An Inferentialist Semantics for Propositional Dynamic Logic

Gabriele Brancati Abate*
UCL Computer Science
gabriele.brancati.23@ucl.ac.uk

David Pym
UCL Computer Science & Philosophy
Institute of Philosophy, University of London
d.pym@ucl.ac.uk

Abstract

We give an inferentialist account of propositional dynamic logic (PDL) through proof-theoretic semantics. Specifically, building on the base-extension semantics for classical propositional logic and for classical propositional modal logics, we give a base-extension semantics for the validity of formulae, as opposed to an account of the validity of proofs. One key step is the use of 'modal relations' on bases, as employed in giving base-extension semantics for modal logics. Another is the introduction of a notion of 'inferentialist labelled transition system', for which the main point is the use in its definition of modal relations on bases. We establish soundness and completeness for PDL provability and the base-extension semantics. We illustrate the use and benefit of the semantics through an example of modelling hardware circuits.

1 Introduction

Understood as a modelling tool, logic has found widespread application in computing, enabling precise and rigorous reasoning about the behaviour of computational and dynamic systems. Modal logics, in particular, emerged as useful tools for system modelling and verification, offering formal languages for specifying and verifying properties of systems. Among these, temporal logics (TL) [21], [12] and propositional dynamic logic (PDL) [13] stand out as foundational frameworks. While TLs focus on reasoning about the progression of states over time, making it particularly suitable for analysing concurrent and reactive systems, PDL captures the dynamic structure of sequential programs by modelling program actions and their effects on system states. Together, these logics provide complementary tools for formally understanding and mathematically ensuring program correctness, safety and reliability.

The development of PDL was deeply influenced by earlier works in logic and program verification, by Engeler [7], Floyd [9], and most notably Hoare [15]. Hoare's seminal paper formalized program reasoning by using Hoare triples, $\{\phi\}\alpha\{\psi\}$, which express that if precondition ϕ holds before executing command α, then postcondition ψ will hold afterwards. In 1976, Pratt [23] extended Hoare's ideas by embedding program actions as modal operators, bridging the gap between logics of programs and modal logics. This laid the groundwork for PDL, formally developed by Fischer and Ladner [8], who introduced a Kripke-style semantics and proved the finite-model property, therefore decidability. Further foundational results included the standard Hilbert-type axiomatic system [38] and the first proofs of soundness and completeness by Parikh [18].

*Corresponding author.

As programs are interpreted as transitions between states, the notion of a program's execution, or run, is conceptually central to PDL, which is represented as a sequence of transitions in a labelled transition system (LTS). In this interpretation, the meaning of a program can be seen as being captured by the totality of its runs: all possible sequences of states and actions that a program can produce when executed. Such a position has been articulated in the case of programs that are understood as 'executable models' in [16].

This interpretation of programs sits within the philosophical position of *inferentialism* [1, 2, 14] (with some significant earlier history, including Frege, Carnap, and Wittgenstein). The essence of this position is that the meaning of expressions (in language, in logic) is explained not in terms of denotation but in terms of the inferential connections between sentences and the role of terms in those sentences in explaining such connections.

So, we have an informal understanding of the inferential nature of program logics and of PDL, in particular. In the work reported herein, we give a formal account of this understanding using the logical realization of inferentialism known as proof-theoretic semantics (see, e.g., [36, 35, 20]). Specifically, we give a base-extension semantics for PDL. One the way, we open up questions around the idea of inferentialist transition systems. Having developed our semantics, we are able to demonstrate its value as modelling tool through an example of modelling hardware circuits. As with other uses of base-extension semantics, such as [6, 5], our examples illustrate the need for, and value of, making the assumptions required for reasoning about the target domain explicit, echoing Brandom's articulation of inferentialism [1, 2, 14].

As indicated above, the logical realization of an inferentialist theory of meaning known as *proof-theoretic semantics* (P-tS) is our technical starting point. P-tS has its origins in the work of Prawitz [24, 27, 24, 25, 28] and Dummett [4, 3]. Proof-theoretic semantics stands in contrast to model-theoretic semantics, in which the meaning of formulae and of proofs is given denotationally. In this paper, we develop a proof-theoretic semantics for PDL with tests, grounded, in the sense described above, in the notion of derivability in an atomic base. Thus we demonstrate that P-tS is able to account for a program logic of substantive applicability.

We can usefully consider P-tS to have two main logical lines of development. For this paper, we shall be using the second of the two, but, we first place this in the context of the first. The first, the earlier approach, is directly in the spirit of Prawitz and Dummett and is concerned with the validity of proofs. We refer to this as proof-theoretic validity (P-tV).

How can the validity of a proof be judged? One answer is given by model-theoretic semantics (M-tS). Following Schroeder-Heister [35], we remark that this view lies in the tradition of Tarskian semantics, which works through the transmission of truth, as follows, for a model $\mathfrak{M}$: $\Gamma \models_{\mathfrak{M}} \phi$ if, and only if, if, for all $\psi \in \Gamma$, if $\models_{\mathfrak{M}} \psi$, then $\models_{\mathfrak{M}} \phi$. The validity of a consequence is then given as follows:

$$\Gamma \models \phi \quad \text{iff} \quad \text{for all models } \mathfrak{M}, \, \Gamma \models_{\mathfrak{M}} \phi$$

In this setting, proofs can be understood denotationally; that is, as operations on the interpretation of formulae in models, by which consequences are yielded; here, for example, an interpretation of a proof of ϕ from proofs of the ψ in Γ.

In proof-theoretic semantics, we seek to give meaning to proofs purely in terms of inference. More specifically, meaning is given in terms of inferences in systems of pre-logical atomic rules [24]. Such a set is called a *base*.

An atomic rule is one in which an atomic proposition is inferred from atomic propositions and which makes no reference to logical constants (connectives or other operators). For example, we have that if, say, Tammy is a vixen, we can infer that it is a fox, and that it is female. In the standard format of proof rules, we have:

$$\frac{\text{Tammy is a vixen}}{\text{Tammy is a fox}} \quad \text{and} \quad \frac{\text{Tammy is a vixen}}{\text{Tammy is female}}$$

Similarly, if we have that Tammy is a fox and that Tammy is female, then we can infer that Tammy is a vixen:

$$\frac{\text{Tammy is female} \qquad \text{Tammy is a fox}}{\text{Tammy is a vixen}}$$

In general, atomic rules have forms such as

$$\frac{}{p} \qquad \text{and} \qquad \frac{p_1 \ldots p_k}{p} \qquad \text{and} \qquad \frac{\begin{matrix}[P_1] \\ p_1\end{matrix} \quad \ldots \quad \begin{matrix}[P_k] \\ p_k\end{matrix}}{p}$$

and more (e.g., [33]). The latter allows each of the p_is to be proved from dischargeable hypotheses [24, 32, 19, 34]. The choice of the form of atomic rules in bases has a profound effect on the strength of the semantics that is obtained [33]. Note that base rules are *pre-logical* — they do not refer to the logical constants.

Prawitz [24] introduces the idea of S-validity of proofs for a base S. To set up such a semantics of proofs relative to a base S, we need a few auxiliary ideas:

- Let S be a base of atomic rules.
- Let $\mathcal{D}$ be a system of proof rules.
- Suppose that the 'correct' proofs of $\mathcal{D}$ are those Φ that are elements of the set of $\mathbf{C}(\mathcal{D})$.
- Let $\mathcal{S}$ denote the class of proof-structures, regulated by the rules of $\mathcal{D}$.
- Let $\mathbf{J}$ be a procedure on proof-structures that yields proof-structures.

Given these, we can define validity relative to a base S and a justification $\mathbf{J}$, $\langle \mathbf{J}, S \rangle$-validity, as follows: a proof-structure Φ is $\langle \mathbf{J}, S \rangle$-valid — that is, represents a proof — if either $\Phi \in \mathbf{C}(\mathcal{D})$ or if $\mathbf{J}$ can be applied to Φ to yield an element of $\mathbf{C}(\mathcal{D})$.

If $\mathbf{J}$ specifies a reduction system — as with normalization in natural deduction or BHK-semantics, as described in [24, 35] — $\langle \mathbf{J}, S \rangle$-valid proofs can be defined inductively on their component structure.

The semantics of proofs of implications presents a particular issue [24, 37, 34]. We expect a construction of an implicational formula $\phi \supset \psi$ to be a construction that given a construction of ϕ yields a construction of ψ. However, such a condition on a construction of $\phi \supset \psi$, as formulated above, would be satisfied vacuously if there be no construction of ϕ relative to the system S in question. It follows (cf. Kripke's semantics of implication) that we must give the semantics of proofs of implications relative to all possible extensions $S \subseteq S'$.[1]

While P-tV deals with the validity of proofs, it does so without considering directly the validity of formulae. The second line of development, which we call *base-extension semantics* (B-eS), is concerned with the validity of formulae. Sandqvist [32] has given an elegant B-eS for

[1] In fact, Prawitz [24] points out that the extensions considered can be restricted to those required by $\mathbf{J}$ for the construction of ψ.

intuitionistic propositional logic. This analysis demonstrates very clearly the basic principles of B-eS.

The basic idea can be articulated conveniently by comparison with the model-theoretic definition of the validity of formulae. In the Kripke semantics of intuitionistic propositional logic, based on ordered worlds, meaning is defined inductively, as follows: the truth of an atomic formula at a world is determined by its interpretation in a model; that is, the atom is true at the world just in case the world is an element of the valuation of the atom. The meaning of the remaining connectives is then defined inductively, with the meaning of implicational formulae requiring, analogously to the requirement for base-extensions described above, judgements relative to worlds higher in the ordering.

By contrast, base-extension semantics gives the meaning ($\Vdash_{\mathcal{B}}$) of atomic formulae relative to a base in terms of provability ($\vdash_{\mathcal{B}}$) in a base, $\mathcal{B}$, where the underlying set of atoms is assumed to be denumerably infinite:[2]

$$\Vdash_{\mathcal{B}} p \quad \text{iff} \quad \vdash_{\mathcal{B}} p$$

The meanings of the connectives are then given inductively, as in Kripke's semantics. In more detail, base rules $\mathcal{R}$, application of base rules, and satisfaction of formulae in a (possibly finite) countable base $\mathcal{B}$ of rules $\mathcal{R}$ are defined as summarized in Figure 1.

$$\dfrac{\begin{matrix}[P_1] & & [P_n]\\ q_1 & \dots & q_n\end{matrix}}{r}\,\mathcal{R}$$

(Ref) $P, p \vdash_{\mathcal{B}} p$

($\mathrm{App}_{\mathcal{R}}$) if $((P_1 \Rightarrow q_1), \dots, (P_n \Rightarrow q_n)) \Rightarrow r)$ and, for all $i \in [1, n]$, $P, P_i \vdash_{\mathcal{B}} q_i$, then $P \vdash_{\mathcal{B}} r$

(At) for atomic p, $\Vdash_{\mathcal{B}} p$ iff $\vdash_{\mathcal{B}} p$

($\supset$) $\Vdash_{\mathcal{B}} \phi \supset \psi$ iff $\phi \Vdash_{\mathcal{B}} \psi$

($\wedge$) $\Vdash_{\mathcal{B}} \phi \wedge \psi$ iff $\Vdash_{\mathcal{B}} \phi$ and $\Vdash_{\mathcal{B}} \psi$

($\vee$) $\Vdash_{\mathcal{B}} \phi \vee \psi$ iff, for every atomic p and every $\mathcal{C} \supseteq \mathcal{B}$, if $\phi \Vdash_{\mathcal{C}} p$ and $\psi \Vdash_{\mathcal{C}} p$, then $\Vdash_{\mathcal{C}} p$

($\bot$) $\Vdash_{\mathcal{B}} \bot$ iff, for all atomic p, $\Vdash_{\mathcal{B}} p$

(Inf) for $\Theta \neq \emptyset$, $\Theta \Vdash_{\mathcal{B}} \phi$ iff, for every $\mathcal{C} \supseteq \mathcal{B}$, if $\Vdash_{\mathcal{C}} \theta$, for every $\theta \in \Theta$, then $\Vdash_{\mathcal{C}} \phi$

Figure 1: Sandqvist's B-eS for Intuitionistic Propositional Logic

The following are the key points in Figure 1:

- The base rules are 'Level 2'; that is, allowing dischargeable hypotheses, P_i.
- The 'Ref' and 'App' rules govern the application of base rules to construct atomic proofs.
- The clauses defining the satisfaction, or *support*, relation are mostly familiar, but note the following:
  - The use of base-extension in the 'Inf' clause reflects Prawitz's argument for its necessity in giving meaning to implication.
  - Base-extension is transmitted to implication through the clause for implication.
  - The form of the clause for $\bot$ may seem a little odd at first sight, but it gives the usual intuitionistic introduction and elimination rules for negation [26], defined as $\neg\phi = \phi \supset \bot$, as well as Ex Falso Quodlibet [3, 32, 34]. Note that the set of atoms (ps) is assumed to be denumerably infinite.

[2] We move to the notation $\mathcal{B}$, etc., for bases to stress the move from P-tV to B-eS.

– The form of the clause for disjunction is critical: it is this form that allows the completeness theorem (q.v., below) — see, for example, [32, 11, 29, **?**] for explanations of this, which are beyond the scope of this brief introduction.

Sandqvist [32] establishes the soundness and completeness of the natural deduction calculus NJ [10, 26] with respect to this semantics. Although PDL, as discussed below, is grounded in classical logic, we judge that the basic ideas of B-eS are efficiently introduced and illustrated in the setting of IPL.

In this paper, we develop a proof-theoretic semantics for PDL with tests through base-extension semantics (B-eS), grounded, in the sense described above, in the notion of derivability in an atomic base. We illustrate a richer notion of state, where the meaning of a formula is not a set of points in a graph; rather, it is dependent on what atomic rules we accept at each state. Thus we demonstrate that P-tS is able to account for a program logic of substantive applicability. Section 2 provides a concise exposition of PDL with tests, introduces a Hilbert-type axiomatic system, and formally specifies the notion of LTS. Section 3 introduces a B-eS for classical and normal modal logics, as it can be found in [17], [31], and [5]. The main contribution of this paper, detailed in Section 4, extends B-eS to PDL with tests, setting up inferentialist labelled transition systems (in Section 4.1) establishing soundness (in Section 3) and completeness (in Section 3). We conclude, Section 5, with a brief discussion of some directions for further research suggested by the ideas presented herein.

2 Propositional Dynamic Logic with Tests

In this section, we give a brief exposition on PDL with tests [13, 18, 38]. As a multi-modal logic, we will have boxed formulae of the form $[\alpha]\varphi$, where α is a program. Its intended meaning is that, after all terminating executions of program α, φ holds. The program α can be either atomic, in which case we will use letters a, b, c, and so on, or complex. Complex programs are built out of other programs, by means of a fixed set of program constructors, or of other formulae, through the test constructor ?, that takes in a formula and outputs a program. We will then have an infinite supply of programs and, therefore, of modalities, and the sets of formulae and programs must be constructed by mutual induction.

2.1 The Syntax of PDL and a Hilbert-type Proof System

Let $\mathcal{L}_{\text{PDL}}$ be a language with fixed countable sets Φ_0 and Π_0 of propositional letters and atomic programs, respectively. Then we define formulae and programs by mutual induction:

$$\begin{array}{lcl} \varphi & ::= & p \mid \bot \mid \varphi \supset \varphi \mid \varphi \wedge \varphi \mid [\alpha]\varphi \\ \alpha & ::= & a \mid \alpha\,;\alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \varphi? \end{array}$$

The sets of formulae and of programs will be denoted Φ and Π, respectively. Of course, each program construct has a different meaning. The semicolon ; denotes sequential composition, so that $\alpha;\beta$ denotes the program obtained by first executing α, then β. Non deterministic choice is expressed by $\cup$, so that $\alpha \cup \beta$ signifies that we execute either α or β. The star operator *, is the *Kleene star*, so that α^* denotes an indeterministic yet finite number of iterations of program α. Finally, we have the test operator $? : \Phi \to \Pi$, that takes in a formula and outputs a program. It is meant to check if condition φ holds at the current state.

