket t is presented and the information carried by it is used. An inferonic atom $\langle\langle O(o), \mathbf{1} \rangle\rangle$, capturing the successful scanning of the passenger is also used. Information $N(a)$ that no harmful materials are airside is generated (in a more detailed model, an invariant would be maintained).

Site $S_3 = S_1 \cup \{\langle\langle T(t), \mathbf{1} \rangle\rangle, \langle\langle O(o), \mathbf{1} \rangle\rangle\}$ contains both the information obtained from l_1 and the information obtained from the security scanning process. There's a channel $S_1 \rightsquigarrow S_3$. The atom

$\langle\langle A(h, t, p), \mathbf{1} \rangle\rangle$ is not in S_3 , representing the fact that this information does not flow to, and so is not used at, l_3 . Base \mathcal{B}_3 for the security agent at l_3 contains

$$\frac{\langle\langle T(t), \mathbf{1} \rangle\rangle \quad \langle\langle O(o), \mathbf{1} \rangle\rangle}{\langle\langle N(a), \mathbf{1} \rangle\rangle}$$

so that we have $S_3 \vdash_{\mathcal{B}_3} \langle\langle N(a), \mathbf{1} \rangle\rangle$ and $\Vdash_{\emptyset}^{S_3} \langle\langle N(a), \mathcal{B}_3, \mathbf{1} \rangle\rangle$.

l_4 (**passport control**): The only relevant information here is that carried by the passport (and that carried by the passenger, in his biometric form, which we do not here model separately). In information-processing terms, for assuring the aircraft is secure, this process could be carried out in parallel with those at l_1 and l_3 . It does have to be sequenced before the processes at l_5 and l_6 . We ignore other forms of information required to successfully cross passport control in practice, such as absence of various ‘flags’. We take site $S_4 = S_3 \cup \{\langle\langle N(a), \mathbf{1} \rangle\rangle\}$ and note that there is a channel $S_3 \xrightarrow{\lambda_4} S_4$.

The agent at this location makes use of all the information in the passport, and so base \mathcal{B}_4 has a rule

$$\frac{\langle\langle I^W(p), \mathbf{1} \rangle\rangle \quad \langle\langle I^S(p), \mathbf{1} \rangle\rangle}{\langle\langle P^S(p), \mathbf{1} \rangle\rangle}$$

generating an inferonic atom $\langle\langle P^S(p), \mathbf{1} \rangle\rangle$ to the effect that the passport passes these strong checks. This yields $S_4 \vdash_{\mathcal{B}_4} \langle\langle P^S(p), \mathbf{1} \rangle\rangle$ and therefore $\Vdash_{\emptyset}^{S_4} \langle\langle P^S(p), \mathcal{B}_4, \mathbf{1} \rangle\rangle$.

l_5 (**departure gate**): At the departure gate, an agent checks the ticket again. This, together with information that airside is secure and the passenger has passed through passport control, is taken to be sufficient to allow the passenger to board. (In practice, another passport check is often conducted.)

We take site $S_5 = S_4 \cup \{\langle\langle N(a), \mathbf{1} \rangle\rangle\}$ and \mathcal{B}_5 to include rule representing the ability of the agent at l_5 to make a decision for the passenger with p and t to board:

$$\frac{\langle\langle P^S(p), \mathbf{1} \rangle\rangle \quad \langle\langle T(t), \mathbf{1} \rangle\rangle \quad \langle\langle N(a), \mathbf{1} \rangle\rangle}{\langle\langle B(p, t), \mathbf{1} \rangle\rangle}$$

We have $\vdash_{\mathcal{B}_5}^{S_5} \langle\langle B(p, t), \mathbf{1} \rangle\rangle$ and $\Vdash_{\emptyset}^{S_5} \langle\langle B(p, t), \mathcal{B}_5, \mathbf{1} \rangle\rangle$.

l_6 (**aircraft**): A final process occurs in the cabin, where it is checked that the passengers boarded and baggage loaded match. This requires input of information from a lead member (purser) of the cabin crew and a lead staff member of the baggage-handling team. We describe a simplified version of such a process, in which the purser conducts the check using boarding information provided by the gate and baggage information from the baggage-handler. Rather than modelling the taking of the decision as something that consumes some information and produces new information, we rather focus on the fact that there is sufficient information in place to support such a decision.

