Bifurcation Logic: Separation Through Ordering

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We introduce Bifurcation Logic, BL, which combines a basic classical modality with separating conjunction, $*$, together with its naturally associated multiplicative implication, $-\!\!*$, that is defined using the modal ordering. Specifically, a formula $\phi_1 * \phi_2$ is true at a world w if and only if each ϕ_i holds at worlds w_i that are each above w, on separate branches of the order, and have no common upper bound. We provide a labelled tableaux calculus for BL and establish soundness and completeness relative to its relational semantics. The standard finite model property fails for BL. However, we show that, in the absence of $-\!\!*$, but in the presence of $*$, every model has an equivalent finite representation and that this is sufficient to obtain decidability. We illustrate the use of BL through an example of modelling multi-agent access control that is quite generic in its form, suggesting many applications.

1 Introduction

We introduce Bifurcation Logic, BL, which combines a basic classical modality with separating conjunction, $*$, together with its naturally associated multiplicative implication, $-\!\!*$, that is defined using the modal ordering. We provide a relational semantics and a labelled tableaux calculus for BL and establish soundness and completeness relative to BL's relational semantics. The standard finite model property fails for BL. However, we show that, in the absence of $-\!\!*$, but in the presence of $*$, every model has a finite representation and that this is sufficient to obtain decidability. We illustrate the use of BL in logical modelling through an example of multi-agent access control.

The key property of interest in BL is the semantics of its multiplicatives, the 'separating' $*$ and $-\!\!*$, which is given in terms of the ordering that is used to define the classical modality. This stands in contrast to the set-up in, say, bunched implications (BI — e.g., [16, 18, 12]) in which specific relational structure is used for their definition — see [13] for a through discussion of BI's semantics. Kamide's account of Kripke semantics for modal substructural logics [15] also employs a binary operation on worlds to give a treatment of the multiplicative conjunction that is similar to that of BI's elementary semantics (e.g., [16, 18, 12, 13]). Galmiche, Kimmel, and Pym [10] consider an epistemic modal extension of boolean BI in which the semantics of the multiplicative conjunction employs a monoidal product on worlds. Došen [7] considers a range of issues in the relationship between modal and substructural logics from the perspective of translations between proof systems, and Ono [17] has also considered the proof theory of modal and substructural logics.

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The basic idea in BL is that a formula $\phi_1 * \phi_2$ is true at a world w if and only if each ϕ_i holds at worlds w_i that are each above w, on separate branches of the order, and have no common upper bound — that is, they are bifurcated. Consequently, the semantics of the multiplicative implication has the property that the implicational formula $\phi \mathrel{-\!\!*} \psi$ and its subformula ϕ are required to hold at bifurcated worlds above the world at which ψ holds. This use of this feature is illustrated in a substantive modelling example given in Section 3. The semantics of the classical connectives and modality is standard.

In Section 2, we introduce the language of Bifurcation Logic and its models, based on frames with a ternary relation structure for the bifurcation semantics. In Section 3, we give an extended, quite generic — i.e., evidently mappable to other settings — example of the use of BL in modelling access control, suggesting wider application in knowledge representation and reasoning. This example, albeit somewhat idealized, illustrates the interaction between the classical modality and the multiplicative connectives, especially the somewhat unusual semantic form of the implication, in a simple and direct way. We also discuss some related work in this section.

In Section 4, we give a system of labelled tableaux for BL. The form of the calculus follows the pattern established in, for example, [12, 10] and allows, in Section 5, soundness and completeness results to be established (cf. [12, 10]). The proofs are provided in the appendix.

While the usual finite model property fails for BL, a modified form of it does hold. Section 6 explains, through a counterexample, why the standard form fails and introduces a modified form through the concept of 'model with back links'. Intuitively, BL is a logic with the subformula property — in the sense evident from the tableaux system — and is about paths in finitely branching trees. Back links describe how the paths go back to already-seen configurations (see Section 6 for a formal explanation). Using this modified finite model property, the decidability of BL is obtained.

Finally, before proceeding to our formal development, we consider a few interesting outstanding questions, among many others, for further work:

- we would aim to give a (Hilbert-type) axiomatization of BL;
- we would explore natural deduction and sequent calculus presentations of BL;
- we would seek to establish the complexity of deciding BL;
- we would seek to explore the addition to BL of multi-agent and epistemic modalities, as well as quantifiers, so extending its potential as a modelling tool (cf. [10], for example).

The question of whether there is an interesting intuitionistic version of BL seems quite challenging, as it would combine the difficulties of intuitionistic modal logics [21] with the need to handle the multiplicatives in a coherent way.

2 Bifurcation Logic

In this section, we introduce Bifurcation Logic, BL, by giving a definition in terms of ternary relational semantics in the style of Routley-Meyer [20], with some similarity to the work of Fuhrmann and Mares [9].

Definition 1 (Language)**.** *Let P be a countable set of propositional letters. The formulae of BL, the set of which is denoted by Φ, are given by the following grammar:*

$$\phi ::= p (\in P) \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid \phi * \phi \mid \phi \mathrel{-\!\!*} \phi$$

The connectives $\vee$, $\rightarrow$, $\leftrightarrow$, $\Box$ *and the units* $\top$, $\bot$ *are defined as follows:* $\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$, $\phi \rightarrow \psi = \neg\phi \vee \psi$, $\phi \leftrightarrow \psi = (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, $\top = \phi \vee \neg\phi$, $\bot = \phi \wedge \neg\phi$, $\Diamond\phi = \neg\Box\neg\phi$. □

To minimize the use of parentheses; we use the following strict order of precedence (with right associativity): $\Box, \Diamond, \neg > * > \wedge, \vee > -\!\!* > \rightarrow > \leftrightarrow$.