The behaviour of the modal operators and of the program constructors is captured by a Hilbert-type proof system over the following set of axioms (cf. [38]):

$$[\alpha](\varphi \supset \psi) \supset ([\alpha]\varphi \supset [\alpha]\psi) \tag{\textbf{A1}}$$

$$[\alpha](\varphi \wedge \psi) \subset\supset ([\alpha]\varphi \wedge [\alpha]\psi] \tag{\textbf{A2}}$$

$$[\alpha \cup \beta]\varphi \subset\supset [\alpha]\varphi \wedge [\beta]\varphi \tag{\textbf{A3}}$$

$$[\alpha \ ; \ \beta]\varphi \subset\supset [\alpha][\beta]\varphi \tag{\textbf{A4}}$$

$$[\psi?]\varphi \subset\supset (\psi \supset \varphi) \tag{\textbf{A5}}$$

$$\varphi \wedge [\alpha][\alpha^*]\varphi \subset\supset [\alpha^*]\varphi \tag{\textbf{A6}}$$

$$\varphi \wedge [\alpha^*](\varphi \supset [\alpha]\varphi) \supset [\alpha^*]\varphi \tag{\textbf{IND}}$$

and rules:

$$\frac{\varphi \quad \varphi \supset \psi}{\psi} \tag{\textbf{MP}}$$

$$\frac{\varphi}{[\alpha]\varphi} \tag{\textbf{GEN}}$$

A brief commentary is in order. Axioms **(A1)**, **(A2)**, and rule **(GEN)** are familiar from normal modal logics. The others are peculiar to PDL, and their meaning will be clear when we will introduce labelled transition systems. Just notice the structural similarity between **(IND)** and the induction scheme in Peano arithmetic:

$$A(0) \wedge \forall n(A(n) \supset A(n+1)) \supset \forall n \ A(n)$$

The induction scheme states that, if some condition holds for 0 and for the successor of any integer n, then it holds for any integer n. The axiom **(IND)**, instead, says that if all φ-states, that can be obtained by running a non-deterministic number of times a program α starting from a φ-state, are such that, by running the program α one more time, we obtain again a φ-state, then no matter how many times we run the program α from our current state, we will always reach a φ-one.

The provability relation generated by the Hilbert-type system given above is denoted $\vdash_{PDL}$.

2.2 PDL's Standard Semantics

Several semantics have been offered for regular PDL (see, for example, [22]). Yet, the most common one is a relational semantics generalising Kripke structures for normal modal logics. A labelled transition system (LTS) is a structure that captures the interplay between states and actions. Here, the notion of 'state' is analogous to that of 'possible world' a Kripke frame. States represent the configurations or conditions of a system, while transitions represent the execution of actions or programs that move the system from one state to another. This connection allows modal formulae to be interpreted over these systems, providing a concrete semantics for reasoning about dynamic behaviours and program correctness.

Formally, a *labelled transition system* (LTS) $\mathfrak{T}$ is a tuple $(S, \{R^\pi \subseteq S \times S : \pi \in \Pi\}, \nu)$. The set S is a non-empty set of states, and $\nu : \Phi_0 \to \mathcal{P}(S)$ is a valuation function. The set of labelled relations $\{R^\pi \subseteq S \times S : \pi \in \Pi\}$ is defined by mutual induction with the consequence relation $\models_{\mathfrak{T}}$ as follows:

$$
\begin{array}{rcl}
s \models_{\mathfrak{T}} p & \text{iff} & s \in \nu(p) \\
s \models_{\mathfrak{T}} \bot & & \text{never} \\
s \models_{\mathfrak{T}} \neg\varphi & \text{iff} & s \not\models_{\mathfrak{T}} \varphi \\
s \models_{\mathfrak{T}} \varphi \wedge \psi & \text{iff} & s \models_{\mathfrak{T}} \varphi \ \&\ s \models_{\mathfrak{T}} \psi \\
s \models_{\mathfrak{T}} \varphi \supset \psi & \text{iff} & s \not\models_{\mathfrak{T}} \varphi \text{ or } s \models_{\mathfrak{T}} \psi \\
s \models_{\mathfrak{T}} [\alpha]\varphi & \text{iff} & t \models_{\mathfrak{T}} \varphi, \text{ for all } t \text{ such that } \langle s,t\rangle \in R^{\alpha} \\
\langle s,t\rangle \in R^{\alpha;\beta} & \text{iff} & \exists u \in S.\ \langle s,u\rangle \in R^{\alpha} \ \&\ \langle u,t\rangle \in R^{\beta} \\
\langle s,t\rangle \in R^{\alpha\cup\beta} & \text{iff} & \langle s,t\rangle \in R^{\alpha} \text{ or } \langle s,t\rangle \in R^{\beta} \\
\langle s,t\rangle \in R^{\alpha^*} & \text{iff} & \langle s,t\rangle \in (R^{\alpha})^* \\
\langle s,t\rangle \in R^{\varphi?} & \text{iff} & s = t \ \&\ t \models_{\mathfrak{T}} \varphi
\end{array}
$$

where R^* denotes the reflexive transitive closure of $R \subseteq X \times X$. If $(s,t) \in R^{\alpha}$, then we will say that t is α-accessible from s. If $(s,t) \in R^{\alpha^*}$, we will say that there exists a α-path $\sigma(\alpha)$ such that $(s,t) \in \sigma(\alpha)$, and that t is α-reachable from s.

In [18], a proof of soundness and completeness for PDL was given, with respect to the standard Kripke semantics. In other terms, it was proved that, for all LTSs $\mathfrak{I} = (S, \{R^{\pi} \subseteq S \times S : \pi \in \Pi\}, \nu)$, and all $s \in S$,

$$s \models_{\mathfrak{I}} \varphi \quad \text{iff} \quad \vdash_{PDL} \varphi$$

If $s \in \nu(\varphi)$ with $s \in S$ in some LTS $\mathfrak{T}$, we will write $\mathfrak{T}, s \models \varphi$. We will omit the LTS being considered if it is clear given the context.

In this section, we gave a brisk exposition on PDL, where we presented both its Hilbert-style proof theory, and its standard semantics. The resulting picture illustrates that as a logic, PDL exhibits a complex structure, given in particular by the expressivity achieved via the operations on actions that it admits. In the following section, we take the first step towards a base-extension semantics for PDL, by first introducing a B-eS for propositional classical logic, followed by an extension for classical modal logic.

3 Towards Base-extension Semantics for PDL

The standard semantics for PDL is based on a generalization of Kripke semantics for normal modal logic, as we saw. It is easy to see how it falls within the model-theoretic tradition that we hinted at in the introduction. In fact, we have that

$$\Gamma \vdash_{PDL} \varphi \quad \text{iff} \quad \text{for all } \psi \in \Gamma \text{ and all } s \in S_{\mathfrak{T}},\ s \models_{\mathfrak{T}} \psi \text{ implies } s \models_{\mathfrak{T}} \varphi$$

for an arbitrary LTS $\mathfrak{T}$. While it is a fruitful approach to semantics, as it delivers important meta-theorems in a rather natural way, it can be deemed as unsatisfactory if we focus on the associated meaning of logical constants. This is apparent if we consider the clause for conditionals, as the model-theoretic semantics does not specify how we deduce the consequent from the antecedent.

As our aim is to present an inferentialist semantics for PDL, we must dispense with the notion of truth, and define the meaning of any given sentence in terms of systems of atomic rules. In particular, we will be working with two kinds of relations. First, we have atomic rules defined as sequents $P \Rightarrow p$ (equivalently, we can also write (P, p)), where the antecedent is a finite and possibly empty set of propositional letters. It is important to stress that atomic rules are not schematic, hence they strictly pertain the propositional letters occurring in them. Furthermore, the Kripkean notion of state that, paired up with a valuation function, encodes complete information (for any state in a given Kripke model decides whether any propositional letter or its negation is true) is substituted by an inferentialist counterpart. In this work, we opted for an interpretation of states as a sets of atomic rules that relate atomic facts, while preserving the intution that a transition is a relation between states as a result of an action. As we will see in what follows, this calls for a careful treatment, as sets of atomic rules can both encode partial information, in case they don't decide every propositional letter, or codify too much information, in case they support all propositional letters.

In the reminder of this section, we introduce a base-extension semantics for propositional classical logic, which is reminiscent even though simpler than the one we saw in the introduction for intuitionistic logic, and a relational inferentialistic semantics for basic modal logic. This material provides the technical foundation for our original contribution.

3.1 B-eS for Classical Logic

We focus first on B-eS for classical logic [30, 31, 17] and then consider normal modal logics.

For now, let $\mathcal{L}$ be a language for propositional classical logic, whose set of logical constants consists of just material implication $\supset$ and absurdity $\bot$.

As our objective is to give semantic clauses within a B-eS for these constants, we first require a base of their inductive definition. As explained for IPL in the introduction, this is given by the notion of *derivability in a base*. Let $X \subseteq \Phi_0$. We say that X is closed under an atomic rule $r = (P, p)$ iff $P \subseteq X$ implies $p \in X$. Moreover, we will say that X is closed under a base $\mathcal{B}$ iff it is closed under all rules in $\mathcal{B}$. We therefore define the relation of derivability in a base $\mathcal{B}$, in symbols $\vdash_{\mathcal{B}}$ as the closure of the empty set under the rules in $\mathcal{B}$, in symbols $\overline{\mathcal{B}}(\emptyset)$.

As an example, let us consider two bases $\mathcal{C} = \{\Rightarrow p_0, p_0 \Rightarrow p_1, p_1 \Rightarrow p_2\}$ and $\mathcal{C}' = \mathcal{C} \setminus \{\Rightarrow p_0\}$. We will therefore have that $\vdash_{\mathcal{C}} p_0$, as the antecedent of $\Rightarrow p_0$ is the empty set (we say it is an *axiom rule*), hence the closure of $\emptyset$ under $\Rightarrow p_0$ contains p_0. The closure under $\Rightarrow p_0$ and $p_0 \Rightarrow p_1$ will contain p_1 as well and, by analogous reasoning, we obtain that $\overline{\mathcal{C}}(\emptyset) = \{p_0, p_1, p_2\}$. Hence, $\vdash_{\mathcal{C}} p_i$, for $i \in \{0, 1, 2\}$. On the other hand, by quick inspection, we see that $\overline{\mathcal{C}'}(\emptyset) = \emptyset$. Hence, for all $p \in \Phi_0$, $\nvdash_{\mathcal{C}'} p$. We can see that this notion of derivability amounts to the existence of a closed atomic derivation tree in a base, where by closed we mean that its leaves are all decorated with the antecedent of an axiom rule (i.e., the empty set). So, for instance, since we can construct the following atomic derivation in $\mathcal{D} := \{\Rightarrow p_0, \Rightarrow p_2, p_0 \Rightarrow p_1, \{p_1, p_2\} \Rightarrow p_3\}$:

$$\dfrac{\dfrac{\dfrac{\varnothing}{p_0}}{p_1} \qquad \dfrac{\varnothing}{p_2}}{p_3}$$

we can conclude that $\vdash_{\mathcal{D}} p_3$ (and the same holds for any other atomic proposition occurring in $\mathcal{D}$). If instead we had $\mathcal{D}' := \{\Rightarrow p_0, p_0 \Rightarrow p_1, \{p_1, p_2\} \Rightarrow p_3\}$, then we would have:

$$\dfrac{\dfrac{\dfrac{\varnothing}{p_0}}{p_1} \qquad \dfrac{\times}{p_2}}{p_3}$$

As we can see, the rightmost branch is not closed (here indicated by affixing a $\times$ above the rightmost branch of the derivation), and there is no other possible atomic derivation of p_3 in $\mathcal{D}'$, hence $\nvdash_{\mathcal{D}'} p_3$.

We can now inductively define the support relation $\Vdash_{\mathcal{B}}$.

$$\begin{aligned}
\Vdash_{\mathcal{B}} p \quad &\text{iff} \quad p \in \bar{\mathcal{B}}(\emptyset) \\
\Vdash_{\mathcal{B}} \bot \quad &\text{iff} \quad \Vdash_{\mathcal{B}} p, \text{ for all } p \in \Phi_0 \\
\Vdash_{\mathcal{B}} \varphi \supset \psi \quad &\text{iff} \quad \varphi \Vdash_{\mathcal{B}} \psi \\
\Gamma \Vdash_{\mathcal{B}} \varphi \quad &\text{iff} \quad \forall \mathcal{B}' \supseteq \mathcal{B},\ \forall \psi \in \Gamma.\ \Vdash_{\mathcal{B}'} \psi \text{ implies } \Vdash_{\mathcal{B}'} \varphi
\end{aligned}$$

A formula φ is valid iff $\Vdash_{\mathcal{B}} \varphi$ for all $\mathcal{B}$.

In the model-theoretic semantics for classical propositional logic, we have that any interpretation function $I : \Phi \to \{0, 1\}$ maps $\bot$ to 0. In B-eS, instead, we encode the principle of explosion at the propositional level directly in the semantic clause for bottom. In other words, we have inconsistent bases that support any atomic proposition — in fact, any formula. This clause is taken because some classical validities, such as double-negation elimination, would fail to hold at the empty base [17]. Conditionals, on the other hand, are not evaluated locally in a base, but rather in all of its extensions, as explained in the introduction. It follows that the interpretation of negation as the implication of absurdity can be read as follows: $\neg\varphi$ is satisfied at some base if, by assuming that φ is supported by a base $\mathcal{B}$, we obtain that $\mathcal{B}$ is inconsistent.

It is worth mentioning the notion of maximally consistent base, as it will be relevant to simulating possible worlds in a B-eS. A *maximally consistent base* $\mathcal{M}$ is a set of atomic rules such that, for any rule r, either $r \in \mathcal{M}$, or $\Vdash_{\mathcal{M} \cup \{r\}} \bot$. In [17], it is shown that maximally consistent bases are classically well-behaved, in the sense that they are not inconsistent, and that $\Vdash_{\mathcal{M}} \varphi \supset \psi$ iff $\nVdash_{\mathcal{M}} \varphi$ or $\Vdash_{\mathcal{M}} \psi$. Moreover, in the same paper, they are used to prove completeness. Soundness can be found in [30, 31].

3.2 B-eS for Modal Logic

A first B-eS for modal logic was given in [6], yet the proposed semantics was shown to be not complete with respect to Euclidean modal logic. Here we briefly present a B-eS for S5, as can be found in [5], as our B-eS for PDL is largely based on an extension of this work.

Just as in the case of Kripke semantics, a B-eS for modal logic is obtained by considering a non-empty domain $\Omega \subseteq \mathbb{B}$ and a binary relation $\mathfrak{R} \subseteq \Omega \times \Omega$, where $\mathbb{B}$ is the set of all atomic rules. It is natural, though, to expect $\mathfrak{R}$ to have some structure-preserving constraints, given the semantic clause for the material conditional is crucially dependent on the inhabitants of the up-set generated by a base, and the existence of inconsistent bases that support all formulae of the language.

First, we impose Ω be an upward closed set of bases, that is, if $\mathcal{B} \in \Omega$ and $\mathcal{B}' \supseteq \mathcal{B}$, then $\mathcal{B}' \in \Omega$. We define a *modal relation* $\mathfrak{R}$ to be any binary relation on Ω satisfying the following conditions:

(a) $\Vdash_{\mathcal{B}} \bot$ implies that there exists $\mathcal{C}$ such that $\langle \mathcal{B}, \mathcal{C} \rangle \in \mathfrak{R}$ and, for all $\mathcal{D}$, if $\langle \mathcal{B}, \mathcal{D} \rangle \in \mathfrak{R}$, then $\Vdash_{\mathcal{D}} \bot$.

(b) $\nVdash_{\mathcal{B}} \bot$ and $\langle \mathcal{B}, \mathcal{C} \rangle \in \mathfrak{R}$ imply $\nVdash_{\mathcal{C}} \bot$.

(c) For all $\mathcal{C}$, $\langle \mathcal{B}, \mathcal{C} \rangle \in \mathfrak{R}$ implies $\forall \mathcal{B}' \supseteq \mathcal{B} \;\; \exists \, \mathcal{C}' \supseteq \mathcal{C}$ such that $\langle \mathcal{B}', \mathcal{C}' \rangle \in \mathfrak{R}$.