We are not interested presently in the information produced by the purser, so for this model have a free choice for what to take as their base, \mathcal{B}_6 . We are interested in the information that they have available. We take site $S_6 := S_2 \cup S_5 \cup \{\langle\langle B(p, t), \mathbf{1} \rangle\rangle\}$. This does not include the inferons $\langle\langle L(h), \mathbf{1} \rangle\rangle$ and $\langle\langle A(h, t, p), \mathbf{1} \rangle\rangle$ generated by the baggage handler.

We have, by monotonicity, that $S_6 \vdash_{\mathcal{B}_2 \cup \mathcal{B}_6} \langle\langle B(p, t), \mathbf{1} \rangle\rangle$, that $S_6 \vdash_{\mathcal{B}_2 \cup \mathcal{B}_6} \langle\langle L(h), \mathbf{1} \rangle\rangle$, and that $S_6 \vdash_{\mathcal{B}_2 \cup \mathcal{B}_6} \langle\langle A(p, t, h), \mathbf{1} \rangle\rangle$. We therefore have $\Vdash_{\mathcal{B}_2}^{S_6} \langle\langle B(p, t), \mathcal{B}_6, \mathbf{1} \rangle\rangle \wedge \langle\langle L(h), \mathcal{B}_6, \mathbf{1} \rangle\rangle \wedge \langle\langle A(p, t, h), \mathcal{B}_6, \mathbf{1} \rangle\rangle$. Equivalently, $\Vdash_{\mathcal{B}_2}^{S_6} \langle\langle B(p, t) \wedge L(h) \wedge A(p, t, h), \mathcal{B}_6, \mathbf{1} \rangle\rangle$. The conjunction of three inferons is supported at the site S_6 using shared inferential capacity from the baggage handler (who uses base \mathcal{B}_2). Each of the three inferons involves the inferential capacity of the purser (\mathcal{B}_6) who accepts information passed along from the departure gate. The inferential capacity of the baggage-handler appears as the base \mathcal{B}_2 that provides context in the support relation. \square

In the view above, flows of information are not between distributed parts of an information system; rather they are between a part and another part containing it. However, we can use a technique similar to the methods of modelling systems using inf(er)omorphisms with a common codomain. Suppose we have a pair of channels $\mathfrak{P}_1 \rightsquigarrow \mathfrak{P}_3$ and $\mathfrak{P}_2 \rightsquigarrow \mathfrak{P}_3$. That is, $\mathfrak{P}_3 = \mathfrak{P}_1 \cup \mathfrak{P}'_1 = \mathfrak{P}_2 \cup \mathfrak{P}'_2$, for some \mathfrak{P}'_1 and \mathfrak{P}'_2 . Suppose ϕ and ψ are contradictory inferons: in particular $\Vdash_{\mathcal{B}}^{\mathfrak{P}_3} (\phi \wedge \psi) \supset \perp$. If $\Vdash_{\mathcal{B}}^{\mathfrak{P}_1} \phi$, then it cannot be the case that $\Vdash_{\mathcal{B}}^{\mathfrak{P}_2} \psi$.

Example 25 is modelled in a framework for understanding distributed systems, depicted more generally than for the airport security examples in Figure 12. It should be clear — at least in principle — that the approach adopted for modelling airport security suggests the availability of a more generic approach.

These are systems with physically distributed, interacting components, embedded in some operating environment (see, for example, [25, 56]). Interaction happens via flows of resources of some kind, where the flow of resources may be constrained by an interface. In the present paper, we are interested in the case where ‘resource’ is a form of information, flowing via ‘channels’ and with logical ‘constraints’ specifying the interface.