Definition 2 (Tree)**.** *Let* $(W, \leq)$ *be a partial order. Two elements* $w_1, w_2 \in W$ *are* separated (or disjoint)*, denoted* $w_1 \bot w_2$*, if neither* $w_1 \leq w_2$ *nor* $w_2 \leq w_1$*. We call* $(W, \leq)$ rooted *if there exists* $w \in W$ *such that* $w \leq w'$ *for all* $w' \in W$*.* $(W, \leq)$ *has the* persistent separation property *if it satisfies the following condition:*

$$(P) \quad \textit{for all } w_1, w_2, w_1' \in W, \textit{ if } w_1 \bot w_2 \textit{ and } w_1 \leq w_1', \textit{ then } w_1' \bot w_2.$$

$(W, \leq)$ *is called a* tree *if it is rooted and has the persistent separation property.* □

Definition 3 (Frame)**.** *A BL* frame *is a structure* $\mathcal{F} = (W, \leq, R)$*, where* $(W, \leq)$ *is a tree of elements called* worlds *and* R *is the ternary relation on worlds defined as follows:*

$$(B) \quad \textit{for all } w, w_1, w_2 \in W,\ R(w, w_1, w_2) \textit{ iff } w \leq w_1, w \leq w_2 \textit{ and } w_1 \bot w_2.$$

That is, w_1 *and* w_2 *belong to distinct futures of* w*.* □

Definition 4 (Model)**.** *A BL* model *is a triple* $\mathcal{M} = (\mathcal{F}, V, \Vdash)$*, where* $\mathcal{F}$ *is a BL frame and* V *is a* valuation function *from* W *to* $\wp(P)$*. The satisfaction relation* $\Vdash$ *is inductively defined as the smallest relation on* $W \times \Phi$ *such that*

$$\begin{array}{rll}
\mathcal{M}, w \Vdash p & \textit{iff} & p \in V(w), \textit{ for all } p \in P \\
\mathcal{M}, w \Vdash \neg\phi & \textit{iff} & \mathcal{M}, w \nVdash \phi \\
\mathcal{M}, w \Vdash \phi \wedge \psi & \textit{iff} & \mathcal{M}, w \Vdash \phi \textit{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \Box\phi & \textit{iff} & \textit{for all } w' \in W, \textit{ if } w \leq w' \textit{ then } \mathcal{M}, w' \Vdash \phi \\
\mathcal{M}, w \Vdash \phi * \psi & \textit{iff} & \textit{for some } w_1, w_2 \in W \textit{ such that } R(w, w_1, w_2), \\
 & & \mathcal{M}, w_1 \Vdash \phi \textit{ and } \mathcal{M}, w_2 \Vdash \psi \\
\mathcal{M}, w \Vdash \phi -\!\!* \psi & \textit{iff} & \textit{for all } w_1, w_2 \in W \textit{ such that } R(w_2, w, w_1), \\
 & & \textit{if } \mathcal{M}, w_1 \Vdash \phi \textit{ then } \mathcal{M}, w_2 \Vdash \psi
\end{array}$$

A formula ϕ *is* satisfied in a model $\mathcal{M}$*, denoted* $\mathcal{M} \Vdash \phi$*, if* $\mathcal{M}, w \Vdash \phi$ *for all worlds* w *in* $\mathcal{M}$*. We write* $w \Vdash \phi$ *instead of* $\mathcal{M}, w \Vdash \phi$ *whenever the model is clear from the context.* ϕ *is* satisfiable *if it is satisfied in some model* $\mathcal{M}$*, and* valid*, denoted* $\Vdash \phi$*, if it is satisfied in all models.* □

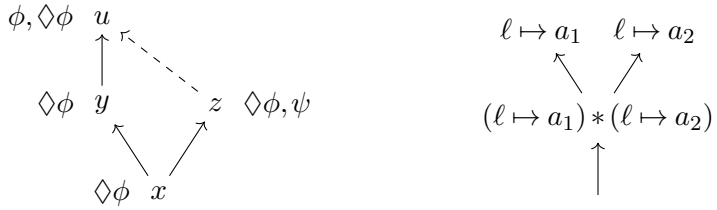


Figure 1: Examples of BL structures

BL uses frames that obey the persistent separation property as we believe that they better correspond to an intuitive understanding of bifurcation. We then define non-persistent (or lax)

BL as the extension of BL that deals with frames that are not required to obey the persistent separation property. For example, the formula $\varphi = (\Diamond\phi * \psi) \leftrightarrow (\phi * \psi)$ is valid in BL (a tableau proof is given in Section 4), but it is not valid in lax BL as it can be falsified in the direct acyclic graph (DAG) given on the left-hand side of Figure 1. Indeed, we have $x \Vdash \Diamond\phi * \psi$ because we have $R(x, y, z)$, $y \Vdash \Diamond\phi$ and $z \Vdash \psi$, but we do not have $x \Vdash \phi * \psi$ because the only world satisfying ϕ is u and u does not belong to a distinct future of z, which is the only world satisfying ψ.

Proposition 5. *BL is a conservative extension of the modal logic $S4$.*

Proof. This follows immediately from observing that the only conditions imposed on the order relation are for the purpose of defining $*$ and $-\!\!*$. □

Although BL is a conservative extension of $S4$, it differs from both bunched and other separating logics. First, we remark that since BL addresses separation as an ordering problem rather than a resource composition problem (as in, for example, bunched logics, where $*$ usually corresponds to a product in a monoid), we do not include a unit $\top^*$ for the multiplicative conjunction $*$. Having the multiplicative unit $\top^*$ would unnecessarily complicate our definition of the bifurcation relation or rule out many partial order structures. Indeed, we would need to satisfy $w \Vdash \phi * \top^*$ iff $w \Vdash \phi$ for all worlds w and all formulae ϕ. Hence, by definition of $*$, we would need worlds w_1, w_2 such that $R(w, w_1, w_2)$, $w_1 \Vdash \phi$ and $w_2 \Vdash \top^*$. In particular, consider a BL frame with only one world w and set $\phi = \top$. We have $w \Vdash \top$, but not $w \Vdash \top * \top^*$ because $R(w, w, w)$ is impossible to achieve as it would imply both $w \leq w$ and $w \not\leq w$ by definition of R.

Second, BL also differs from Separation Logic, as illustrated in the right-hand side of Figure 1. In Separation Logic [14, 19], the built-in points-to predicate $(\ell \mapsto a)$ intuitively denotes a memory heap with only one cell whose location (address) is ℓ and whose value is a. Heaps are defined as partial functions from locations to values and composition of heaps is given by the union of functions with disjoint domains. Therefore, the formula $(\ell \mapsto a_1) * (\ell \mapsto a_2)$ is not satisfiable in Separation Logic as it denotes the disjoint composition of two one-cell heaps that share the same location ℓ. In BL, as $*$ represents bifurcation, a node satisfying $(\ell \mapsto a_1) * (\ell \mapsto a_2)$ simply implies that the location ℓ might have two distinct futures, one in which it points to the value a_1 and the other one in which it points to the value a_2. More interestingly, the formula $\Box(\ell \mapsto a)$, when satisfied by some world w, would imply that in all possible futures of w, the location ℓ should point to the same value a. This would be useful, for example, to state that an interrupt vector always points to the address where its legitimate handler resides.