(d) For all $\mathcal{C}$, if $\nVdash_{\mathcal{C}} \bot$ then $\langle \mathcal{B}, \mathcal{C} \rangle \in \mathfrak{R}$ implies that for all $\mathcal{B}' \subseteq \mathcal{B}$ there exists $\mathcal{C}' \supseteq \mathcal{C}$ such that $\langle \mathcal{B}', \mathcal{C}' \rangle \in \mathfrak{R}$.

Conditions (a) and (b) grant that inconsistent bases are isolated, and the first conjunct of (a) guarantees that inconsistent bases support all formulae, even the boxed ones. Conditions (c) and (d) are, instead, analogous to the zig-zag conditions for bisimilarity. They are meant to ensure that if a boxed formula is supported by some base, then it is supported at any extension thereof. In other words, their purpose is to enforce that the support relation is monotone.

As for the definition of the support relation, to the clauses we saw in the previous section we add one for boxed formulae:

$$\Vdash_{\mathcal{B}} \Box \varphi \quad \Leftrightarrow \quad \forall \mathcal{B}' \supseteq \mathcal{B} \text{ and } \mathcal{C}' \text{ s.t. } \langle \mathcal{B}', \mathcal{C}' \rangle \in \mathfrak{R}. \Vdash_{\mathcal{C}'} \varphi$$

Proofs of soundness and completeness for this semantics can be found in [5].

In this section we introduced a B-eS for classical and normal modal logics, briefly discussing some pivotal conceptual points that will be extended in the following section. Indeed, now we tackle the main topic of this work, by first introducing the notion inferentialist labelled transition system (ILTS), and giving a thorough exposition on modal relations between atomic bases for complex programs. Next, we prove soundness by a straightforward argument. Finally, completeness is proved by means of a countermodel construction, that associates to every non-theorem of PDL a counter-ILTS and base that does not support it.

4 Base-extension Semantics for PDL

As we have seen, the development of base-extension semantics for modal logics is a recent endeavour [6], [5]. What follows builds on these previous works, yet it deviates in significant ways. Just as in [6] and [5], we import some key ideas from Kripke semantics; for instance, the presence of relations between points of evaluation (in our case, between atomic bases). However, we prove both soundness and completeness directly, that is without indirect means such as encoding of Kripke semantics in our own approach. Moreover, the proof of completeness here provided represents an element of novelty, as we give a method to construct, given a formula, an *inferentialist labelled transition system* (ILTS), which we will define below in Section 4.1. If the given formula is not a theorem of PDL, we obtain a counter-ILTS for it. If, instead, the given formula is a theorem of PDL, the procedure outputs an ILTS that contains just inconsistent bases.

In Section 4.1, we introduce ILTSs and discuss some of their important aspects. One of these is of course the notion of modal relation, as it is not as straightforward as in Kripke-style semantics, as it will be clear in due course. Moreover, as we are considering labelled edges and operations on them, some further subtleties are considered. Next, we give a direct proof of soundness. Finally, a thorough discussion of the completeness theorem and its proof is offered.

4.1 Inferentialist Labelled Transition Systems

Henceforth, we work with an augmented language $\mathcal{L}'_{PDL} \supseteq \mathcal{L}_{PDL}$, which contains both the material conditional $\supset$ and the conjunction $\wedge$ as primitive binary connectives.

An inferentialist labelled transition system (ILTS) is an upper set of bases together with a function that associates atomic programs to modal relations on bases. Formally,

Definition 1 (Inferentialist LTS)**.** *An ILTS $\mathfrak{I}$ is a pair (Ω, ρ), where:*

- *Ω is an upward-closed set of bases.*
- *$\rho : \Pi_0 \to \Omega^2$ is a labelling function that associates, to each atomic program in the language a modal relation on bases.*

A pointed ILTS is a triple $(\Omega, \rho, \mathcal{B})$, where (Ω, ρ) is an ILTS and $\mathcal{B} \in \Omega$.

Just as in the case of the B-eS for modal logic, in ILTSs we have a specific notion of modal relation, which we define in Section 4.2 (the next subsection). □

We extend ρ to the full set of programs Π as follows:

$$\begin{aligned}
\langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\alpha; \beta) \quad &\text{iff} \quad \exists \mathcal{Z} \in \Omega \text{ s.t. } \langle \mathcal{X}, \mathcal{Z} \rangle \in \rho(\alpha) \;\&\; \langle \mathcal{Z}, \mathcal{Y} \rangle \in \rho(\beta) \\
\langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\alpha \cup \beta) \quad &\text{iff} \quad \langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\alpha) \text{ or } \langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\beta) \\
\langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\alpha^*) \quad &\text{iff} \quad \exists n \geq 0 \; \exists \mathcal{X}_0, \ldots, \mathcal{X}_n \text{ s.t. } \mathcal{X} = \mathcal{X}_0,\; \mathcal{Y} = \mathcal{X}_n \;\& \\
&\qquad \forall k = 0, \ldots, n-1,\; \langle \mathcal{X}_k, \mathcal{X}_{k+1} \rangle \in \rho(\alpha) \\
\langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\varphi?) \quad &\text{iff} \quad \mathcal{X} = \mathcal{Y} \;\&\; \Vdash^{\rho}_{\mathcal{Y}} \varphi.
\end{aligned}$$

It is easy to show that $\langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\alpha^*)$ if and only if $\langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\alpha)^*$.

The support relation $\Vdash^{\rho}_{\mathcal{B}}$, with respect to a given ILTS $\mathfrak{I} = (\Omega, \rho)$, is defined as follows:

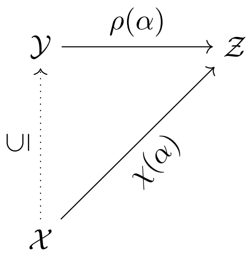
$$\begin{aligned}
\Vdash^{\rho}_{\mathcal{B}} p \quad &\text{iff} \quad p \in \bar{\mathcal{B}}(\emptyset) \\
\Vdash^{\rho}_{\mathcal{B}} \bot \quad &\text{iff} \quad \Vdash^{\rho}_{\mathcal{B}} p, \text{ for every basic sentence } p \\
\Gamma \Vdash^{\rho}_{\mathcal{B}} \varphi \quad &\text{iff} \quad \Vdash^{\rho}_{\mathcal{C}} \varphi \text{ for all } \mathcal{C} \supseteq \mathcal{B} \text{ s.t. } \Vdash^{\rho}_{\mathcal{C}} \psi \text{ for all } \psi \in \Gamma \\
\Vdash^{\rho}_{\mathcal{B}} \varphi \supset \psi \quad &\text{iff} \quad \varphi \Vdash^{\rho}_{\mathcal{B}} \psi \\
\Vdash^{\rho}_{\mathcal{B}} \varphi \wedge \psi \quad &\text{iff} \quad \Vdash^{\rho}_{\mathcal{B}} \varphi \;\&\; \Vdash^{\rho}_{\mathcal{B}} \psi \\
\Vdash^{\rho}_{\mathcal{B}} [\alpha]\varphi \quad &\text{iff} \quad \text{for all } \mathcal{B}' \supseteq \mathcal{B} \text{ and } \mathcal{C}' \text{ s.t. } \mathcal{B}' \rho(\alpha) \mathcal{C}',\; \Vdash^{\rho}_{\mathcal{C}'} \varphi
\end{aligned}$$

where $\mathcal{B} \in \Omega$, and $\overline{\mathcal{B}}(\emptyset)$ is the closure of the empty set under the rules in $\mathcal{B}$. The semantic clause for bottom mirrors the bottom elimination rule in natural deduction. If a formula is supported by all bases in any ILTS, we say that it is *valid.*

A few comments are in order here. The labelling function plays a role just in the semantic clause for boxed formulae, as expected. Here, we consider the set of bases that are accessible not only from a given base $\mathcal{B}$, but from all of its extensions. An atomic base can be seen as a partial information state, As it will come in handy later, let $\circ$ be the relation composition operator. Hence, define $\chi(\alpha) \subseteq \Omega^2$ as follows:

$$\begin{aligned}
\chi(\alpha) \;&:=\; \subseteq \;\circ\; \rho(\alpha) \\
&=\; \{(\mathcal{X}, \mathcal{Z}) \in \Omega^2 :\; \exists \mathcal{Y} \in \Omega.\; \mathcal{X} \subseteq \mathcal{Y} \;\&\; (\mathcal{Y}, \mathcal{Z}) \in \rho(\alpha)\} ko
\end{aligned}$$

Diagrammatically, $\chi(\alpha)$ can be represented as follows:



4.2 Modal Relations

The intuition behind Definition 2, is the incorporation of bisimulation into the definition of modal relation. Modal relations of this kind were introduced in [6, 5] in giving base-extension semantics for propositional classical modal logics.

Definition 2 (Modal relation)**.** *Let Ω be the set of bases. A relation $\rho(\alpha) \subseteq \Omega \times \Omega$, for some $\alpha \in \Pi$, is a* modal relation *iff:*

(a) *If $\Vdash_{\mathcal{B}} \bot$ then there is a $\mathcal{C}$ s.t. $\mathcal{B}\rho(\alpha)\mathcal{C}$ and $\Vdash_{\mathcal{C}} \bot$ and, for all $\mathcal{D}$, if $\mathcal{B}\rho(\alpha)\mathcal{D}$ then $\Vdash_{\mathcal{D}} \bot$.*

(b) *If $\nVdash_{\mathcal{B}} \bot$ then, for all $\mathcal{C}$, if $\mathcal{B}\rho(\alpha)\mathcal{C}$ then $\nVdash_{\mathcal{C}} \bot$.*

(c) *For all $\mathcal{C}$, if $\nVdash_{\mathcal{B}} \bot$ and $\mathcal{B}\rho(\alpha)\mathcal{C}$, then either* (c1) *$\mathcal{B}$ is maximally consistent or* (c2)*, for any proper extension $\mathcal{B}^+$ of $\mathcal{B}$, there is a base $\mathcal{C}' \supseteq \mathcal{C}$ such that $\mathcal{B}^+\rho(\alpha)\mathcal{C}'$.*

(d) *For all $\mathcal{C}$, if $\mathcal{B}\rho(\alpha)\mathcal{C}$ then for all $\mathcal{B}' \subseteq \mathcal{B}$, there exists $\mathcal{C}' \subseteq \mathcal{C}$ such that $\mathcal{B}'\rho(\alpha)\mathcal{C}'$.* □

Given an ILTS $\mathfrak{I} = (\Omega, \rho)$, we call $\mathcal{R}_\Omega$ the set of modal relations on Ω. Conditions (a) and (b) are meant to segregate inconsistent bases, so that no consistent base can access an inconsistent one. This is key when evaluating boxed formulae. Condition (c) grants that extensions of consistent bases can only access other consistent bases, and corresponds to the zig condition for bisimilarity. Condition (d) corresponds, instead, to the zag condition for bisimilarity.

A similar definition can be found in [5] and, crucially, [6]. The definition of modal relation in the former is, in a way, a refinement of the latter, as it enables the handling of Euclidean modal logics. In the current context, this is not necessary, as a semantical treatment of PDL does not require any extra constraint on the transition relations.

While in Kripke-style semantics it is trivial that the composition and the union of modal relations is a modal relation, in our semantics it is not, for we impose extra conditions. Therefore, we must check that $\mathcal{R}_\Omega$, for any given ILTS, is closed under relational composition and union, to make sure that our clauses for program composition and nondeterministic choice make sense.

Lemma 1. *For any $\alpha, \alpha' \in \Pi$, if $\rho(\alpha), \rho(\alpha') \in \mathcal{R}_\Omega$, then $\rho(\alpha) \cup \rho(\alpha') \in \mathcal{R}_\Omega$.*

Proof. Assume that $\rho(\alpha), \rho(\beta) \in \mathcal{R}_\Omega$. Let $dom : (A \to B) \to A$ and $rn : (A \to B) \to B$ be two functions that, given a function f, return the domain and the range of f respectively.

(a) Assume $\mathcal{B} \in dom((\rho(\alpha) \cup \rho(\beta))$ is inconsistent. Towards a contradiction, suppose that either **(i)** there is no $\mathcal{C}$ such that $\mathcal{B}\rho(\alpha) \cup \rho(\beta)\mathcal{C}$ and $\Vdash^{\rho}_{\mathcal{C}} \bot$, or **(ii)** there is a $\mathcal{D}$ such that $\mathcal{B}\rho(\alpha) \cup \rho(\beta)\mathcal{D}$ and $\nVdash^{\rho}_{\mathcal{D}} \bot$.

(i) Since $\mathcal{B} \in dom((\rho(\alpha) \cup \rho(\beta))$, either $\mathcal{B} \in dom((\rho(\alpha)$ or $\mathcal{B} \in dom(\rho(\beta))$. Let $\mathcal{C}' \in rn(\rho(\alpha))$. As $\rho(\alpha) \in \mathcal{R}_\Omega$, $\Vdash^\rho_{\mathcal{C}'} \bot$. The same goes for the other case. Hence, contradiction.

(ii) If such a $\mathcal{D}$ exists, then $\mathcal{D} \in rn(\rho(\alpha))$ or $\mathcal{D} \in rn(\rho(\beta))$. Since $\mathcal{B}$ is inconsistent, and either $\mathcal{B} \in dom(\rho(\alpha))$ or $\mathcal{B} \in dom(\rho(\beta))$, $\mathcal{D}$ is inconsistent as well. Contradiction.

Contradiction.

(b) Suppose that there exists $(\mathcal{B}, \mathcal{C}) \in \rho(\alpha) \cup \rho(\beta)$ such that $\nVdash^\rho_{\mathcal{B}} \bot$ and $\mathcal{B}\rho(\alpha) \cup \rho(\beta)\mathcal{C}$ and $\Vdash^\rho_{\mathcal{C}} \bot$. Then $(\mathcal{B}, \mathcal{C}) \in \rho(\alpha)$ or $(\mathcal{B}, \mathcal{C}) \in \rho(\beta)$. Since $\rho(\alpha) \in \mathcal{R}_\Omega$, $\mathcal{C} \notin rn(\rho(\alpha)$. By the same reasoning, $\mathcal{C} \notin rn(\rho(\beta)$. Given the definition of $\rho(\alpha) \cup \rho(\beta)$, we have that $\mathcal{C} \notin rn(\rho(\alpha) \cup \rho(\beta))$. Contradiction.

(c) Let $(\mathcal{B}, \mathcal{C}) \in \rho(\alpha) \cup \rho(\beta)$ and $\mathcal{B}$ consistent base. Suppose that $\mathcal{B}$ is not maximally consistent. If $\mathcal{B} \in dom(\rho(\alpha))$, then $\rho(\alpha) \cup \rho(\beta)$ satisfies condition (c2) of Definition 2, for $\rho(\alpha) \in \mathcal{R}_\Omega$. The same goes for the other case. Hence, $\rho(\alpha) \cup \rho(\beta)$ satisfies condition (c2) of Definition 2.

Now, suppose that $\mathcal{B}$ does not meet condition (c2) of Definition 2. Then there is $\mathcal{B}^+ \supset \mathcal{B}$ such that, for any base $\mathcal{C}' \supseteq \mathcal{C}$, $(\mathcal{B}^+, \mathcal{C}') \notin \rho(\alpha) \cup \rho(\beta)$. If it is the case, then $(\mathcal{B}^+, \mathcal{C}') \notin \rho(\alpha)$ and $(\mathcal{B}^+, \mathcal{C}') \notin \rho(\beta)$. Hence, neither $\rho(\alpha)$ nor $\rho(\beta)$ meet condition (c2), but since $\rho(\alpha)$, $\rho(\beta) \in \mathcal{R}_\Omega$, and $\mathcal{B} \in dom(\rho(\alpha) \cup \rho(\beta))$, we have that $\mathcal{B}$ is maximally consistent. Therefore, $\rho(\alpha) \cup \rho(\beta)$ meets condition (c1).

(d) Let $(\mathcal{B}, \mathcal{C}) \in \rho(\alpha) \cup \rho(\beta)$. Suppose $\mathcal{B} \in dom(\rho(\alpha))$. As $\rho(\alpha) \in \mathcal{R}_\Omega$, for any $\mathcal{B}' \subseteq \mathcal{B}$, there exists $\mathcal{C}' \subseteq \mathcal{C}$ such that $\mathcal{B}'\rho(\alpha)\mathcal{C}'$, hence $\mathcal{B}'\rho(\alpha) \cup \rho(\beta)\mathcal{C}'$ holds. The same goes in case $\mathcal{B} \in dom(\rho(\beta))$. Therefore, $\rho(\alpha) \cup \rho(\beta)$ meets condition (d) of Definition 2.