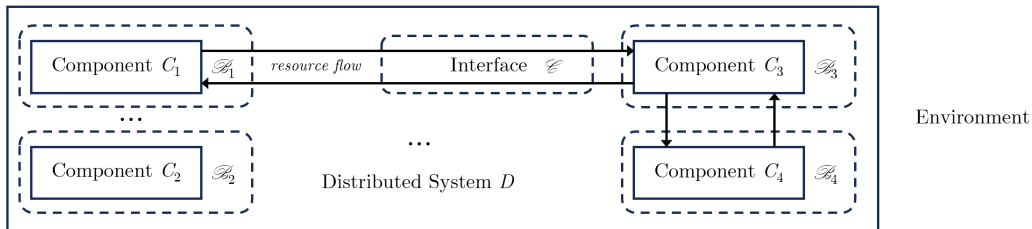


FIGURE 12. A generic distributed system (taken from [47], cf. [56])

Certain substructural logics are well-known to be useful in producing richer models than can be expressed purely in intuitionistic logic. For example, multiplicative linear logics have a number-of-uses reading, while bunched logics have a sharing interpretation. It is known how to give a base extension semantics to both the linear logic IMALL and the bunched logic BI [47, 53].⁸ In both cases, the support relation takes the form $\Gamma \Vdash^S \phi$ in which S has structure (and is a multiset or bunch) containing atoms, rather than simply a set as we have it for intuitionistic logic. The setup with sites and channels generalizes to these support relations. Sites consist of these structured collections of atoms, channels correspond to an evident operations on sites (which can be formulated as filling the hole in a context with a single hole), and constraints work by the respective versions of the Inf clause. We mention in particular the generalized form discussed in [47]:

$$(\text{Gen-Inf}) \quad \Gamma \Vdash_{\mathcal{B}}^{S(\cdot)} \phi \quad \text{iff} \quad \text{for all } \mathcal{C} \supseteq \mathcal{B} \text{ and all } U \in R(A), \text{ if } \Vdash_{\mathcal{C}}^U \Gamma, \text{ then } \Vdash_{\mathcal{C}}^{S(U)} \phi$$

where we would take $R(A)$ to be the set of sites and $S(\cdot)$ is a context with a hole. Work remains to be done in fully developing the underlying theory in the case of inferonic atoms and in exploring richer examples.

9. DISCUSSION

We have presented a first step towards an inferentialist account of information that is grounded in proof-theoretic semantics. Through such a theory we seek to capture the fundamentally inferential nature of information in a way that is both direct and a strong foundation for further development.

We can identify scope for further development in three main directions: metaphysics, logic, and systems. First, in metaphysics, we might seek to characterize the essence of the philosophy of information through employing the razor of proof-theoretic semantics. One important aspect, as mentioned in Section 2, is an inferentialist veridicality thesis and its influence on our understanding of information.

Second, in the logical theory, it is clear that there is a wide range of logical variants to be explored, including classical, second-order, substructural, modal, and epistemic logics of inferons. Comparisons with model-theoretic approaches such as discussed in [61, 73] may be useful. While the basic analyses of such logics in terms of base-extension semantics are now mostly available, their integration in logics of inferons remains to be explored. As indicated in Section 2, we foresee, in future work, substantive connections with van Benthem’s work on logical dynamics and information [86, 87]. Furthermore, the computational-logic-related ideas (in the context

⁸Note that BI has a model-theoretic semantics in terms of ternary relations (e.g., [49] and references therein), perhaps offering an interesting comparison with [61, 73].

of base-extension semantics, see [48]) of closed world assumptions, negation-as-failure, and their interaction assertion/denial in inferons would seem to offer a useful perspective. Moreover, the role of proof-relevance, and so proof-theoretic validity, in the setting of logics of inferons remains unexplored.

Third, the use of the ideas we have introduced as tools for modelling systems seems to offer both theoretical and pragmatic challenges. The starting point for an inferonic account of system modelling would be to capture systems at the level of generality suggested by Figure 12, supported by (Gen-Inf). The inferentialist account of information flow we have begun to develop would seem to be well-aligned with this objective, and would seem to have good prospects of integrating well with inferentialist accounts of the meaning of models, such as those suggested by Kuorikoski and Reijula [59] and Bueno [20].

Acknowledgements. This work has been partially supported by a gift from Amazon Web Services. We thank many of the attendees of the 6th Symposium on Proof-theoretic Semantics (London, February 2026) for their comments, to Johan van Benthem for his encouragement of our exploration of an inferentialist account of information, and to Tim Button and Gabriele Brancati for discussing these ideas with us. Pym is grateful to University College London for its support of his sabbatical leave at the University of London’s Institute of Philosophy, and to the Institute for its generous hosting.

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