3 Modelling With Bifurcation Logic

Logics can be used not merely to describe reasoning itself, but also to describe reasoning about systems. This use of logics as modelling tools has delivered significant advances in many areas — too numerous to describe here — including program analysis and verification, with one leading example, making essential use of multiplicative conjunction, being Separation Logic [14, 19]. Another highly effective example in the same spirit is Context Logic [3, 4]. More abstractly, substructural modal logics provide reasoning tools for models of systems in the 'distributed systems metaphor' (e.g., [1, 5, 11]). Demri and Deter [6] have surveyed connections between modality and separation, but they do not consider separating connectives defined through ordering. More detailed connections with Separation Logic, as mentioned above, are beyond the scope of this paper; so, instead of describing how connections with that might work, we give a quite generic

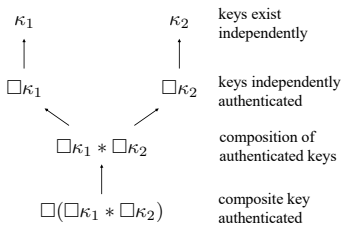


Figure 2: Joint access

example — many examples involving obtaining, representing, and verifying knowledge and information will be very similar — of how BL's modality and multiplicatives interact through an example in the context of multi-agent access control.

Example 6 (Crimson Tide [22] Part 1)**.** *The plot of the film* Crimson Tide *[22] takes place mainly on board a United States Navy Ballistic Missile Submarine (the USS Alabama). During a period of international tension, the submarine receives an order to launch nuclear-armed missiles. For the release of the weapons to be authorized, a composite key must be authenticated:*

1. *An order to release weapons is received in a message that contains a code.*
2. *Two senior officers (neither the Captain nor the Executive Officer) must independently authenticate the message by verifying the code against a local authentication device. Authentication (think 'necessarily correct') is denoted using* $\Box$*.*
3. *This yields a composite —* $\Box\kappa_1 * \Box\kappa_2$ *in Figure 2 — of authenticated keys that is passed (verbally) to the Executive Officer and the Captain.*
4. *The Executive Officer confirms that the correct protocol has been followed and so authenticates the composite key, thereby yielding* $\Box(\Box\kappa_1 * \Box\kappa_2)$ *in Figure 2.*
5. *The Captain then confirms the authentication and may order the release of the weapons.*

This concludes the first part of our example. $\Box$

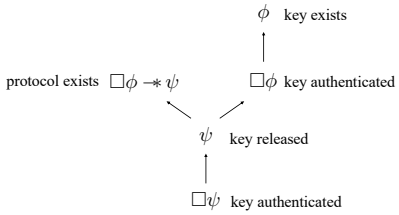


Figure 3: Protocol for obtaining a key

Example 7 (Crimson Tide [22] Part 2)**.** *The use of a protocol can also be represented in Bifurcation Logic. This is illustrated in Figure 3. Here the idea is that a protocol is modelled by an implicational formula that, given an authenticated key, yields a key, which may then need to be authenticated. (Informally, the implicational formula may be thought of as the type of a function that returns a key.) We can see how in an inessential, slightly more detailed, variant of the set-up, Figure 4 represents the use of protocols in* Crimson Tide *as follows:*

1. *In the discussion based on Figure 2, the role of protocols is suppressed.*

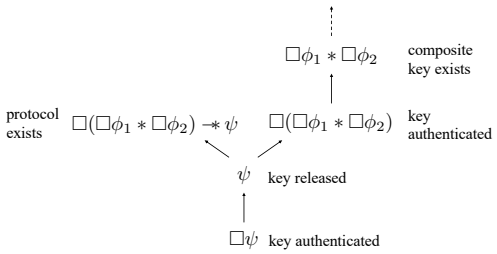


Figure 4: Protocol for obtaining a composite key

2. *Figure 3 illustrates the use of protocols in general.*
3. *In the case of our example, we represent the Captain's key permitting the release of weapons by ψ. For the key to be authenticated, it must have been released through the protocol.*
4. *The protocol is given, in Figure 4, by the implicational formula $\Box(\Box\phi_1 * \Box\phi_2) \mathrel{-\!\!*} \psi$. That is, the authenticated composite key allows access to the Captain's key, ψ, which, when authenticated, allows the weapons to be released.*

Here we use the modality $\Box$ to denote authentication, but what is the role of separation? The role of $*$ should be clear: it enforces the independence of multiple authentications. But what of $\mathrel{-\!\!*}$? It ensures that there is no interference between the authentication of the composite key and its use to make available the Captain's key, ψ (which must itself be authenticated for use). Note the essential use of the semantics of $\mathrel{-\!\!*}$: first, the separation between worlds afforded by $\mathrel{-\!\!*}$, as opposed to $\rightarrow$, ensures no interference between the existence of the protocol and existence of the (authenticated) key to which it applies. Second, the particular semantic form of this implication — in that its application 'looks back' down the ordering — captures exactly the release of the Captains's key, ψ, through access to the authenticated composite key. □

4 A Tableaux Calculus for BL: T_{BL}

The tableaux calculus for classical propositional logic [8] can be adapted systematically to calculi for many non-classical logics by the addition of labelling. The basic idea is that the structure of a Kripke model for a given non-classical logic is used to define a tableaux calculus for that logic by reflecting its structure in an algebra of labels that is used to impose side-conditions on the tableaux rules. Through this mechanism, the basic classical and/or tableaux figures are modified to capture non-classical connectives.

T_{BL} is presented in Figure 5. T_{BL} has logical rules that capture the meaning of the connectives, structural rules that capture the properties of BL models, and closure rules (whose conclusion is a cross mark) that capture (logical or structural) inconsistencies. As usual, closure rules put an end to the expansion of a branch. T_{BL} can address either BL or its lax variant depending on the inclusion or not of the optional persistency rule $\| P$. Let us remark that all of the results presented in this section for BL also hold for lax BL.