We conclude that $\rho(\alpha) \cup \rho(\beta) \in \mathcal{R}_\Omega$. □

Lemma 2. *For any $\alpha, \alpha' \in \Pi$, if $\rho(\alpha)$, $\rho(\alpha') \in \mathcal{R}_\Omega$ then $\rho(\alpha) \circ \rho(\alpha') \in \mathcal{R}_\Omega$.*

Proof. Assume that $\rho(\alpha)$, $\rho(\alpha') \in \mathcal{R}_\Omega$. Suppose $\sigma(\alpha, \alpha') = \rho(\alpha) \circ \rho(\alpha') \notin \mathcal{R}_\Omega$.

(a) Assume $\Vdash^\rho_{\mathcal{B}} \bot$. Towards a contradiction, suppose that either (i) there is no $\mathcal{C}$ such that $\mathcal{B}\sigma(\alpha, \alpha')\mathcal{C}$ and $\Vdash^\rho_{\mathcal{C}} \bot$, or (ii) there is a $\mathcal{D}$ such that $\mathcal{B}\sigma(\alpha, \alpha')\mathcal{D}$ and $\nVdash^\rho_{\mathcal{D}} \bot$.

(i) Let $\mathcal{B}' \in rn(\rho(\alpha)) \cap dom(\rho(\alpha'))$. By assumption on $\rho(\alpha)$, $\Vdash^\rho_{\mathcal{B}'} \bot$. Again, by assumption on $\rho(\alpha')$, $\Vdash^\rho_{\mathcal{C}} \bot$. Hence, contradiction.

(ii) By the same argument as in (i), this cannot be the case.

(b) Assume $\nVdash^\rho_{\mathcal{B}} \bot$. Suppose that there is a $\mathcal{C}$ such that $\mathcal{B}\sigma(\alpha, \alpha')\mathcal{C}$ and $\Vdash^\rho_{\mathcal{C}} \bot$. Then there is $\mathcal{D}$ such that $\mathcal{B}\rho(\alpha)\mathcal{D}$ and $\mathcal{D}\rho(\alpha')\mathcal{C}$. By assumption on $\rho(\alpha)$, we have that $\nVdash^\rho_{\mathcal{D}} \bot$. Again, by assumption on $\rho(\alpha')$, we have that $\nVdash^\rho_{\mathcal{C}} \bot$. Contradiction.

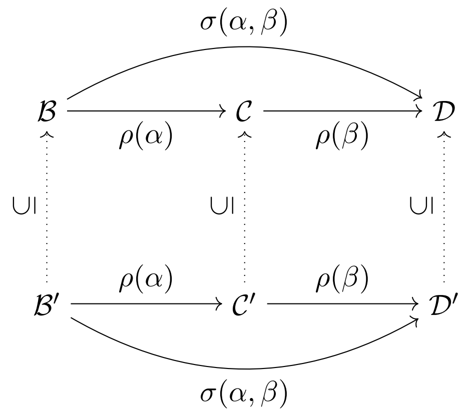
(c) Assume that $\mathcal{B}$ is consistent. Towards a contradiction, suppose that $\mathcal{B}$ is not maximally consistent, and that there is a base $\mathcal{X}$ such that $\mathcal{B}\sigma(\alpha, \beta)\mathcal{X}$ and it is not the case that, for any proper extension $\mathcal{B}^+ \supset \mathcal{B}$, there is a base $\mathcal{X}' \supseteq \mathcal{X}$ such that $\mathcal{B}^+\sigma(\alpha, \beta)\mathcal{X}'$. Let $(\mathcal{B}, \mathcal{C}) \in \rho(\alpha)$ and $(\mathcal{C}, \mathcal{D}) \in \rho(\beta)$. Since $\rho(\alpha) \in \mathcal{R}_\Omega$ and $\mathcal{B}$ is not maximally consistent, for any proper extension $\mathcal{B}^+ \supset \mathcal{B}$, there is a base $\mathcal{C}' \supseteq \mathcal{C}$ such that $\mathcal{B}^+\rho(\alpha)\mathcal{C}'$. Since $\mathcal{B}$ is consistent, by condition (b) of Definition 2, we have that $\mathcal{C}$ is consistent as well.

Now, since $(\mathcal{C},\mathcal{D}) \in \rho(\beta)$ and $\rho(\beta) \in \mathcal{R}_\Omega$, either $\mathcal{C}$ is maximally consistent, or for any $\mathcal{C}^+ \supset \mathcal{C}$ there is $\mathcal{D}' \supseteq \mathcal{D}$ such that $\mathcal{C}^+\rho(\beta)\mathcal{D}'$. In the first case, take $\mathcal{X}' = \mathcal{D}$. So $(\mathcal{B}^+,\mathcal{D}) \in \sigma(\alpha,\beta)$. Contradiction. In the second case, take $\mathcal{X}' = \mathcal{D}'$. So $(\mathcal{B}^+,\mathcal{D}') \in \sigma(\alpha,\beta)$. Contradiction. We conclude that either $\mathcal{B}$ is maximally consistent, or that for any $\mathcal{B}^+ \supset \mathcal{B}$, there exists $\mathcal{X}' \supseteq \mathcal{X}$ such that $\mathcal{B}^+\sigma(\alpha,\beta)\mathcal{X}'$. Hence, $\sigma(\alpha,\beta) \in \mathcal{R}_\Omega$.
Diagrammatically, we have the following scenarios:

(a) First case (b) Second case

(d) Let $\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E} \in \Omega$, with $\mathcal{B}\rho(\alpha)\mathcal{C}$, $\mathcal{C}\rho(\beta)\mathcal{D}$. By assumption on $\sigma(\alpha,\beta)$, there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that there is no $\mathcal{E}' \subseteq \mathcal{E}$ such that $\mathcal{B}'\sigma(\alpha,\beta)\mathcal{E}'$. By assumption on $\rho(\beta)$, since $\mathcal{C}\rho(\beta)\mathcal{D}$, for each $\mathcal{C} \subseteq \mathcal{C}'$, there exists $\mathcal{D} \subseteq \mathcal{D}'$ such that $\mathcal{C}'\rho(\beta)\mathcal{D}'$. By the same line of reasoning, for any $\mathcal{B}' \subseteq \mathcal{B}$ there exists $\mathcal{C}' \subseteq \mathcal{C}$ such that $\mathcal{B}'\rho(\beta)\mathcal{C}'$. Now, for $\mathcal{E},\mathcal{E}'$, pick $\mathcal{D},\mathcal{D}'$, respectively. We then have that for any $\mathcal{B}' \subseteq \mathcal{B}$ there exists $\mathcal{E}' \subseteq \mathcal{E}$ such that $\mathcal{B}'\sigma(\alpha,\beta)\mathcal{E}'$. Hence, contradiction.

Diagrammatically, we have the following:



(1)

Conclude that $\rho(\alpha) \circ \rho(\beta) \in \mathcal{R}_\Omega$. □

We conclude that our definition of modal relation is appropriate with respect to our interpretation of program composition and nondeterministic choice.

What we are still missing is the semantic clause for program nondeterministic repetition. We must show that $\langle \mathcal{X},\mathcal{Y}\rangle \in \rho(\alpha^*)$ is equivalent to showing that $(\mathcal{X},\mathcal{Y}) \in \rho(\alpha)^*$. Therefore, we must first prove the following lemma:

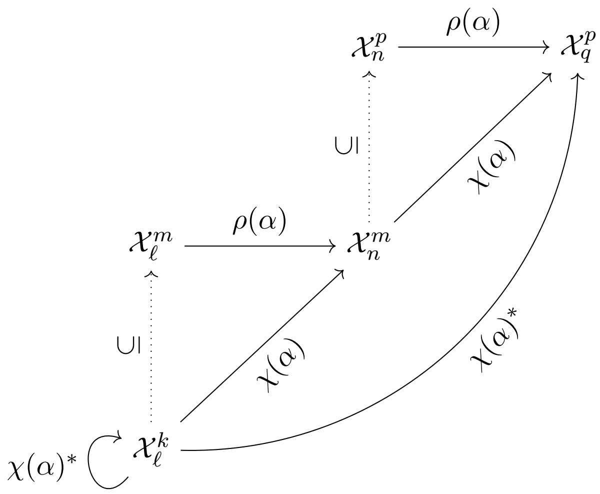
Lemma 3. *For $R_1, R_2 \subseteq A \times A$, $(R_1 \circ R_2)^* = R_1^* \circ R_2^*$.*

Proof. $\boxed{\Rightarrow}$ Let $(a, c) \in (R_1 \circ R_2)^*$. Then $\exists(x_1, \ldots, x_n) \in \prod_{i=1}^{n} A$ such that $(x_i, x_{i+1}) \in R_1 \circ R_2$ for $i \in \{1, \ldots, n-1\}$ with $x_1 = a, x_n = c$. Now, for each $(x_i, x_{i+1}) \in R_1 \circ R_2$ there exists $b_i \in A$ such that $(x_i, b_i) \in R_1$ and $(b_i, x_{i+1}) \in R_2$. Hence, $(x_i, b_i) \in R_1^*$ and $(b_i, x_{i+1}) \in R_2^*$. It follows that $(x_i, x_{i+1}) \in R_1^* \circ R_2^*$. In particular, $(a, b) \in R_1^* \circ R_2^*$.

$\boxed{\Leftarrow}$ Let $(a, c) \in R_1^* \circ R_2^*$. Then there exists $b \in A$ such that (i) $(a, b) \in R_1^*$ and (ii) $(b, c) \in R_2^*$. From (i), it follows that for some $n \in \mathbb{N}$, there exists $x_1, \ldots, x_n \in A$ such that $(x_i, x_{i+1}) \in R_1$ and $x_1 = a, x_n = b$, for $i \in \{1, \ldots, n\}$. From (ii), it follows that for some $m \in \mathbb{N}$, there exists $y_1, \ldots, y_m \in A$ such that $(y_j, y_{j+1}) \in R_2$ and $y_1 = b, y_m = c$. As a result, $(a, c) \in R_1 \circ R_2$. We conclude that $(a, c) \in (R_1 \circ R_2)^*$. □

We will use this lemma when proving that all instances of **(A6)** are valid with respect to our semantics, and that rule **(IND)** preserves validity.

As it is notationally convenient, let $\mathcal{X}, \mathcal{Y} \in \Omega$, for some ILTS (Ω, ρ). Let $\ell, k, m, n \in \mathbb{N}$, with $k \leq m$, $\ell \leq n$. Then, if $\mathcal{X} \subseteq \mathcal{Y}$, then we will write $\mathcal{X}_\ell^k$, $\mathcal{Y}_\ell^m$, and if $\langle \mathcal{X}, \mathcal{Y} \rangle \in \rho(\alpha)$, we will write $\mathcal{X}_\ell^k$, $\mathcal{Y}_n^k$. In other words, we use indexes and subscripts to keep track of the position of bases, relatively to set inclusion and accessibility. Therefore, we can represent $\chi(\alpha)^*$ as follows:



with $k \leq m \leq p$ and $\ell \leq n \leq q$. For convenience, only one reflexive arrow has been drawn. Here, $\chi(\alpha)^*$ is the reflexive transitive closure of $\chi(\alpha)$, that is, it is the smallest relation that contains $\chi(\alpha)$ and is both reflexive and transitive.

4.3 Soundness Theorem

The soundness theorem ensures that all theorems of a given logic are valid with respect to the semantics under scrutiny. Before proceeding with the soundness proof for inferentialist PDL, we need the following lemma:

Lemma 4. $\chi(\alpha)^* = \{\langle \mathcal{X}_\ell^k, \mathcal{X}_n^m \rangle \in \Omega \times \Omega : \ \mathcal{X}_\ell^k \subseteq \mathcal{X}_\ell^m, \mathcal{X}_\ell^k \rho(\alpha)^* \mathcal{X}_n^k, \ \textit{for } k \leq m, \ \ell \leq n\}$.

Proof. Since $\chi(\alpha)^*$ is the smallest reflexive transitive closure of $\chi(\alpha)$, and $Z =_{df} \{\langle \mathcal{X}^k_\ell, \mathcal{X}^m_n\rangle \in \Omega \times \Omega : \mathcal{X}^k_\ell \subseteq \mathcal{X}^m_\ell, \mathcal{X}^k_\ell \rho(\alpha)^* \mathcal{X}^k_n, \text{ for } k \leq m, \ \ell \leq n\}$ is reflexive and transitive by definition, we just need to check that for any pair $\langle \mathcal{X}, \mathcal{Y}\rangle$ of bases, if $\langle \mathcal{X}, \mathcal{Y}\rangle \in Z$, then $\langle \mathcal{X}, \mathcal{Y}\rangle \in \chi(\alpha)^*$.

Let $\langle \mathcal{B}, \mathcal{C}\rangle \in \Omega \times \Omega$ such that (i) $\langle \mathcal{B}, \mathcal{C}\rangle \in Z$ and (ii) $(\mathcal{B}, \mathcal{C}) \notin \chi(\alpha)^*$. From (ii), it follows that there is no $\chi(\alpha)$-path from $\mathcal{B}$ to $\mathcal{C}$. In particular, $\langle \mathcal{B}, \mathcal{C}\rangle \notin \rho(\alpha)^*$ and $\mathcal{B} \not\subseteq \mathcal{C}$. From (i), it follows that either $\mathcal{B} \subseteq \mathcal{C}$ or $\mathcal{B}\rho(\alpha)^*\mathcal{C}$. In both cases, we have a contradiction. □

Theorem 1. *For any $\mathcal{B} \in \Omega, \rho \in \mathcal{R}_\Omega$, if $\vdash_{PDL} \varphi$ then $\Vdash^\rho_\mathcal{B} \varphi$.*

Proof. We prove that every PDL-axiom is valid in our B-eS and that rules preserve validity.

Axiom (A1). Assume $\Vdash^\rho_\mathcal{B} [\alpha](\varphi \supset \psi)$ and $\Vdash^\rho_\mathcal{B} [\alpha]\varphi$. By the semantic clause for boxed formulae, we have that for any $\mathcal{B}' \supseteq \mathcal{B}$ and $\mathcal{C}'$ such that $\mathcal{B}'\rho(\alpha)\mathcal{C}'$, $\Vdash^\rho_{\mathcal{C}'} \varphi \supset \psi$ and $\Vdash^\rho_{\mathcal{C}'} \varphi$. Hence, $\Vdash^\rho_{\mathcal{C}'} \psi$. Since $\mathcal{C}'$ is arbitrary and $\mathcal{B}'\rho(\alpha)\mathcal{C}'$ for any $\mathcal{B}' \supseteq \mathcal{B}$, we also have that $\Vdash^\rho_\mathcal{B} [\alpha]\psi$.

Axiom (A2). Left to right, assume that $\Vdash^\rho_\mathcal{B} [\alpha](\varphi \wedge \psi)$. By the semantic clause for boxed formulae and for conjunctions, we have that for any $\mathcal{B}' \supseteq \mathcal{B}$ and $\mathcal{C}'$ such that $\mathcal{B}'\rho(\alpha)\mathcal{C}'$, $\Vdash^\rho_{\mathcal{C}'} \varphi$ and $\Vdash^\rho_{\mathcal{C}'} \psi$. Since $\mathcal{C}'$ is arbitrary and $\mathcal{B}'\rho(\alpha)\mathcal{C}'$ for any $\mathcal{B}' \supseteq \mathcal{B}$, we also have that $\Vdash^\rho_\mathcal{B} [\alpha]\varphi$ and $\Vdash^\rho_\mathcal{B} [\alpha]\psi$, hence $\Vdash^\rho_\mathcal{B} [\alpha]\varphi \wedge [\alpha]\psi$.

