Reductive Logic, Proof-search, and Coalgebra (Extended Abstract)*

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Traditionally, logic proceeds by inferring a conclusion from established premisses using *inference rules*. This is the paradigm of *deductive logic*:

$$\frac{\text{Established Premiss}_1 \quad \ldots \quad \text{Established Premiss}_n}{\text{Conclusion}} \Downarrow$$

The dual of deductive logic is the paradigm known as *reductive logic* (RL). Here one proceeds from a putative conclusion, called the *goal*, to a collection of premisses that suffice to witness the conclusion by means of a *reduction operator*,

$$\frac{\text{Sufficient Premiss}_1 \quad \ldots \quad \text{Sufficent Premiss}_n}{\text{Putative Conclusion}} \Uparrow$$

Reductions may correspond to inference rules, read from conclusion to premisses, or may have other forms (e.g., see [6]). The process of constructing a proof in RL is known as *proof-search*.

Reductive logic more closely resembles the way in which mathematicians actually prove theorems and, more generally, the way in which people solve problems, especially when using formal representations. For example, it encompasses diverse applications of logic to computer science such as, *inter alia*, logic programming (LP), program verification, and model-checking. Despite the ubiquity of reductive reasoning, it currently has little unified meta-theory. Developing a general metatheory of RL (i.e., proof theory and semantics, with results such as soundness and completeness) is an ongoing project. Some models have been considered, especially for classical and intuitionistic logic (IL) (e.g., see [7]).[1]

In general, the proof-search space for a goal can be regarded as a *state* space whose one-step dynamics is given by the reduction operators. It follows that an appropriate model of reduction is provided by a coalgebraic construction; specifically, let $\wp_f$ be the finite powerset functor, then reduction operators are coalgebras $\rho : \mathsf{GOALS} \to \wp_f \wp_f(\mathsf{GOALS})$. Using this perspective, the authors [2] have developed a general coalgebraic model of reduction in sequent calculi, generalizing earlier work in [4] on Horn clause LP (HcLP).

Let I be the identity function on GOALS, and Y_k and ρ_k be defined as follows: $Y_0 := \mathsf{GOALS}$, $Y_{\alpha+1} := \mathsf{GOALS} \times \wp_f \wp_f(\mathsf{Y}_\alpha)$ and $\rho_0 := I$, $\rho_{\alpha+1} := I \times \wp_f \wp_f(\rho_\alpha \circ \rho)$.

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[1] For example, uniform proof [5], while complete for the hereditary Harrop fragment of IL, does not specify an operational semantics (OS) for proof-search. Rather, it gives an RL basis relative to which an OS, specifying controls such as clause selection and backtracking, can be defined.

Let $\mathcal{C}$ be the co-free comonad of the $\wp_f \wp_f$ functor. The resulting coalgebra (e.g, see [1]) $\lambda : \mathsf{GOALS} \to \mathcal{C}(\mathsf{GOALS})$ maps a goal to its proof-search space (a more accurate *bi*-algebraic model was given for HcLP in [1]).

This coalgebraic semantics is a model of reduction, but not of *proof-search*. Here, proof-search is distinguished from reduction by a *control* régime determining precisely what reductions are made at what time. In general, control manifests as a *choice* (e.g., to backtrack). One control problem that can be handled in this coalgebraic semantics is *choice of premisses*. One applies a choice function σ after applying a reduction operator: $\mathsf{GOALS} \xrightarrow{\rho} \wp_f \wp_f(\mathsf{GOALS}) \xrightarrow{\sigma} \wp_f(\mathsf{GOALS})$ Choice could also be described using a structural OS for proof-search. In general, such systems admit co- and bi-algebraic models; see, for example, [8].

One approach to a general theory of control is to simulate proof-search in one logic as proof-search in another logic that is enriched by some algebra such that solutions to equations on the algebra represent various control choices; for example, this is the approach in [3] for the context-management problem of proof-search in linear, bunched, and relevant logics. At the abstract level, this approach could represent control as the algebra component of a bialgebra, whose coalgebra components are essentially the reduction operators provided. This coheres with the bialgebraic model of structural OS provided in [8] (regarding controls as constructors for explorations of a proof-search space).

In conclusion, RL proof-searches are important phenomena within philosophy, mathematics, and computing, but currently lack a uniform meta-theory. The proof-search space for a goal can be understood as a state space, for which coalgebra provides an suitably general technology for a mathematical theory of reduction. Further work is to characterize fully control in this setting.

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