Definition 8. *Let L be a countable set of symbols called* labels. *A* labelled formula *is a pair (ϕ, x), written $\phi{:}x$, where ϕ is a formula and x is a label. A* label constraint *is an expression of the form $x \prec y$, where x, y are labels.* □

Definition 9. *Let Sg be the set $\{T, F\}$ of* signs. *A* signed labelled formula *is a triple (S, ϕ, x), written $S\,\phi{:}x$, where S is a sign and $\phi{:}x$ is a labelled formula. Similarly, a* signed label constraint *is a label constraint prefixed with a sign.* □

We define $T\,x \sim y$ as a shorthand for the expression $T\,x \prec y, T\,y \prec x$. Similarly, $T\,x \parallel y$ is a shorthand for $F\,x \prec y, F\,y \prec x$ and $T\,x \prec y \parallel z$ is a shorthand for $T\,x \prec y, T\,x \prec z, T\,y \parallel z$.

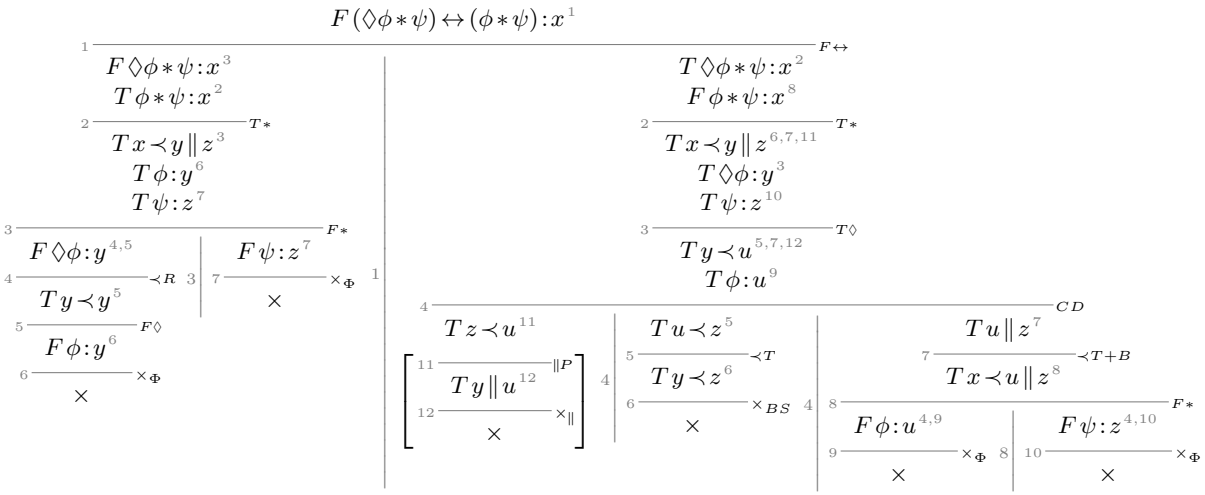
Definition 10. *A* tableau for $\phi{:}x$ *is a finitely branching rooted tree built inductively according to the rules given in Figure 5 and the root node of which is the signed labelled formula $F\,\phi{:}x$.* □

Definition 11. *A tableau branch b is* closed *if it ends with a closure rule. A tableau t is* closed *if all of its branches are closed.* □

Definition 12. *Let $\phi{:}x$ be a labelled formula. A T_{BL}-*proof of $\phi{:}x$ *is a* closed tableau for $\phi{:}x$. *We write $\vdash \phi{:}x$ if $\phi{:}x$ is* provable in T_{BL}, *that is if there exists a T_{BL}-proof of $\phi{:}x$. Similarly, a formula ϕ is* provable *in T_{BL}, written $\vdash \phi$, if $\vdash \phi{:}x$ for some label x.* □

Example 13. *The tableau depicted on the left-hand side of Figure 6 is a T_{BL}-proof of $\phi * \psi \to \Diamond\phi \wedge \Diamond\psi$. Step 3 is only given to improve readability and is not really necessary as it is just the explicit expansion of the shorthand for $T\,x \prec y \parallel z$.* □

Example 14. *A T_{BL}-proof of $\varphi \equiv (\Diamond\phi * \psi) \leftrightarrow (\phi * \psi)$ is given in the tableau below.*



After Step 1, the tableau splits into two parts, the interesting one being the second one (on the right-hand side of the first vertical rule) that eventually leads to four closed branches. Step 4 is an example of the case distinction rule. We remark that the first case only leads to a closed branch when the optional rule $\parallel P$ is used (resulting in the optional steps Step 11 *and Step* 12*). Indeed, without persistence of separation, one can build a countermodel of φ as illustrated in Figure 1. Hence, φ is a formula that distinguishes persistent from non-persistent BL.* □

Example 15. *The tableau depicted on the right-hand side of Figure 6 is an example of an infinite open tableau for $\neg\Box(\top * \top)$. The signed formula $T\,\Box(\top * \top) : x$ introduced in Step 1 must be expanded for all labels u such that $T\,x \prec u$ occurs in the branch. The first such label is x, so that $T\,\top * \top : x$ is introduced in Step 3. The expansion of $T\,\top * \top : x$ in Step 4 generates two new successors of x, namely y_1, z_1, and then induces Step 5, where two new expansions of*

Logical rules

$$\frac{F\neg\phi{:}x}{T\phi{:}x}\,F\neg \qquad \frac{T\neg\phi{:}x}{F\phi{:}x}\,T\neg \qquad \frac{T\phi\wedge\psi{:}x}{\begin{array}{l}T\phi{:}x\\T\psi{:}x\end{array}}\,T\wedge \qquad \frac{F\phi\wedge\psi{:}x}{F\phi{:}x \mid F\psi{:}x}\,F\wedge$$

$$\frac{F\Box\phi{:}x}{\begin{array}{l}Tx\prec u\\F\phi{:}u\end{array}}\,F\Box \qquad \frac{\begin{array}{l}T\Box\phi{:}x\\Tx\prec y\end{array}}{T\phi{:}x}\,T\Box$$

$$\frac{T\phi*\psi{:}x}{\begin{array}{l}Tx\prec u\,\|\,v\\T\phi{:}u\\T\psi{:}v\end{array}}\,T* \qquad \frac{\begin{array}{l}F\phi*\psi{:}x\\Tx\prec y\,\|\,z\end{array}}{F\phi{:}y \mid F\psi{:}z}\,F* \qquad \frac{\begin{array}{l}T\phi\mathrel{-\!\!*}\psi{:}x\\Tz\prec x\,\|\,y\end{array}}{F\phi{:}y \mid T\psi{:}z}\,T\mathrel{-\!\!*} \qquad \frac{F\phi\mathrel{-\!\!*}\psi{:}x}{\begin{array}{l}Tv\prec x\,\|\,u\\T\phi{:}u\\F\psi{:}v\end{array}}\,F\mathrel{-\!\!*} \qquad \frac{\begin{array}{l}S\phi{:}x\\Tx\sim y\end{array}}{S\phi{:}y}\,S\sim$$