Right to left, assume that $\Vdash^\rho_\mathcal{B} [\alpha]\varphi \wedge [\alpha]\psi$. By the semantic clause for conjunctions and boxed formulae, we have that for any $\mathcal{B}' \supseteq \mathcal{B}$ and $\mathcal{C}'$ such that $\mathcal{B}'\rho(\alpha)\mathcal{C}'$, $\Vdash^\rho_{\mathcal{C}'} \varphi \wedge \psi$. Since $\mathcal{C}'$ is arbitrary and $\mathcal{B}'\rho(\alpha)\mathcal{C}'$ for any $\mathcal{B}' \supseteq \mathcal{B}$, we also have that $\Vdash^\rho_\mathcal{B} [\alpha](\varphi \wedge \psi)$.

Axiom (A3). Left to right, assume that $\Vdash^\rho_\mathcal{B} [\alpha \cup \beta]\varphi$. This means that, for any $\mathcal{B}' \supseteq \mathcal{B}$, if $\mathcal{B}'\rho(\alpha \cup \beta)\mathcal{C}'$, then $\Vdash^\rho_{\mathcal{C}'} \varphi$. In other words, there is no $\mathcal{C}'$ such that $\langle \mathcal{B}', \mathcal{C}'\rangle \in \rho(\alpha) \cup \rho(\beta)$ and $\nVdash^\rho_{\mathcal{C}'} \varphi$. Hence, $\Vdash^\rho_\mathcal{B} [\alpha]\varphi$ and $\Vdash^\rho_\mathcal{B} [\beta]\varphi$. We conclude that $\Vdash^\rho_\mathcal{B} [\alpha]\varphi \wedge [\beta]\varphi$.

Right to left. Assume that $\Vdash^\rho_\mathcal{B} [\alpha]\varphi \wedge [\beta]\varphi$. By semantic clause for conjunctions and boxed formulae, we have that for any extension $\mathcal{B}'$ of $\mathcal{B}$, if $\langle \mathcal{B}', \mathcal{C}'\rangle \in \rho(\alpha)$ then $\Vdash^\rho_{\mathcal{C}'} \varphi$ and, if $\langle \mathcal{B}', \mathcal{C}'\rangle \in \rho(\beta)$, then $\Vdash^\rho_{\mathcal{C}'} \varphi$. Hence, if $\langle \mathcal{B}', \mathcal{C}'\rangle \in \rho(\alpha) \cup \rho(\beta)$ then $\Vdash^\rho_{\mathcal{C}'} \varphi$. We conclude that $\Vdash^\rho_\mathcal{B} [\alpha \cup \beta]\varphi$.

Axiom (A4). Assume that $\Vdash^\rho_\mathcal{B} [\alpha; \beta]\varphi$. Hence, for any $\mathcal{B}' \supseteq \mathcal{B}$, if $\langle \mathcal{B}', \mathcal{D}'\rangle \in \rho(\alpha; \beta)$, then $\Vdash^\rho_{\mathcal{D}'} \varphi$. This is the case if and only if for $\mathcal{B}' \supseteq \mathcal{B}$, $\langle \mathcal{B}', \mathcal{D}'\rangle \in \rho(\alpha) \circ \rho(\beta)$. It follows that there is $\mathcal{C}'$ such that $\langle \mathcal{B}', \mathcal{C}'\rangle \in \rho(\alpha)$ and $\langle \mathcal{C}', \mathcal{D}'\rangle \in \rho(\beta)$. Hence, $\Vdash^\rho_{\mathcal{C}'} [\beta]\varphi$. From this, we conclude that $\Vdash^\rho_{\mathcal{B}'} [\alpha][\beta]\varphi$.

Axiom (A5). Assume $\Vdash^\rho_\mathcal{B} [\psi?]\varphi$. This is the case if and only if, for all $\mathcal{B}' \supseteq \mathcal{B}$, if $\mathcal{B}'\rho(\psi?)\mathcal{C}'$, then $\Vdash_{\mathcal{C}'} \varphi$. Now, $\mathcal{B}'\rho(\psi?)\mathcal{C}'$ if and only if $\mathcal{B}' = \mathcal{C}'$ and $\Vdash^\rho_{\mathcal{C}'} \psi$. Since $\mathcal{B}' = \mathcal{C}'$, we also have that $\Vdash^\rho_{\mathcal{B}'} \psi$. Therefore, we conclude that $\Vdash^\rho_{\mathcal{C}'} \psi \supset \varphi$.

Axiom (A6). Left to right, assume $\Vdash^\rho_\mathcal{B} \varphi \wedge [\alpha][\alpha^*]\varphi$. By semantic clause for conjunctions, we have $\Vdash^\rho_\mathcal{B} \varphi$ and $\Vdash^\rho_\mathcal{B} [\alpha][\alpha^*]\varphi$. By semantic clause for boxed formulae, we have that for any $\mathcal{B}' \supseteq \mathcal{B}$, if $\mathcal{B}'\rho(\alpha)\mathcal{C}'$, then $\Vdash^\rho_{\mathcal{C}'} [\alpha^*]\varphi$. Again, for any $\mathcal{C}'' \supseteq \mathcal{C}'$, if $\langle \mathcal{C}'', \mathcal{D}''\rangle \in \rho(\alpha^*)$, then $\Vdash^\rho_{\mathcal{D}''} \varphi$. This means that there is a finite path $\langle \mathcal{X}_0, .., \mathcal{X}_n\rangle \in \prod^n_{i=0} \Omega$ such that $\mathcal{X}_0 = \mathcal{C}'$, $\mathcal{X}_n = \mathcal{D}''$, and for any $k = 0, \ldots, n-1$, $\langle \mathcal{X}_k, \mathcal{X}_{k+1}\rangle \in \chi(\alpha)$. Consider the path $\langle \mathcal{B}, \mathcal{C}', \ldots, \mathcal{D}''\rangle$ such that $\langle \mathcal{B}, \mathcal{D}''\rangle \in \chi(\alpha)^*$, where $\chi(\alpha)^*$ is the reflexive transitive closure of $\chi(\alpha)$. We conclude that $\Vdash^\rho_\mathcal{B} [\alpha^*]\varphi$.

Right to left, assume that $\Vdash^\rho_{\mathcal{B}^k_\ell} [\alpha^*]\varphi$. It follows that for all $m \geq k$, $n \geq \ell$, $\Vdash^\rho_{\mathcal{B}^m_n} \varphi$. Now, as $k \geq k$ and $\ell \geq \ell$, we have that $\Vdash^\rho_{\mathcal{B}^k_\ell} \varphi$. We also have that for any $p > m$, $q > n$, $\Vdash^\rho_{\mathcal{B}^p_q} \varphi$. Hence, $\Vdash^\rho_{\mathcal{B}^m_n} [\alpha^*]\varphi$. As $m \geq k$ and $n \geq \ell$, $\Vdash^\rho_{\mathcal{B}^k_\ell} [\alpha][\alpha^*]\varphi$. By semantic clause for conjunction, we conclude that $\Vdash^\rho_{\mathcal{B}^k_\ell} \varphi \wedge [\alpha][\alpha^*]\varphi$.

Axiom (IND). For this proof, let us use the following notation. Given two bases $\mathcal{X}_\ell^k, \mathcal{X}_n^m$, if $k \leq m$ $\mathcal{X}_\ell^k \subseteq \mathcal{X}_n^m$, and if $\ell \leq n$, $\mathcal{X}_\ell^k \rho(\alpha^*) \mathcal{X}_n^m$, if no confusion arises as to what program α we are considering.

Assume that $\Vdash^\rho_{\mathcal{X}_\ell^k} \varphi \wedge [\alpha^*](\varphi \supset [\alpha]\varphi)$. By semantic clause for conjunction, we have that (i) $\Vdash^\rho_{\mathcal{X}_\ell^k} \varphi$ and (ii) $\Vdash^\rho_{\mathcal{X}_\ell^k} [\alpha^*](\varphi \supset [\alpha]\varphi)$. From (ii), it follows that for any $\mathcal{X}_\ell^m \in \Omega$ with $k \leq m$, if $\langle \mathcal{X}_\ell^m, \mathcal{X}_p^m \rangle \in \rho(\alpha^*)$ then $\Vdash_{\mathcal{X}_p^m} \varphi \supset [\alpha]\varphi$. Since $\langle \mathcal{X}_\ell^m, \mathcal{X}_p^m \rangle \in \rho(\alpha^*)$, there exists $\langle X_0, \ldots, X_n \rangle \in \prod_{i=0}^n$ such that $\langle \mathcal{X}_i, \mathcal{X}_{i+1} \rangle \in \rho(\alpha)$ & $\mathcal{X}_0 = \mathcal{X}_\ell^m, \mathcal{X}_n = \mathcal{X}_p^m$. Hence, $\ell \leq p$. We also have that $\varphi \Vdash^\rho_{\mathcal{X}_p^m} [\alpha]\varphi$. By semantic clause, we have that for all $\mathcal{X}_p^q$ with $m \leq q$, $\Vdash^\rho_{\mathcal{X}_p^q} \varphi$. Since $\ell \leq p, k \leq q$, by Lemma 4, $\langle \mathcal{X}_\ell^k, \mathcal{X}_p^q \rangle \in \chi(\alpha)^*$. By Lemma 3, $\langle \mathcal{X}_\ell^k, \mathcal{X}_p^q \rangle \in \ (\subseteq \circ \rho(\alpha)^*)$. We conclude that $\Vdash^\rho_{\mathcal{X}_\ell^k} [\alpha^*]\varphi$.

MP. Let $\varphi, \varphi \supset \psi$ be valid formulae, that is, for any ILTS (Ω, ρ) and any base $\mathcal{X} \in \Omega$, $\Vdash^\rho_{\mathcal{X}} \varphi$ and $\Vdash^\rho_{\mathcal{X}} \varphi \supset \psi$. Now, pick an arbitrary ILTS $\mathbb{I}$ and base $\mathcal{B}_{\mathbb{I}}$. Then (i) $\Vdash^\rho_{\mathcal{B}} \varphi$ and (ii) $\Vdash^\rho_{\mathcal{B}} \varphi \supset \psi$. From (ii), it follows that $\Vdash^\rho_{\mathcal{C}} \psi$ for all $\mathcal{C} \supseteq \mathcal{B}$ such that $\Vdash^\rho_{\mathcal{C}} \varphi$. As φ is valid, we have that $\Vdash^\rho_{\mathcal{C}} \psi$.

GEN. Let φ be a valid formula. Pick an arbitrary base $\mathcal{B}$. Then, $\Vdash^\rho_{\mathcal{B}} \varphi$. Since φ is valid, for any $\mathcal{B}' \supseteq \mathcal{B}$ and $\mathcal{C}'$ such that $\mathcal{B}' \rho(\alpha) \mathcal{C}'$, $\Vdash^\rho_{\mathcal{C}'} \varphi$. Hence, $\Vdash^\rho_{\mathcal{B}} [\alpha]\varphi$.

This concludes the proof of soundness. □

4.4 Completeness Theorem

In this section, we prove that validity in the base-extension semantics for PDL corresponds exactly to provability in PDL's Hilbert-type proof system. We do so by showing that for any formula we can construct a specialized pointed ILTS that does not support the given formula, just in case it is not a theorem. Our objective then will be proving the following claim:

Theorem 2. *For all $\varphi \in \Phi$, if $\nvdash_{PDL} \varphi$ then there exists an ILTS (Ω, ρ) and a base $\mathcal{B} \in \Omega$ such that $\nVdash^\rho_{\mathcal{B}} \varphi$.* □

We now explain how the proof of Theorem 2 proceeds. We define two functions, σ and τ, that will be needed for the construction of a pointed counter-ILTS $(\Omega_\varphi, \rho_\varphi, \mathcal{B}_\varphi)$ for any given PDL-formula φ that is not a theorem. Both functions will take a formula φ and a triple $(\mathcal{F}, f, \mathcal{B})$, such that $\mathcal{B}$ is an atomic base, $\mathcal{F}$ is a family of atomic bases, and $f : \Pi_0 \to \mathcal{F}^2$ is a function that associates to any atomic program in φ a pair of bases in $\mathcal{F}$.

The definitions of both σ and τ are given recursively on the structure of φ.

C1 $\varphi = p$.

$$\begin{aligned}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= (\mathcal{F}, f, \mathcal{B})\\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= (\mathcal{F}, f, \mathcal{B} \cup \{\Rightarrow p\})\end{aligned}$$

C2 $\varphi = \bot$.

$$\begin{aligned}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= (\mathcal{F}, f, \mathcal{B})\\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= (\mathcal{F}, f, \mathcal{B} \cup \{\Rightarrow p : \forall p \in \Phi_0\})\end{aligned}$$

C3 $\varphi = \psi_1 \wedge \psi_2$.

$$\begin{aligned}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= \sigma(\sigma((\mathcal{F}, f, \mathcal{B}), \psi_1), \psi_2)\\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= \tau(\tau((\mathcal{F}, f, \mathcal{B}), \psi_1), \psi_2)\end{aligned}$$

C4 $\varphi = \psi_1 \supset \psi_2$.

$$\begin{array}{rcl}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \sigma(\tau((\mathcal{F}, f, \mathcal{B}), \psi_1), \psi_2) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \tau((\mathcal{F}, f, \mathcal{B}), \psi_2)\end{array}$$

C5 $\varphi = [\alpha]\psi$. Here we need to consider the structure of α as well.

(a) $\alpha = a$.

$$\begin{array}{rcl}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \sigma((\mathcal{F}, f', \mathcal{C}), \psi) \\ & & \text{where } f'(\alpha') = f(\alpha') \text{ for all } \alpha' \neq \alpha, \text{ and } f'(a) = f(a) \cup \{\langle \mathcal{B}, \mathcal{C} \rangle\}. \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \tau((\mathcal{F}, f, \mathcal{C}), \varphi), \text{ for all } \mathcal{C} \in \mathcal{F} \text{ s.t. } \langle \mathcal{B}, \mathcal{C} \rangle \in f(\alpha)\end{array}$$

(b) $\alpha = \psi?$.

$$\begin{array}{rcl}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \sigma((\mathcal{F}, f, \mathcal{C}), \chi \supset \psi) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \tau((\mathcal{F}, f, \mathcal{C}), \chi \supset \psi)\end{array}$$

(c) $\alpha = \beta \cup \gamma$.

$$\begin{array}{rcl}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \sigma((\mathcal{F}, f, \mathcal{B}), [\beta]\psi \wedge [\gamma]\psi) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \tau((\mathcal{F}, f, \mathcal{B}), [\beta]\psi \wedge [\gamma]\psi)\end{array}$$

(d) $\alpha = \beta; \gamma$.

$$\begin{array}{rcl}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \sigma((\mathcal{F}, f, \mathcal{B}), [\beta]([\gamma]\psi)) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \tau((\mathcal{F}, f, \mathcal{B}), [\beta]([\gamma]\psi))\end{array}$$

(e) $\alpha = \beta^*$.

$$\begin{array}{rcl}\sigma((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \sigma((\mathcal{F}, f, \mathcal{B}), \psi) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \tau((\mathcal{F}, f, \mathcal{B}), \psi \wedge [\beta]([\beta^*]\psi))\end{array}$$

Notice that both functions do not yield, in general, an ILTS, as we put no constraint on either the domain of the resulting triple nor on the corresponding labelling function. Therefore, it is necessary to modify the output of both τ and σ as follows. For $\circ \in \{\sigma, \tau\}$, let $\circ((\mathcal{F}, f, \mathcal{B}), \varphi) = (\mathcal{F}', f', \mathcal{B}')$. Then take the upward closure of $\mathcal{F}'$, that is $\{\mathcal{Y} \in \mathbb{B} : \mathcal{Y} \supseteq \mathcal{X}, \text{ for } \mathcal{X} \in \mathcal{F}'\}$. To ensure that f' yields modal relations when applied to atomic programs, we impose the conditions (a)-(d) of Definition 2.

Say we have a pointed ILTS $(\Omega, \rho, \mathcal{B})$, with $\vdash^{\rho}_{\mathcal{B}} p$. Then $\sigma((\Omega, \rho, \mathcal{B}), p)$ does not produce a counterbase for p, as it will output the same ILTS it was given. This feature of function σ, paired up with an interesting feature of atomic bases — that is, they can express partial information, contrary to points in a Kripke model — will give us exactly the countermodel we need. Hence, we will apply σ to a specific case of pointed ILTS, that contains no information — that is, $(\{\varnothing\}, f, \varnothing)$, where $f(a) = \varnothing$ for all $a \in \Pi_0$.

To conclude, the proof of Theorem 2 requires two lemmas. Lemma 5 is needed to grant that function τ behaves as intended, whereas Lemma 6 guarantees that the pointed ILTS constructed using function σ is indeed a counter-ILTS to the given formula.

Lemma 5. *Let* $(\Omega, \rho, \mathcal{B}) = \tau((\{\varnothing\}, f, \varnothing), \varphi)$, *for* φ *arbitrary PDL-formula. Then* $\Vdash^{\rho}_{\mathcal{B}} \varphi$.

Proof. The proof is by induction on the structure of φ.

$\boxed{\varphi = p}$ By definition, we have that $\Omega = \{\mathcal{B}\}$, $\rho(a) = \varnothing$ for all $a \in \Pi_0$, and $\mathcal{B} = \{\Rightarrow p\}$. Therefore $\Vdash^{\rho}_{\mathcal{B}} \varphi$.

$\boxed{\varphi = \bot}$ By definition, we have that $\Omega = \{\mathcal{B}\}$, $\rho(a) = \varnothing$ for all $a \in \Pi_0$, and $\mathcal{B} = \{\Rightarrow p : \forall p \in \Phi_0\}$. Therefore $\Vdash^{\rho}_{\mathcal{B}} \varphi$.

$\boxed{\varphi = \psi_1 \wedge \psi_2}$ By definition, $(\Omega, \rho, \mathcal{B}) = \tau(\tau((\{\varnothing\}, f, \varnothing), \psi_1), \psi_2)$. Let $\tau((\{\varnothing\}, f, \varnothing), \psi_1) = (\Omega', \rho', \mathcal{B}')$. By the inductive hypothesis, we have that $\Vdash^{\rho'}_{\mathcal{B}'} \psi_1$ and $\Vdash^{\rho}_{\mathcal{B}} \psi_2$. By semantic clause for conjunction and monotonicity, we conclude that $\Vdash^{\rho}_{\mathcal{B}} \varphi$.

$\boxed{\varphi = \psi_1 \supset \psi_2}$ By definition, $(\Omega, \rho, \mathcal{B}) = \tau((\{\varnothing\}, f, \varnothing), \psi_2)$. By the inductive hypothesis, we have that $\Vdash^{\rho}_{\mathcal{B}} \psi_2$. By semantic clause for implication and monotonicity, we conclude that $\Vdash^{\rho}_{\mathcal{B}} \varphi$.

$\boxed{\varphi = [\alpha]\psi}$ We have to consider all possible subcases. We must show that $\Vdash^{\rho}_{\mathcal{B}} \varphi$, that is

$$\Vdash^{\rho}_{\mathcal{C}'} \psi \quad \forall \mathcal{B}' \supseteq \mathcal{B}, \forall \mathcal{C}' \text{ s.t. } \langle \mathcal{B}', \mathcal{C}' \rangle \in \rho(\alpha)$$

- $\alpha = a$**.** By definition, τ recurses on ψ at each $\mathcal{C}'$ already in $f(a) = \rho(a)$. As $f(a) = \varnothing$, we have that $\rho(a) = f(a) = \varnothing$. It follows that there is no $\rho(a)$-successor of any $\mathcal{B}'$ superset of $\mathcal{B}$. Therefore, vacuously, $\Vdash^{\rho}_{\mathcal{B}} \varphi$.
- $\alpha = \chi?$**.** By definition, $\tau((\{\varnothing\}, f, \varnothing), \varphi) = \tau((\{\varnothing\}, f, \varnothing), \chi \supset \psi)$. By the same line of reasoning as in the case where $\varphi = \psi_1 \supset \psi_2$, we conclude that $\Vdash^{\rho}_{\mathcal{B}} \varphi$.
- $\alpha = \beta;\gamma$**.** By definition, $\tau((\{\varnothing\}, f, \varnothing), \varphi) = \tau((\{\varnothing\}, f, \varnothing), [\beta]([\gamma]\psi))$. By the inductive hypothesis, $\Vdash^{\rho}_{\mathcal{B}} [\beta]([\gamma]\psi)$. By soundness, $\Vdash^{\rho}_{\mathcal{B}} [\beta;\gamma]\psi \leftrightarrow [\beta]([\gamma]\psi)$. We conclude that $\Vdash^{\rho}_{\mathcal{B}} [\beta;\gamma]\psi$.
- $\alpha = \beta \cup \gamma$**.** By definition, $\tau((\{\varnothing\}, f, \varnothing), \varphi) = \tau((\{\varnothing\}, f, \varnothing), [\beta]\psi \wedge [\gamma]\psi)$. By the inductive hypothesis, $\Vdash^{\rho}_{\mathcal{B}} [\beta]\psi$ and $\Vdash^{\rho}_{\mathcal{B}} [\gamma]\psi$. We conclude that $\Vdash^{\rho}_{\mathcal{B}} \varphi$.
- $\alpha = \beta^*$**.** By definition, $\tau((\{\varnothing\}, f, \varnothing), \varphi) = \tau((\{\varnothing\}, f, \varnothing), \psi \wedge [\beta]([\beta^*]\psi))$. By the inductive hypothesis, $\Vdash^{\rho}_{\mathcal{B}} \psi$ and $\Vdash^{\rho}_{\mathcal{B}} [\beta]([\beta^*]\psi)$. Hence, $\Vdash^{\rho}_{\mathcal{B}} \psi \wedge [\beta]([\beta^*]\psi)$. By soundness, we have that $\Vdash^{\rho}_{\mathcal{B}} [\beta^*]\psi \leftrightarrow \psi \wedge [\beta]([\beta^*]\psi)$. We conclude that $\Vdash^{\rho}_{\mathcal{B}} \varphi$.

This concludes our proof. □

Lemma 5 is needed because the definition of the function σ depends upon it. Specifically, the definition of σ, as given in the discussion of the proof of Theorem 2 above, requires an application of function τ in case **C4**.

Completeness now follows immediately from the following lemma:

Lemma 6. *Let φ be a formula that is not a PDL-theorem, and $(\Omega, \rho, \mathcal{B}) = \sigma((\{\varnothing\}, f, \varnothing), \varphi)$, with $f(a) = \varnothing$ for all $a \in \Pi_0$. Then $\nVdash^{\rho}_{\mathcal{B}} \varphi$.*

Proof. The proof is again by structural induction on φ.

$\boxed{\varphi = p}$ By definition, $(\Omega, \rho, \mathcal{B}) = \sigma(\{\varnothing\}, f, \varnothing)$. It immediately follows that $\nVdash^{\rho}_{\mathcal{B}} \varphi$.

$\boxed{\varphi = \bot}$ By definition, $(\Omega, \rho, \mathcal{B}) = \sigma(\{\varnothing\}, f, \varnothing)$. It immediately follows that $\nVdash^{\rho}_{\mathcal{B}} \varphi$.

$\boxed{\varphi = \psi_1 \wedge \psi_2}$ By definition, $(\Omega, \rho, \mathcal{B}) = \sigma(\sigma((\{\varnothing\}, f, \varnothing), \psi_1), \psi_2)$. By the inductive hypothesis, we have that $\sigma((\{\varnothing\}, f, \varnothing), \psi_1)$ does not support ψ_1. We then conclude that $\nVdash^{\rho}_{\mathcal{B}} \varphi$.

$\boxed{\varphi = \psi_1 \supset \psi_2}$ By definition, $(\Omega, \rho, \mathcal{B}) = \sigma(\tau((\{\varnothing\}, f, \varnothing), \psi_1), \psi_2)$. By the inductive hypothesis, $(\Omega, \rho, \mathcal{B})$ does not support ψ_2. Moreover, by Lemma 5, we have that $\Vdash^{\rho}_{\mathcal{B}} \psi_1$. By monotonicity and semantic clause for implication, we conclude that $\nVdash^{\rho}_{\mathcal{B}} \varphi$.

$\boxed{\varphi = [\alpha]\psi}$ We have to consider all subcases. Let $(\Omega, \rho, \mathcal{B}) = \sigma((\{\varnothing\}, f, \varnothing), \varphi)$. We must show that $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} \varphi$, that is that there exists a pair $\langle \mathcal{B}', \mathcal{C}' \rangle \in \rho(\alpha)$, with $\mathcal{B}' \supseteq \mathcal{B}$, such that $\Vdash\!\!\!\!/^{\rho}_{\mathcal{C}'} \psi$.

- $\alpha = a$. By definition, $\sigma((\{\varnothing\}, f, \varnothing), \varphi)$ only adds one pair $\langle \mathcal{B}, \mathcal{C} \rangle$, where $\Vdash\!\!\!\!/^{\rho}_{\mathcal{C}} \psi$ by the inductive hypothesis. Hence, there is a $\rho(\alpha)$-successor of $\mathcal{B}$ that does not support ψ. We conclude that $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} \varphi$.
- $\alpha = \chi?$. By definition, $\sigma((\{\varnothing\}, f, \varnothing), \varphi) = \sigma((\{\varnothing\}, f, \varnothing), \chi \supset \psi)$. By the same line of reasoning as in the case where $\varphi = \psi_1 \supset \psi_2$, we conclude that $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} \varphi$.
- $\alpha = \beta; \gamma$. By definition, $\sigma((\{\varnothing\}, f, \varnothing), \varphi) = \sigma((\{\varnothing\}, f, \varnothing), [\beta]([\gamma]\psi))$. By the inductive hypothesis, $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} [\beta]([\gamma]\psi)$. By soundness, $\Vdash^{\rho}_{\mathcal{B}} [\beta;\gamma]\psi \leftrightarrow [\beta]([\gamma]\psi)$. We conclude that $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} [\beta;\gamma]\psi$.
- $\alpha = \beta \cup \gamma$. By definition, $\sigma((\{\varnothing\}, f, \varnothing), \varphi) = \sigma((\{\varnothing\}, f, \varnothing), [\beta]\psi \wedge [\gamma]\psi)$. By the inductive hypothesis, $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} [\beta]\psi$ and $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} [\gamma]\psi$. We conclude that $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} \varphi$.
- $\alpha = \beta^*$. By definition, $\sigma((\{\varnothing\}, f, \varnothing), \varphi) = \sigma((\{\varnothing\}, f, \varnothing), \psi)$. By the inductive hypothesis, $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} \psi$. Hence, $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} \psi \wedge [\beta]([\beta^*]\psi)$. By soundness, we have that $\Vdash^{\rho}_{\mathcal{B}}$ $[\beta^*]\psi \leftrightarrow \psi \wedge [\beta]([\beta^*]\psi)$. We conclude that $\Vdash\!\!\!\!/^{\rho}_{\mathcal{B}} \varphi$.

This concludes the proof. □

Theorem 2 follows from Lemma 6 as an immediate corollary, as we have proved that we can always construct a pointed counter-ILTS for any non-theorem of PDL.

In the final section, we present an application of our B-eS to a modelling task. We use atomic bases and ILTSs to model the logical and dynamic behaviour of a simple hardware circuit with registers.

4.5 Modelling With Bases: Hardware Circuits

We explore the application of bases in the context of system modelling, where modal logics and their Kripke-style semantics serve as the standard. The present work builds upon these foundations, extending the notion of labelled transition system by adding structure, as witnessed by the definition of modal relation, and the closure results necessary to prove the adequacy of our semantics. While this adds some complexity, bases offer a truly novel perspective on formal semantics and, as a consequence, on the representation of complex systems. Here we will give some simple examples of system modelling, in the hope that future work will shed light on the usefulness of base-extension semantics in system modelling and model checking.

Atomic bases and production rules are rather interesting objects. Contrary to the standard semantics for classical logic, for instance, bases do not force us to commit to the truth of some facts, implicitly compelling us to decide what facts are true or false. In what follows, we will give several examples of how to encode a sequential hardware circuit using atomic bases and ILTSs. Our approach is based on a fundamental intuition: production rules of the form $P \Rightarrow p$ mimic the input/output behaviour of functions, hence we can make use of them to talk about Boolean functions describing hardware circuits and, possibly, past values of registers.

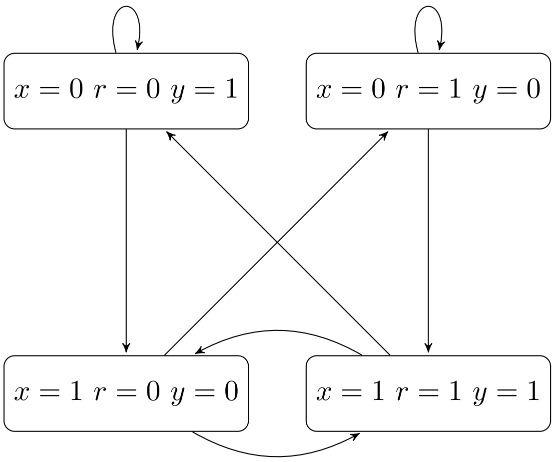
Say we have a simple sequential hardware circuit described by the following Boolean function:

$$y = \neg(x \oplus r) \tag{2}$$

where $\neg$ corresponds to the NOT gate, and $\oplus$ corresponds to XOR, or exclusive or. The value of the register r changes according to the following equality:

$$r' = x \oplus r \tag{3}$$

Hence, the value of the register r at any given moment depends on the value of x and of r at the previous clock signal. If we wanted to model the behaviour of the circuit using a labelled transition system, we would have to identify the possible states, and use transitions to capture the dynamic behaviour of the register. As the independent variables in (2) are x and r, and they can only take values in $\{0, 1\}$, we will have 2^2 states. Each state is identified by the values of the three variables, where the value of y at any given state is computed according to (2). Next, we draw edges between vertices according to (3). So, for instance, the top left state can access itself and the bottom left one, as $r' = x \oplus r = 0 \oplus 0 = 0$. Each state of this LTS correspond to a possible state of the circuit, depending on the values of x and r. Notice, though, that we have no way of reconstructing the Boolean function describing the circuit, as a state is not an explicit description of the logical behaviour of the circuit. Moreover, we are implicitly modelling the clock signal using transitions between states, so it is not explicitly given in the corresponding structure.



Now, we focus on a few ways of modelling this scenario using atomic bases. As each propositional variable can have either value 1 or 0, we will use p to represent the case where p has value 1, and $\overline{p}$ to represent the case where p has value 0. Given a base $\mathcal{B}$, we say that $p \wedge \overline{p}$ is a *conflict* for $\mathcal{B}$ if $\Vdash_{\mathcal{B}} p \wedge \overline{p}$, and that $\mathcal{B}$ is *weakly inconsistent*[3].

In what follows, we will give three different ways of encoding the circuit using atomic bases, to showcase the modelling capabilities of the proposed framework.

[3]We say 'weakly' as an inconsistent base is also weakly so, whereas the other direction does not hold in general.

First attempt: We can give a 'shallow' encoding, where we consider four bases as follows:

$$\begin{aligned}
\mathcal{B}_1 &:= \{\Rightarrow \overline{x}, \Rightarrow \overline{r}, \Rightarrow y\} \\
\mathcal{B}_2 &:= \{\Rightarrow \overline{x}, \Rightarrow r, \Rightarrow \overline{y}\} \\
\mathcal{B}_3 &:= \{\Rightarrow x, \Rightarrow \overline{r}, \Rightarrow \overline{y}\} \\
\mathcal{B}_4 &:= \{\Rightarrow x, \Rightarrow r, \Rightarrow y\}
\end{aligned}$$

so that, for instance, $\Vdash_{\mathcal{B}_1} \overline{x} \wedge \overline{r} \wedge y$, just as the state in the top left corner of the previous LTS. Next, we define the corresponding ILTS as one might expect. Arguably, this attempt is not really interesting, as it just mimics an LTS, with all the extra structure imposed by using a base-extension semantics.

Second attempt: In this second attempt, we leverage the expressivity of bases to reason on the structure of a logical circuit (and of Boolean expressions more generally), so as to encode its logical behaviour directly. The idea is that rules can be used to represent the I/O behaviour of Boolean functions. First, we will give a general method, then we will apply it to our specific logical circuit.