Structural rules

$$\frac{S\phi{:}x,\ S\psi{:}y}{Tx\prec y \mid Ty\prec x \mid Tx\|y}\,CD \qquad \frac{S\phi{:}x}{Tx\prec x}\,{\prec}R \qquad \frac{\begin{array}{l}Tx\prec y\\Ty\prec z\end{array}}{Tx\prec z}\,{\prec}T \qquad \left[\frac{\begin{array}{l}Tx_1\,\|\,x_2\\Tx_i\prec y\end{array}}{Tx_j\,\|\,y}\,\|P\right]$$

$$\frac{Tx\|z}{Fx\prec z,\ Fz\prec x}\,\| \qquad \frac{Tx\prec y\|z}{Tx\prec y,\ Tx\prec z,\ Ty\|z}\,B \qquad \frac{\begin{array}{l}Tx\prec y\\Ty\prec x\end{array}}{Tx\sim y}\,\sim$$

Closure rules

$$\frac{\begin{array}{l}T\phi{:}x\\F\phi{:}x\end{array}}{\times}\,\times_\Phi \qquad \frac{\begin{array}{l}Tx\prec y\\Fx\prec y\end{array}}{\times}\,\times_\prec$$

Derivable rules

$$\frac{F\phi\vee\psi{:}x}{\begin{array}{l}F\phi{:}x\\F\psi{:}x\end{array}}\,T\vee \qquad \frac{T\phi\vee\psi{:}x}{T\phi{:}x \mid T\psi{:}x}\,T\vee \qquad \frac{T\phi\to\psi{:}x}{F\phi{:}x \mid T\psi{:}x}\,T\to \qquad \frac{F\phi\to\psi{:}x}{\begin{array}{l}T\phi{:}x\\F\psi{:}x\end{array}}\,F\to$$

$$\frac{T\phi\leftrightarrow\psi{:}x}{\begin{array}{l|l}T\phi{:}x & F\phi{:}x\\T\psi{:}x & F\psi{:}x\end{array}}\,T\leftrightarrow \qquad \frac{F\phi\leftrightarrow\psi{:}x}{\begin{array}{l|l}F\phi{:}x & T\phi{:}x\\T\psi{:}x & F\psi{:}x\end{array}}\,F\leftrightarrow \qquad \frac{T\Diamond\phi{:}x}{\begin{array}{l}Tx\prec u\\T\phi{:}u\end{array}}\,T\Diamond \qquad \frac{\begin{array}{l}F\Diamond\phi{:}x\\Tx\prec y\end{array}}{F\phi{:}x}\,F\Diamond \qquad \frac{Tx\|y}{Ty\|x}\,\|C$$

$$\frac{F\top{:}x}{\times}\,\times_\top \qquad \frac{T\bot{:}x}{\times}\,\times_\bot \qquad \frac{\begin{array}{l}Tx_1\,\|\,x_2\\Tx_i\prec x_j\end{array}}{\times}\,\times_\| \qquad \frac{\begin{array}{l}Tx\prec y_1\,\|\,y_2\\Fx\prec y_i\end{array}}{\times}\,\times_{BA} \qquad \frac{\begin{array}{l}Tx\prec y_1\,\|\,y_2\\Ty_i\prec y_j\end{array}}{\times}\,\times_{BS}$$

Side-conditions and comments

- In rules introducing u and/or v, u and v are distinct labels that are fresh in the branch.
- In rules involving i and/or j, $i\in\{1,2\}$ and $j=3-i$.
- Double lines indicate rules that can be used bottom-up or top-down.
- The brackets indicate that the rule is optional (included for BL, excluded for lax BL).

Figure 5: Rules of the T_{BL} calculus

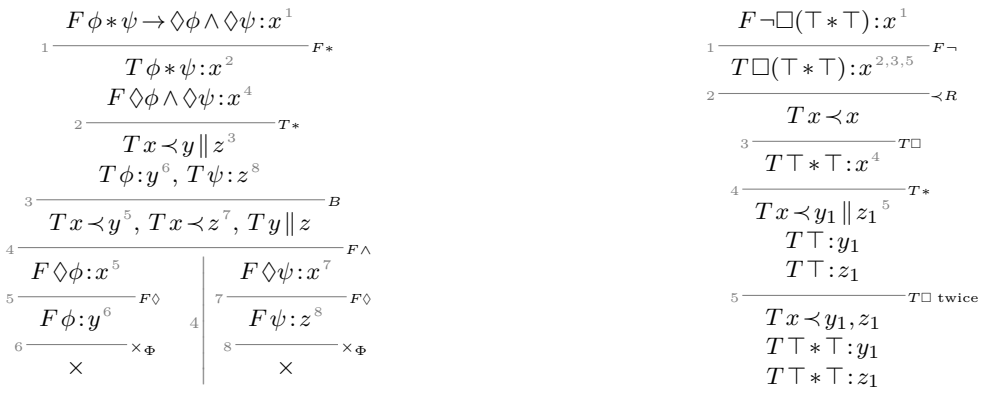


Figure 6: Tableaux examples

$T\,\Box(\top * \top) : x$ *are performed (for simplicity, in one step for both* y_1 *and* z_1*). Step 5 results in the introduction of* $T\,\top * \top : y_1$ *and* $T\,\top * \top : z_1$*, the expansion of which creates four new successors of* x*, two of* y_1 *and two of* z_1*. The whole process described previously then repeats itself infinitely often as the BL models such that* $x \Vdash \Box(\top * \top)$ *are the ones in which all sucessors of* x *have at least two distinct successors. This point is further discussed at the beginning of Section 6.* □

5 Soundness and Completeness

Definition 16. *The* domain of a tableau branch b*, denoted* $D(b)$*, is the set* $\{\, x \mid (S\,\phi : x) \in b \,\}$ *of all labels occurring in* b*.* □

Definition 17 (Realization)**.** *Let* b *be a tableau branch. A* realization *of* b *in a BL model* $\mathcal{M} = (W, \leq, R, V, \Vdash)$ *is a function* ρ *from* $D(b)$ *to* W *such that*

- *if* $(T\,\phi : x) \in b$*, then* $\rho(x) \Vdash \phi$ *and if* $(F\,\phi : x) \in b$*, then* $\rho(x) \nVdash \phi$
- *if* $(T\,x \prec y) \in b$*, then* $\rho(x) \leq \rho(y)$*, and if* $(F\,x \prec y) \in b$*, then* $\rho(x) \not\leq \rho(y)$*.*

A tableau branch is realizable *if has at least one realization in some BL model. A tableau is* realizable *if has at least one realizable branch.* □

Lemma 18. *If a tableau branch is closed, then it is not realizable.*

Proof. Let b be a closed tableau branch. Assume that b is realizable. Then, there exists a realization ρ of b in some BL model.

- If b is closed because both $(T\,\phi : x) \in b$ and $(F\,\phi : x) \in b$, then by definition of a realization, we have both $\rho(x) \Vdash \phi$ and $\rho(x) \nVdash \phi$, which is a contradiction.
- If b is closed because both $(T\,x \prec y) \in b$ and $(F\,x \prec y) \in b$, then by definition of a realization, we have both $\rho(x) \leq \rho(y)$ and $\rho(x) \not\leq \rho(y)$, which is a contradiction.

Therefore, b cannot be realizable. □

Lemma 19. *If a tableau* t *is realizable, then expanding* t *using one of the tableau expansion rules given in Figure 5 results in a realizable tableau* t'*.*

Proof. Suppose that t is realizable. Then, it has at least one realizable branch b. If t' is obtained from t by expanding a branch that is distinct from b then t' remains realizable since it still contains the unchanged realizable branch b. Otherwise, t' is obtained by expanding b into b'. We then proceed by case analysis on the tableau rule expanding b. Let ρ be a realization of b is some BL model.

We consider just a few illustrative cases, the others being similar.

– Case $T{-\!\!*}$:

If $(T\, z \prec x \,\|\, y) \in b$ and $(T\, \phi -\!\!* \psi : x) \in b$, then by Definition 17, we get $R(\rho(z), \rho(x), \rho(y))$ and $\rho(x) \Vdash \phi -\!\!* \psi$. It then follows from Definition 4 that $\rho(y) \Vdash \phi$ and $\rho(z) \nVdash \psi$. Therefore, ρ realizes b'.

– Case $F{-\!\!*}$:

If $(F\, \phi -\!\!* \psi : x) \in b$, then by Definition 17, we get $\rho(x) \nVdash \phi -\!\!* \psi$. By Definition 4, there exist $w_1, w_2 \in W$ such that $R(w_2, \rho(x), w_1)$, $w_1 \Vdash \phi$ and $w_2 \nVdash \psi$. Since b' extends b with $(T\, v \prec x \,\|\, u)$, $(T\, \phi : u)$ and $(F\, \psi : v)$, where u and b are distinct fresh labels, we can extend ρ into a realization of b' by setting $\rho(u) = w_1$ and $\rho(v) = w_2$.

The other cases are similar. □

Theorem 20 (Soundness)**.** *If $\vdash \phi$, then $\vDash \phi$.*

Proof. If $\vdash \phi$ then we have closed tableau t for $\phi : x$ for some label x. Assume that $\nvDash \phi$. Any tableau construction procedure that results in t begins with the tableau t_0 that consists in the single node $F\, \phi : x$. Since t_0 is realizable, Lemma 19 implies that t should also be realizable and should therefore contain at least one realizable branch b. By Lemma 18, b cannot be closed. It then follows that t is open, which is a contradiction. Thus, we have $\vDash \phi$. □

Definition 21. *A tableau branch b is* saturated *if it satisfies all of the following conditions:*

1. *if $(S\, \phi : x) \in b$, then $(T\, x \prec x) \in b$*
2. *if $(S\, \phi : x, S\, \psi : y) \in b$, then $(T\, x \prec y) \in b$ or $(T\, y \prec x) \in b$ or $(T\, x \,\|\, y) \in b$*
3. *if $(T\, x \prec y) \in b$ and $(T\, y \prec z) \in b$, then $(T\, x \prec z) \in b$*
4. *if $(T\, x \sim y) \in b$ and $(S\, \phi : x) \in b$, then $(S\, \phi : y) \in b$*
5. *if $(T\, x \,\|\, y) \in b$ and $(T\, x \prec z) \in b$, then $(T\, z \,\|\, y) \in b$*
6. *if $(T\, \neg\phi : x) \in b$, then $(F\, \phi : x) \in b$*
7. *if $(F\, \neg\phi : x) \in b$, then $(T\, \phi : x) \in b$*
8. *if $(T\, \phi \wedge \psi : x) \in b$, then $(T\, \phi : x) \in b$ and $(T\, \psi : x) \in b$*
9. *if $(F\, \phi \wedge \psi : x) \in b$, then $(F\, \phi : x) \in b$ or $(F\, \psi : x) \in b$*
10. *if $(T\, \Box\phi : x) \in b$, then for all $y \in D(b)$, if $(T\, x \prec y) \in b$, then $(T\, \phi : y) \in b$*
11. *if $(F\, \Box\phi : x) \in b$, then for some $y \in D(b)$, $(T\, x \prec y) \in b$ and $(F\, \phi : y) \in b$*
12. *if $(T\, \phi * \psi : x) \in b$, then, for some $y, z \in D(b)$, $(T\, x \prec y \,\|\, z) \in b$, $(T\, \phi : y) \in b$, and $(T\, \psi : z) \in b$*
13. *if $(F\, \phi * \psi : x) \in b$, then, for all $y, z \in D(b)$, if $(T\, x \prec y \,\|\, z) \in b$, then $(F\, \phi : y) \in b$ or $(F\, \psi : z) \in b$*
14. *if $(T\, \phi -\!\!* \psi : x) \in b$, then, for all $y, z \in D(b)$, if $(T\, z \prec x \,\|\, y) \in b$, then $(F\, \phi : y) \in b$ or $(T\, \psi : z) \in b$*
15. *if $(F\, \phi -\!\!* \psi : x) \in b$, then, for some $y, z \in D(b)$, $(T\, z \prec x \,\|\, y) \in b$, $(T\, \phi : y) \in b$, and $(F\, \psi : z) \in b$.* □