Let φ be a Boolean formula. Let $\psi_0, \ldots, \psi_n$ be an enumeration of its subformulae (hence, $\psi_n = \varphi$)[4]. Next, define an injective assignment $\alpha_\varphi : \mathit{Form} \to \mathbb{A}$ as follows:

$$\alpha_\varphi(\psi_i) = \begin{cases} \psi_i & \text{if } \psi_i \text{ is a propositional letter} \\ p_i & \text{otherwise, with } p_i \notin \mathit{prop}(\varphi) \end{cases}$$

We define a function $\tau : \mathbb{A} \to \mathbb{B}$ by cases:

$$\tau(p_i) = \begin{cases}
\emptyset & \text{if } p_i = \alpha_\varphi(\psi_i) = \psi_i \\
\{\overline{\alpha(\psi_j)} \Rightarrow p_i; \quad \alpha(\psi_j) \Rightarrow \overline{p_i}\} & \text{if } p_i = \alpha(\neg\psi_j) \\
\{\alpha(\psi_j),\ \alpha(\psi_{j+1}) \Rightarrow p_i; \quad \overline{\alpha(\psi_j)} \Rightarrow \overline{p_i}; \quad \overline{\alpha(\psi_{j+1})} \Rightarrow \overline{p_i}\} & \text{if } p_i = \alpha(\psi_j \wedge \psi_{j+1}) \\
\{\alpha(\psi_j) \Rightarrow p_i; \quad \alpha(\psi_{j+1}) \Rightarrow p_i; \quad \overline{\alpha(\psi_j)},\ \overline{\alpha(\psi_{j+1})} \Rightarrow \overline{p_i}\} & \text{if } p_i = \alpha(\psi_j \vee \psi_{j+1}) \\
\{\overline{\alpha(\psi_j)} \Rightarrow p_i; \quad \alpha(\psi_{j+1}) \Rightarrow p_i; \quad \alpha(\psi_j),\ \overline{\alpha(\psi_{j+1})} \Rightarrow \overline{p_i}\} & \text{if } p_i = \alpha(\psi_j \supset \psi_{j+1}) \\
\{\alpha(\psi_j), \alpha(\psi_{j+1}) \Rightarrow \overline{p_i}; \quad \overline{\alpha(\psi_j)}, \overline{\alpha(\psi_{j+1})} \Rightarrow \overline{p_i}; & \\
\overline{\alpha(\psi_j)}, \alpha(\psi_{j+1}) \Rightarrow p_i; \quad \alpha(\psi_j), \overline{\alpha(\psi_{j+1})} \Rightarrow p_i\} & \text{if } p_i = \alpha(\psi_j \oplus \psi_{j+1}) \\
\{\alpha(\psi_j), \alpha(\psi_{j+1}) \Rightarrow \overline{p_i}; \quad \overline{\alpha(\psi_j)}, \overline{\alpha(\psi_{j+1})} \Rightarrow p_i; & \\
\overline{\alpha(\psi_j)}, \alpha(\psi_{j+1}) \Rightarrow p_i; \quad \alpha(\psi_j), \overline{\alpha(\psi_{j+1})} \Rightarrow p_i\} & \text{if } p_i = \alpha(\psi_j \otimes \psi_{j+1})
\end{cases}$$

Finally, define $\mathcal{B}_\varphi := \bigcup_{i=0}^{n} \tau(\alpha(\psi))$, for every ψ subformula of φ.

The base so obtained does not support any specific propositional letter, as it contains no axiom rules. This mirrors the fact that we are representing the logical behaviour of a Boolean formula, without making any assumptions on the truth values of its propositional letters. Moreover, our approach is fully compositional, in the sense that if φ is a subformula of φ', then $\mathcal{B}_\varphi \subseteq \mathcal{B}_{\varphi'}$ (with appropriate definition of $\alpha_{\varphi'}$).

[4]Just take its construction tree, and enumerate top down left to right its nodes, leaves included.

We give a proof of the fact that a base so defined does indeed captures the logical behaviour of a Boolean formula. The following claim states that we can indeed simulate an interpretation function within our modelling framework.

Claim 1. *$I(\varphi) = x$ iff $\Gamma \Vdash_{\mathcal{B}_\varphi} p^x_\varphi$, where $\Gamma := \{\alpha(\psi) \mid I(\psi) = 1\} \cup \{\overline{\alpha(\psi)} \mid I(\psi) = 0\}$, $i \in \{0,1\}$, $p^0_\varphi := \overline{p_\varphi}$ and $p^1_\varphi := p_\varphi$.*

Proof. The proof is by induction on the structure of φ. In the inductive step, we will just consider the cases where $\varphi = \neg\psi$ and $\varphi = \psi_1 \supset \psi_2$, as all the other cases are analogous.
Base case: $\varphi = p$. By definition, $\alpha(\varphi) = p$ and $\mathcal{B}_\varphi = \varnothing$. Now, suppose that $I(\varphi) = 1$. Then we must check whether $p \Vdash_\varnothing p$. This amounts to its being that any base that supports p does support p, which is trivial. The same goes in the case $I(p) = 0$.

As for the other direction, suppose that $p \Vdash_\varnothing p$. By definition of $\Gamma = \{p\}$, we have that $I(p) = 1$. The same goes in case $\overline{p} \Vdash_\varnothing \overline{p}$.

Inductive step: Here we cover two cases.

- **$\varphi = \neg\psi$.** Suppose $I(\varphi) = 0$. Then, $I(\psi) = 1$ and, by the inductive hypothesis, $\Gamma \Vdash_{\mathcal{B}_\psi} p_\psi$. Also, by definition, $\mathcal{B}_\varphi = \mathcal{B}_\psi \cup \{\overline{p_\psi} \Rightarrow p_\varphi;\ p_\psi \Rightarrow \overline{p_\varphi}\}$. As just propositional letters are involved, we have that all $\mathcal{B} \supseteq \mathcal{B}_\psi$ that support atoms in Γ support p_ψ too. Now, by working with $\mathcal{B}_\varphi$, we must check whether $\Gamma \Vdash_{\mathcal{B}_\varphi} p_\varphi$ holds. As in all extensions of $\mathcal{B}_\psi$ supporting the atoms in Γ, we could construct a closed derivation of p_ψ, it follows that all extensions of $\mathcal{B}_\varphi$ supporting atoms in Γ will support $\overline{p_\varphi}$, as the addition of $p_\psi \Rightarrow \overline{p_\varphi}$ allows. Analogously for the case $I(\varphi) = 1$.

  Now, suppose $\Gamma \Vdash_{\mathcal{B}_\varphi} p_\varphi$. As before, $\mathcal{B}_\varphi = \mathcal{B}_\psi \cup \{\overline{p_\psi} \Rightarrow p_\varphi;\ p_\psi \Rightarrow \overline{p_\varphi}\}$, hence we have that $\Gamma \Vdash_{\mathcal{B}_\psi} \overline{p_\psi}$. By the inductive hypothesis, it follows that $I(\psi) = 0$ and, therefore, we conclude that $I(\varphi) = 1$.

- **$\varphi = \psi_1 \supset \psi_2$.** Suppose $I(\varphi) = 1$. This means that either $I(\psi_1) = 0$ or $I(\psi_2) = 1$. In the first case, by the inductive hypothesis, $\Gamma \Vdash_{\mathcal{B}_{\psi_1}} \overline{p_{\psi_1}}$. Now, by definition, $\mathcal{B}_\varphi \supset \mathcal{B}_{\psi_1} \cup \{\overline{p_{\psi_1}} \Rightarrow p_\varphi\}$. By the same line of reasoning as before, we have that all extensions of $\mathcal{B}_\varphi$ supporting atoms in Γ support p_φ as well. The second case is analogous, just observe that $\mathcal{B}_\varphi \supset \mathcal{B}_{\psi_1} \cup \{p_{\psi_2} \Rightarrow p_\varphi\}$. The case where $I(\varphi) = 0$ is analogous.

  As for the other direction, suppose $\Gamma \Vdash_{\mathcal{B}_\varphi} p_\varphi$. Observe that $\mathcal{B}_\varphi = \mathcal{B}_\psi \cup \{\overline{p_{\psi_1}} \Rightarrow p_\varphi;\ p_{\psi_2} \Rightarrow p_\varphi;\ p_{\psi_1}, \overline{p_{\psi_2}} \Rightarrow \overline{p_\varphi}\}$. Hence, we have to consider two cases, $\Gamma \Vdash_{\mathcal{B}_{\psi_1}} \overline{p_{\psi_1}}$, and $\Gamma \Vdash_{\mathcal{B}_{\psi_2}} p_{\psi_2}$. In the first case, by the inductive hypothesis, $I(\psi_1) = 0$, hence $I(\varphi) = 1$. In the second, by the inductive hypothesis, $I(\psi_2) = 1$, hence $I(\varphi) = 1$. We conclude that $I(\varphi) = 1$. The case where $\Gamma \Vdash_{\mathcal{B}_\varphi} \overline{p_\varphi}$ is analogous.

This concludes the proof. □

The Boolean formula $y = \neg(x \oplus r)$ can therefore be encoded as follows:

$$\mathcal{B}_y \ := \quad \{x, r \Rightarrow \overline{p_0};\quad \overline{x}, \overline{r} \Rightarrow \overline{p_0};\quad x, \overline{r} \Rightarrow p_0;\quad \overline{x}, r \Rightarrow p_0;\quad \overline{p_0} \Rightarrow p_1;\quad p_0 \Rightarrow \overline{p_1}\}$$

where $p_0 = \alpha(x \oplus r)$, and $p_1 = \alpha(\neg(x \oplus r))$. While this base captures the logical content of y, it is not sufficient to also express the dynamic behaviour exhibited by the register r. Hence, we need to consider four bases $\mathcal{B}'_1, \ldots, \mathcal{B}'_4$ defined as follows:

$$\begin{aligned}
\mathcal{B}'_1 &:= \mathcal{B}_y \cup \{\Rightarrow \overline{x}, \Rightarrow \overline{r}\} \\
\mathcal{B}'_2 &:= \mathcal{B}_y \cup \{\Rightarrow \overline{x}, \Rightarrow r\} \\
\mathcal{B}'_3 &:= \mathcal{B}_y \cup \{\Rightarrow x, \Rightarrow \overline{r}\} \\
\mathcal{B}'_4 &:= \mathcal{B}_y \cup \{\Rightarrow x, \Rightarrow r\}
\end{aligned}$$

where the addition of an axiom rule to $\mathcal{B}_y$ can be interpreted as 'giving an input' to the corresponding Boolean function: if enough and appropriate inputs are given, $\mathcal{B}_y$ will give (more than) one output, as new closed atomic derivations will be made available. Finally, transitions between bases are decided just as before.

Third attempt: Finally, we present a unified encoding that captures both the logical and dynamic aspects of y. Let $\mathcal{B}_y$ be just as before, and define $\mathcal{B}_{\text{register}}$ as follows:

$$\mathcal{B}_{\text{register}} := \{p_0 \Rightarrow r'; \quad \overline{p_0} \Rightarrow \overline{r'}\}$$

This base captures the register's update behaviour as specified by the circuit's Boolean function. It says that if $x \oplus r$ evaluates to 1 (0), then at the next clock cycle the register r updates to 1 (0). Notice that, conceptually, we have to consider just one action, that is the clock signal. For increased explicitness, one can introduce an atomic proposition (for instance, *tick*), to mark the occurrence of a clock pulse. In that case, the update base becomes:

$$\mathcal{B}'_{\text{register}} := \{p_0, tick \Rightarrow r'; \quad \overline{p_0}, tick \Rightarrow \overline{r'}\}$$

thus making explicit the dependence of the update on the tick. When only one action is involved, however, it is preferable to just omit it for simplicity. Note that this approach is limited to deterministic updates. In contexts where a single action may lead to multiple outcomes, the functional nature of production rules proves insufficient to capture nondeterminism.

The union

$$\mathcal{B} := \mathcal{B}_{\text{register}} \cup \mathcal{B}_y$$

yields a complete description of our circuit, integrating both its static logical structure and its dynamic behaviour. For example, given a state where inputs are fixed (say, $\{\overline{x}, r\}$, one can derive r' directly from $\mathcal{B}$ — formally, $\overline{x}, r \Vdash_{\mathcal{B}} r'$). In practice, the problem of determining the next state of the circuit is transformed into a derivation problem — if a closed derivation of r (or $\overline{r}$) exists in some extension of $\mathcal{B}$, then the correctness of the corresponding state update in $\mathcal{B}$ is guaranteed by monotonicity.

Overall, the methods presented in this section yield a compact yet straightforward encoding of sequential hardware circuits. More importantly, they raise fundamental questions and open exciting prospects for further research in system modelling and verification from a proof-theoretic perspective.

5 Discussion

After introducing the ideas of inferentialism, proof-theoretic semantics and, specifically, base-extension semantics, we have presented a base-extension semantics for PDL, building on base-extension semantics for classical propositional modal logics.

We have explained the need for 'inferentialist labelled transition systems' — a topic that opens up many philosophical and logical questions — and how they employ 'modal relations'. We have obtained soundness and completeness theorems for the given base-extension semantics of PDL and its associated Hilbert-type proof system.

Considering applications of the semantics, we have explored several methods for modelling sequential hardware circuits using atomic bases and ILTSs. We began with a shallow encoding that mirrors traditional state-based approaches via labelled transition systems, then introduced a more expressive, compositional method to encode the logical behaviour of Boolean functions. Finally, we integrated dynamic aspects — state updates via register transitions — into our encoding, showcasing how production rules can capture both static and dynamic properties of circuits.

Our approach demonstrates that ideas from proof-theoretic semantics, and base-extension semantics in particular, can offer multiple clear and interesting alternatives to standard Kripke semantics. By preserving the logical structure of Boolean expressions and directly relating inputs to outputs through atomic rules, our framework provides an intuitive yet rigorous method for system modelling. Unlike the standard state-based approach, it is explicit and auditable, as it offers a complete description of the encoded system.

Several research directions are worth exploring:

- **Handling Nondeterminism**: Our current framework is inherently deterministic. Future work could investigate how to generalize production rules to capture nondeterministic behaviours, which are common both in theoretical and practical contexts.

- **Integration with Model Checking**: Further research could explore how base-extension semantics can bridge the gap between structural proof theory and model checking. In particular, standard techniques and results such as normalization and cut elimination could help reduce the derivation search space, promoting a syntax-driven yet semantic approach to system verification.

- **Theoretical Foundations**: A deeper theoretical investigation into the properties of atomic systems and their generalizations could further clarify their relationship with traditional approaches, as those inspired by Kripke-style semantics. This might suggest how configurations of base-extension semantics might serve as a unifying theoretical framework between model-theoretic and proof-theoretic techniques.

Acknowledgements

We are grateful for the support of UKRI for PhD Brancati's studentship through EPSRC Training Grant EP/S022503/1. Pym has been partially supported by UKRI through EPSRC Research Grants EP/S013008/1 and EP/R006865/1. We thank Yll Buzoku, Timo Eckhardt, Alexander Gheorghiu, and an anonymous referee for helpful comments on this work.

References

[1] Robert Brandom. *Making It Explicit: Reasoning, Representing, and Discursive Commitment.* Harvard University Press, 1994.

[2] Robert B Brandom. *Articulating reasons: An introduction to inferentialism.* Harvard University Press, 2001.

[3] Michael Dummett. *The Logical Basis of Metaphysics.* Harvard University Press, 1991.

[4] Michael Dummett. What Is a Theory of Meaning? In *The Seas of Language.* Oxford University Press, 1996.

[5] Timo Eckhardt and David Pym. Base-extension semantics for S5 modal logic. *Logic Journal of the IGPL*, 2025. doi: 10.1093/jigpal/jzae131.

[6] Timo Eckhardt and David J Pym. Base-extension semantics for modal logic. *Logic Journal of the IGPL*, 2024. doi: 10.1093/jigpal/jzae004.

[7] Erwin Engeler. Algorithmic properties of structures. *Mathematical Systems Theory*, 1(2):183–195, 1967.

[8] MJ Fischer and RE Ladner. Propositional Dynamic Logic of Regular Programs. *Journal of Computer and System Sciences*, 18(2), 1979.

[9] Robert W Floyd. Assigning meanings to programs. In *Program Verification: Fundamental Issues in Computer Science*, pages 65–81. Springer, 1993.

[10] Gerhard Gentzen. Untersuchungen über das logische Schließen. I. *Math. Z.*, 39:176–210, 1935.