Lemma 22. *Let b be a saturated open tableau branch. The binary relation $\prec_b$ over $D(b)$ defined as $x \prec_b y$ iff $(T\, x \prec y) \in b$ is a quasi-order over $D(b)$ such that if $(F\, x \prec y) \in b$ then $x \not\prec_b y$.*

Proof. Transitivity and reflexivity of $\prec_b$ clearly follow from Conditions 1 and 3 of Definition 21. Now, if $(F\,x \prec y) \in b$, then $(T\,x \prec y) \notin b$ because b is open. Hence, $x \not\prec_b y$ by definition of $\prec_b$. □

Lemma 23. *Let b be a saturated open branch. Let $\sim_b$ be the equivalence relation over $D(b)$ defined as $x \sim_b y$ iff $x \prec_b y$ and $y \prec_b x$. For all $x \in D(b)$, let $[x] = \{\, y \in D(b) \mid x \sim_b y \,\}$ denote the equivalence class of x under $\sim_b$ and let W_b denote the quotient $D(b)/\sim_b$. Then, the binary relation $\leq_b$ over W_b defined as $[x] \leq_b [y]$ iff $x \prec_b y$ is a partial order over W_b that satisfies the persistent separation property.*

Proof. First, we show that $\leq_b$ is well defined by showing that the following conditions are equivalent for all $[u], [v] \in W_b$:

(*a*) $u \prec_b v$

(*b*) $x \prec_b y$ for all $x \in [u]$ and all $y \in [v]$

(*c*) $x \prec_b y$ for some $x \in [u]$ and some $y \in [v]$

It is clear that (*a*) implies (*c*) and that (*b*) implies (*c*) (and (*a*)). Therefore, we only have to show that (*c*) implies (*b*). Assume $x \prec_b y$ for some $x \in [u]$ and some $y \in [v]$. Then, $x \sim_b u$ implies $u \prec_b x$ and $y \sim_b v$ implies $y \prec_b v$. Pick an arbitrary $x' \in [u]$, then $x' \sim_b u$ implies $x' \prec_b u$. Since $u \prec_b x$, we get $x' \prec_b x$. Pick an arbitrary $y' \in [v]$, then $y' \sim_b v$ implies $v \prec_b y'$ Since $y \prec_b v$, we get $y \prec_b y'$. Finally, since we assumed $x \prec_b y$, $x' \prec_b x$ and $y \prec_b y'$ imply $x' \prec_b y'$.

Second, we show that $\leq_b$ is a partial order over W_b. Since $\prec_b$ is a quasi-order over $D(b)$ by Lemma 22, it immediately follows that $\leq_b$ is both reflexive and transitive. It remains to show that $\leq_b$ is anti-symmetric. Assume $[x] \leq_b [y]$ and $[y] \leq_b [x]$, then, by definition of $\leq_b$, we have $x \prec_b y$ and $y \prec_b x$, from which we get $x \sim_b y$ by definition of $\sim_b$. Hence, $[x] = [y]$.

Last, we show that $\leq_b$ satisfies the separation persistence property stated in Definition 2. We pick arbitrary $[x], [y], [z] \in W_b$ such that $[x] \perp [y]$ and $[x] \leq_b [z]$ and show that $[z] \perp [y]$. We have the following facts:

(*i*) By definition of $\perp$, $[x] \perp [y]$ implies $[x] \not\leq_b [y]$ and $[y] \not\leq_b [x]$.

(*ii*) By definition of $\leq_b$, $[x] \leq_b [z]$ implies $x \prec_b z$.

Assume that $[z] \leq_b [y]$. Then, $z \prec_b y$ by definition of $\leq_b$. Since $x \prec_b z$ and $z \prec_b y$ imply $x \prec_b y$, we get $[x] \leq_b [y]$ by definition of $\leq_b$, which contradicts $[x] \perp [y]$ (*i*). Hence, $[z] \not\leq_b [y]$.
Assume that $[y] \leq_b [z]$. Then, $y \prec_b z$ by definition of $\leq_b$, from which we get $(T\,y \prec z) \in b$ by definition of $\prec_b$. Besides, $[x] \not\leq_b [y]$ and $[y] \not\leq_b [x]$ imply $x \not\prec_b y$ and $y \not\prec_b x$ by definition of $\leq_b$. By definition of $\prec_b$, we get $(T\,x \prec y) \notin b$ and $(T\,y \prec x) \notin b$. Since b is saturated, $(T\,x \parallel y) \in b$ then follows from Condition 2 of Definition 21. In turn, $(T\,x \parallel y) \in b$ and $(T\,y \prec z) \in b$ imply $(T\,x \parallel z) \in b$ by Condition 5 of Definition 21. $(T\,x \parallel z) \in b$ implies $(F\,x \prec z) \in b$ by definition of $\parallel$, but from $(F\,x \prec z) \in b$, Lemma 22 implies $x \not\prec_b z$, which contradicts $x \prec_b z$ (*ii*). Hence, $[y] \not\leq_b [z]$. From, $[z] \not\leq_b [y]$ and $[y] \not\leq_b [z]$, we conclude $[y] \perp [z]$ by definition of $\perp$. □

Lemma 24 (Model existence)**.** *Let b be a saturated open tableau branch. Then, b induces a BL model $\mathcal{M}_b = (W, \leq, R, V, \Vdash)$, where $W = W_b$ and $\leq = \leq_b$ as per Lemma 23, R is the ternary relation induced by $\leq_b$ as per Definition 3, and, for all worlds $[x] \in W_b$, $V([x]) = \{\, p \mid (T\,p{:}x) \in b \,\}$. Moreover, $\mathcal{M}_b$ is such that if $(T\,\phi{:}x) \in b$ then $[x] \Vdash \phi$ and if $(F\,\phi{:}x) \in b$ then $[x] \nVdash \phi$.*

Proof. By mutual induction on the structure of ϕ for all labels x.

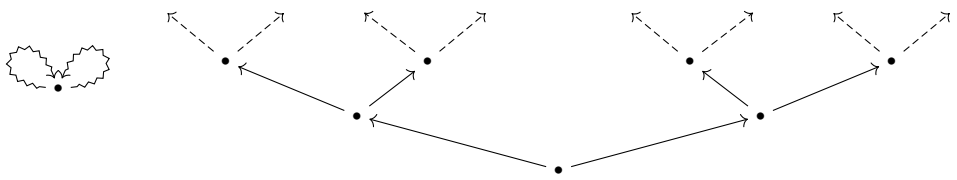


Figure 7: A finite model with backlinks (left) and its infinite unfolding (right)

Base case $\phi = p$:

If $(T\,p{:}x) \in b$, then $p \in V([x])$ by definition of V, which implies $[x] \Vdash p$ by Definition 4.

If $(F\,p{:}x) \in b$, then since b is open, we have $(T\,p{:}x) \notin b$. Hence, we get $p \notin V([x])$ by definition of V, which implies $[x] \nVdash p$ by Definition 4.

Case $\phi = \phi_1 \mathrel{-\!\!*} \phi_2$:

If $(F\,\phi_1 \mathrel{-\!\!*} \phi_2{:}x) \in b$, then for some $y, z \in D(b)$, we have $(T\,z \prec x \parallel y) \in b$, $(T\,\phi_1{:}y) \in b$ and $(F\,\phi_2{:}z) \in b$. $(T\,z \prec x \parallel y) \in b$ implies $R([z],[x],[y])$ by Lemma 23. Moreover, by induction hypothesis, $(T\,\phi_1{:}y) \in b$ and $(F\,\phi_2{:}z) \in b$ imply $[y] \Vdash \phi_1$ and $[z] \nVdash \phi_2$. Hence, by Definition 4, we have $[x] \nVdash \phi_1 \mathrel{-\!\!*} \phi_2$.

If $(T\,\phi_1 \mathrel{-\!\!*} \phi_2{:}x) \in b$, then for all $y, z \in D(b)$, if $(T\,z \prec x \parallel y) \in b$ then $(F\,\phi_1{:}y) \in b$ or $(T\,\phi_2{:}z) \in b$. Pick arbitrary $[u],[v] \in W_b$ such that $R([v],[x],[u])$ and $[u] \Vdash \phi_1$. Since $[v] \leq_b [x]$ and $[v] \leq_b [u]$, we have $(T\,v \prec x) \in b$ and $(T\,v \prec u) \in b$ by definition of $\leq_b$. Similarly, since $[x] \not\leq_b [u]$ and $[u] \not\leq_b [x]$, we have $(T\,x \prec u) \notin b$ and $(T\,u \prec x) \notin b$, which by Condition 2 implies $(T\,x \parallel u) \in b$. It then follows that we have $(T\,v \prec x \parallel u) \in b$, which implies $(F\,\phi_1{:}u) \in b$ or $(T\,\phi_2{:}v) \in b$. By the induction hypothesis, we then get $[u] \nVdash \phi_1$ or $[v] \Vdash \phi_2$. Since we assume $[u] \Vdash \phi_1$, we necessarily have $[v] \Vdash \phi_2$. Therefore, $[x] \Vdash \phi_1 \mathrel{-\!\!*} \phi_2$.

The other cases are similar. □

Corollary 25. *If b is a saturated open tableau branch in a tableau for $\phi{:}x$, then the induced model $\mathcal{M}_b = (W_b, \leq_b, R, V, \Vdash)$ is such that $[x] \nVdash \phi$.*

Proof. Since $(F\,\phi{:}x)$ is the root of any tableau for $\phi{:}x$, The concluding property of Lemma 24 implies that $x \nVdash \phi$. □

Theorem 26 (Completeness)**.** *If $\Vdash \phi$, then $\vdash \phi$.*

Proof. It is standard to define a tableau construction procedure that applies the rules given in Figure 5 with a fair strategy. Such a procedure will either result in a finite closed tableau for ϕ, in which case we get a T_{BL}-proof of ϕ, or build at least one (possibly infinite) complete open branch b, in which case b gives rise to a BL model $\mathcal{M}_b$ such that $\mathcal{M}_b \nVdash \phi$ by Corollary 25. □

6 A Finite Model Property and Decidability

The formula $\Box(\top * \top)$ enforces that every world has at least two distinct successors and can, therefore, only be satisfied in infinite models like the one shown in Figure 7 (right). Consequently, the finite model property in its traditional formulation fails for BL.

However, models such as the one in Figure 7 (right) are sufficiently regular to allow for a finite schematic representation. Consider, for example, the scheme in Figure 7 (left). Here, each squiggly edge represents a *backlink*, indicating that a copy of the graph originating from its endpoint should be attached to its starting point via a single edge. By unfolding such a schematic model, we obtain precisely the full binary tree shown in Figure 7 (right).

For simplicity, we only present an informal description of *models with backlinks*. These models are trees augmented with additional backlinks — edges that point backward in the tree order. Formulas in BL can be interpreted on such models by first unfolding them into (infinite) trees. One can then establish the following result:

Lemma 27. *If φ is $-\!\!*$-free and satisfiable in BL, then it has a finite model with backlinks.*

Proof. Let Σ be the set of subformulae of φ. For a subset $\Sigma_0 \subseteq \Sigma$, consider all immmediate successors of the root node that satisfy exactly those formulae in Σ that are in Σ_0. If we remove all but two of such nodes and the branches stemming from them we obtain a different model that still satisfies φ at the root. Note that we need to retain two nodes–unlike in standard modal logic where one representative suffices–as the root might satisfy a statement $\psi * \psi$ that enforces the existence of two *different* successors, even with equal theories. Applying this idea repeatedly we can create a model of φ that is finitely branching up to any given height.

On the other hand, if a branch is longer than $2^{|\Sigma|}$, we will encounter worlds $w < v$ satisfying exactly the same formulae from Σ. In this case, we can remove the branch stemming from v and instead create a back link from the immediate predecessor of v to w. This too preserves truth of φ at the root. In the end, we obtain a finitely branching model with backlinks that is of bounded height, and therefore finite. □

The proof is a modification of a standard filtration and pruning argument used in other modal logics [2]. These arguments rely on the fact that the truth of a formula depends only on the truth of its subformulae at successor nodes — a property that fails once one considers backwards-looking modalities. This explains the restriction on $-\!\!*$.

Corollary 28. *It is decidable whether a $-\!\!*$-free formula is valid in BL.*

Proof. By Theorem 26, the $-\!\!*$-free formulae of BL are computably enumerable. Furthermore, Lemma 27 implies that and it follows that their complement is computably enumerable too, since we can systematically generate all finite models with backlinks. Thus, decidability follows. □

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