[11] Alexander Gheorghiu and David Pym. From proof-theoretic validity to base-extension semantics for intuitionistic propositional logic. *Studia Logica*, 2025. https://doi.org/10.1007/s11225-024-10163-9.

[12] Valentin Goranko. *Temporal Logics.* Cambridge University Press, 2023.

[13] David Harel, Dexter Kozen, and Jerzy Tiuryn. *Dynamic Logic.* The MIT Press, Massachusetts, USA, 2001.

[14] Ulf Hlobil and Robert B. Brandom. *Reasons for Logic, Logic for Reasons: Pragmatics, Semantics, and Conceptual Roles.* Routledge, 2024.

[15] C. A. R. Hoare. An axiomatic basis for computer programming. *Commun. ACM*, 12(10):576–580, October 1969.

[16] Jaakko Kuorikoski and Samuli Reijula. Making it count: An inferentialist account of computer simulation. *SocArXiv*, 2019. https://doi.org/10.31235/osf.io/v9bmr.

[17] David Makinson. On an inferential semantics for classical logic. *Logic Journal of IGPL*, 22(1):147–154, 2014.

[18] Rohit Parikh. The completeness of propositional dynamic logic. In *International Symposium on Mathematical Foundations of Computer Science*, pages 403–415. Springer, 1978.

[19] Thomas Piecha, Wagner de Campos Sanz, and Peter Schroeder-Heister. Failure of Completeness in Proof-theoretic Semantics. *Journal of Philosophical Logic*, 44(3):321–335, 2015.

[20] Thomas Piecha and Peter Schroeder-Heister, editors. *Advances in Proof-theoretic Semantics*. Springer, 2015.

[21] Amir Pnueli. The temporal logic of programs. In *18th annual symposium on foundations of computer science (sfcs 1977)*, pages 46–57. ieee, 1977.

[22] Vaughan Pratt. Dynamic algebras: Examples, constructions, applications. *Studia Logica*, 50(3):571–605, 1991.

[23] Vaughan R Pratt. Semantical considerations on floyd-hoare logic. In *17th Annual Symposium on Foundations of Computer Science (sfcs 1976)*, pages 109–121. IEEE, 1976.

[24] Dag Prawitz. Ideas and Results in Proof Theory. In *Studies in Logic and the Foundations of Mathematics*, volume 63, pages 235–307. Elsevier, 1971.

[25] Dag Prawitz. Towards a Foundation of a General Proof Theory. In *Studies in Logic and the Foundations of Mathematics*, volume 74, pages 225–250. Elsevier, 1973.

[26] Dag Prawitz. *Natural Deduction: A Proof-theoretical Study*. Dover Publications, 2006 [1965].

[27] Dag Prawitz. Logical Consequence From a Constructivist View. In *The Oxford Handbook of Philosophy of Mathematics and Logic*. Oxford University Press, 2007.

[28] Dag Prawitz. An approach to general proof theory and a conjecture of a kind of completeness of intuitionistic logic revisited . *Advances in Natural Deduction: A Celebration of Dag Prawitz's Work*, pages 269–279, 2014.

[29] David Pym, Eike Ritter, and Edmund Robinson. Categorical proof-theoretic semantics. *Studia Logica*, 2024. https://doi.org/10.1007/s11225-024-10101-9.

[30] T. Sandqvist. *An Inferentialist Interpretation of Classical Logic*. PhD thesis, Unoversity of Uppsala, 2005. Uppsala Prints and Preprints in Philosophy, Uppsala Prints and Preprints in Philosophy, 2005:1, 2005.

[31] Tor Sandqvist. Classical logic without bivalence. *Analysis*, 69(2):211–218, 2009.

[32] Tor Sandqvist. Base-extension semantics for intuitionistic sentential logic. *Logic Journal of the IGPL*, 23(5):719–731, 2015.

[33] Tor Sandqvist. Atomic bases and the validity of peirce's law. Presented at 'The meaning of proofs: Celebrating the World Logic Day', 2022:https://sites.google.com/view/wdl-ucl2022/home, accessed 14 March 2025.

[34] Tor Sandqvist. Base-extension semantics as meaning theory: Some philosophical reflections on negation, disjunction, and quantification. Presented at the 5th Symposium on Proof-theoretic Semantics, University of London, 4-6 February, 2025: https://sites.google.com/view/pts-symposium-uk/home, accessed 14 March 2025.

[35] Peter Schroeder-Heister. Proof-Theoretic versus Model-Theoretic Consequence. In Michal Pelis, editor, *The Logica Yearbook 2007*. Filosofia, 2008.

[36] Peter Schroeder-Heister. Proof-Theoretic Semantics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2018 edition, 2018.

[37] Peter Schroeder-Heister. Proof-theoretic semantics: Some basic ideas. Uinversität Tübingen, Mathematisch-Naturwissenschaftliche Fakultät, 2003. Available at: https://publikationen.uni-tuebingen.de/xmlui/handle/10900/131188, accessed 14 March 2025.

[38] Krister Segerberg. A completeness theorem in the modal logic of programs. *Banach Center Publications*, 9:31–46, 1982.

Lemma 1. Direct proof.

Proof. Assume that $\rho(\alpha), \rho(\alpha') \in \mathcal{R}_\Omega$.

(a) Assume $\mathcal{B} \in dom(\rho(\alpha) \cup \rho(\alpha'))$ is inconsistent.

 (i) Since $\rho(\alpha), \rho(\beta) \in \mathcal{R}_\Omega$, there exists $\mathcal{C}$ such that $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha)$ or $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha')$. We conclude that $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha) \cup \rho(\alpha')$.

 (ii) As $\rho(\alpha), \rho(\alpha') \in \mathcal{R}_\Omega$, all $\mathcal{D}$ such that $\langle \mathcal{B}, \mathcal{D} \rangle \in \rho(\alpha)$ or $\langle \mathcal{B}, \mathcal{D} \rangle \in \rho(\alpha')$ are inconsistent. Hence, $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha) \cup \rho(\alpha')$.

(b) Assume that $\mathcal{B} \in dom(\rho(\alpha) \cup \rho(\alpha'))$ is consistent. Then all of its $\rho(\alpha)$- and $\rho(\alpha')$-images are consistent as well, by assumption on $\rho(\alpha)$ and $\rho(\alpha')$. We conclude that all of its $\rho(\alpha) \cup \rho(\alpha')$-images are consistent as well.

(c) Let $\mathcal{B}$ be a consistent base, and $\mathcal{C}$ be such that $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha) \cup \rho(\alpha')$. Wlog, assume that $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha)$. As $\rho(\alpha) \in \mathcal{R}_\Omega$, either $\mathcal{B}$ is a maximally consistent base, or for any proper extension $\mathcal{B}^+$ of $\mathcal{B}$ there exists some $\mathcal{C}' \supseteq \mathcal{C}$ such that $\langle \mathcal{B}^+, \mathcal{C}' \rangle \in \rho(\alpha)$. In the first case, $\rho(\alpha) \cup \rho(\alpha') \in \mathcal{R}_\Omega$, for it satisfies condition (c1) of **Definition** 2. In the second case, we also have that $\rho(\alpha) \cup \rho(\alpha')$ is a modal relation, for it satisfies condition (c2) of **Definition** 2. We conclude that $\rho(\alpha) \cup \rho(\alpha')$ satisfies condition (c).

(d) Suppose that $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha) \cup \rho(\alpha')$. Wlog, assume that $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha)$. Per hypothesis, $\rho(\alpha)$ satisfies condition (d) of **Definition** 2. It immediately follows that $\rho(\alpha) \cup \rho(\alpha')$ satisfies condition (d) as well.

As we proved that $\rho(\alpha) \cup \rho(\alpha')$ satisfies all conditions of **Definition** 2, we conclude that it is a modal relation. □

Lemma 1 and proof of IND corrected.

Lemma 7. *Let $\chi(\alpha) = \supseteq \circ \rho(\alpha)$. The following equality holds:*

$$\chi(\alpha^*) = \chi(\alpha)^*$$

Proof. First, notice that $\rho(\alpha^*) = \bigcup_{n \geq 0} \rho(\alpha)^n$. Hence we have that

$$\chi(\alpha^*) = \supseteq \circ \bigcup_{n \geq 0} \rho(\alpha)^n = \bigcup_{n \geq 0} (\supseteq \circ \rho(\alpha)^n)$$

`[The following step needs justification. The idea is that condition (c) (and (a)) of Def. 2 guarantees that, whenever we can do a` $\supseteq$`-step, we can do a` $\rho(\alpha)$`-step, hence the two relations effectively commute.]` As $\subseteq$ is reflexive and transitive,

$$\supseteq \circ \rho(\alpha)^n = (\supseteq \circ \rho(\alpha))^n = \chi(\alpha)^n$$

By definition, $\bigcup_{n \geq 0} \chi(\alpha)^n = \chi(\alpha)^*$. We conclude that

$$\chi(\alpha^*) = \bigcup_{n \geq 0} \chi(\alpha)^n = \chi(\alpha)^*$$

□

Lemma 8. ***IND*** *is sound.*

Proof. We begin by rephrasing the box clause using χ, for readability's sake.

$$\Vdash^{\rho}_{\mathcal{B}} [\alpha]\varphi \quad\Longleftrightarrow\quad \forall \mathcal{C}.\langle \mathcal{B}, \mathcal{C}\rangle \in \chi(\alpha) \Rightarrow \Vdash^{\rho}_{\mathcal{C}} \varphi$$

Now, checking the soundness of **IND** boils down to checking if supporting (A1) a formula φ, and (A2) $[\alpha^*](\varphi \supset [\alpha]\varphi)$ forces any base $\mathcal{B}$ in any ILTS (Ω, ρ) to support $[\alpha^*]\varphi$. In other words, any $\chi(\alpha)$-path π of length n with $\pi[0] = \mathcal{B}$ is such that $\Vdash^{\rho}_{\pi[n]} \varphi$. The proof is by induction on the length of π.

Base case. π has length 0. In this case, (A1) guarantees that $\Vdash^{\rho}_{\mathcal{B}} \varphi$.

Step. π has length $n+1$. By inductive hypothesis, $\Vdash^{\rho}_{\pi[n]} \varphi$. By (A2), we have that for any $k \in [0, \dots, n]$, if $\Vdash^{\rho}_{\pi[k]} \varphi$, then $\Vdash^{\rho}_{\pi[k]} [\alpha]\varphi$. Hence, $\Vdash^{\rho}_{\pi[n+1]} \varphi$.

This concludes the proof. □

Lemma 9.
$$(\supseteq \circ \rho(\alpha)^n) = (\supseteq \circ \rho(\alpha))^n$$
for all $n \geq 0$.

Proof. By conditions (a) and (c) in **Definition** 2, we have that $\supseteq$ and $\rho(\alpha)$ commute, that is

$$\supseteq \circ \rho(\alpha) = \rho(\alpha) \circ \supseteq$$

for whenever we can take a $\supseteq$-step, we can also take a $\rho(\alpha)$-step.

The proof is by induction on n.

Base case. $n = 1$.
$$\supseteq \circ \rho(\alpha)^1 = \supseteq \circ \rho(\alpha) = (\supseteq \circ \rho(\alpha))^1$$

Step. $n = m + 1$.

$$\begin{aligned}
\supseteq \circ \rho(\alpha)^{m+1} &= \supseteq \circ (\rho(\alpha)^m \circ \rho(\alpha)) && (4)\\
&= (\supseteq \circ \rho(\alpha)^m) \circ \rho(\alpha) && (5)\\
&= (\supseteq \circ \rho(\alpha))^m \circ (\alpha) && (6)\\
&= (\rho(\alpha) \circ \supseteq)^m \circ \rho(\alpha) && (7)\\
&= (\rho(\alpha) \circ \supseteq)^{m-1} \circ (\rho(\alpha) \circ \supseteq) \circ \rho(\alpha) && (8)\\
&= (\rho(\alpha) \circ \supseteq)^{m-1} \circ (\supseteq \circ \rho(\alpha)) \circ \rho(\alpha) && (9)\\
&= (\rho(\alpha) \circ \supseteq)^{m-1} \circ (\supseteq \circ \rho(\alpha)^2) && (10)\\
&= (\rho(\alpha) \circ \supseteq)^{m-1} \circ (\supseteq^2 \circ \rho(\alpha)^2) && (11)
\end{aligned}$$

□

$$(\rho \circ \supset)^{m-1} \circ (\supset^2 \circ \rho^2) \ldots \circ (\supset \circ \supset \circ \rho \circ \rho) \ldots \circ ((\rho \circ \supset) \circ (\rho \circ \supset)) \ldots \circ$$

Lemma 2, point (c) corrected.

Proof. Suppose that $\mathcal{B}$ is consistent, and that $\langle \mathcal{B}, \mathcal{D} \rangle \in \sigma(\alpha, \alpha')$, with $\sigma(\alpha, \alpha') =_{\text{def}} \rho(\alpha) \circ \rho(\alpha')$, and $\rho(\alpha), \rho(\alpha')$ modal relations. We need to show that $\sigma(\alpha, \alpha')$ satisfies either condition (c1) or (c2) of Definition 2. By definition of $\sigma(\alpha, \alpha')$, there exists a base $\mathcal{C}$ such that $\langle \mathcal{B}, \mathcal{C} \rangle \in \rho(\alpha)$ and $\langle \mathcal{C}, \mathcal{D} \rangle \in \rho(\alpha')$.

We reason by cases. We have four possible cases, as we have two conditions for two modal relations. If $\rho(\alpha)$ satisfies condition (c1), $\sigma(\alpha, \alpha')$ satisfies (c1) as well, for $\mathcal{B}$ is maximally consistent. If instead $\rho(\alpha)$ satisfies (c2), we need to check two subcases. If $\rho(\alpha')$ satisfies (c2), it follows immediately that $\sigma(\alpha, \alpha')$ satisfies (c2) too. Now we consider what happens if $\rho(\alpha')$ satisfies (c1). As we have that $\mathcal{C}$ is maximally consistent, any proper extension of $\mathcal{C}$ is inconsistent. This means that any proper consistent extension of $\mathcal{B}$ α-sees $\mathcal{C}$, for $\rho(\alpha)$ satisfies condition (b). Given the definition of $\sigma(\alpha, \alpha')$, this means that any proper extension of $\mathcal{B}$ α-sees $\mathcal{D}$. We conclude that $\sigma(\alpha, \alpha')$ satisfies (c2).

As we exhausted all possible cases, we conclude that $\sigma(\alpha, \alpha') \in \mathcal{R}_\Omega$. □

Lemma 2, point (d) clarified.

Proof. Let $\langle \mathcal{B}, \mathcal{D} \rangle \in \sigma(\alpha, \alpha')$. By definition, we have that $\sigma(\alpha, \alpha') = \rho(\alpha) \circ \rho(\alpha')$, with both $\rho(\alpha)$ and $\rho(\alpha')$ modal relations. This means that both satisfy condition (d) of Definition 2. So, for all $\mathcal{B}' \subseteq \mathcal{B}$ there exists $\mathcal{C}' \subseteq \mathcal{C}$ such that $\langle \mathcal{B}', \mathcal{C}' \rangle \in \rho(\alpha)$, and for all $\mathcal{C}' \subseteq \mathcal{C}$ there exists $\mathcal{D}' \subseteq \mathcal{D}$ such that $\langle \mathcal{C}', \mathcal{D}' \rangle \in \rho(\alpha')$. Given the definition of $\sigma(\alpha, \alpha')$, it immediately follows that it satisfies condition (d). □

Proof of completeness, clarified. In this section, we prove that validity in the base-extension semantics for PDL corresponds to provability in PDL's Hilbert-style proof system. We do so by showing that for any formula we can construct a specialised pointed ILTS that does not support the given formula, just in case it is not a theorem. Our objective will be proving the following claim:

Theorem 3. *For all $\varphi \in \Phi$, if $\nvdash_{PDL} \varphi$ then there exists a pointed ILTS $(\Omega, \rho, \mathcal{B})$ such that $\nVdash^{\rho}_{\mathcal{B}} \varphi$.*

Our aim is, then, to construct a pointed counter-ILTS, given a formula that is not a theorem of PDL. Effectively, this requires designing a procedure that, given a formula φ, constructs an ILTS $(\Omega_\varphi, \rho_\varphi, \mathcal{B}_\varphi)$ such that $\nVdash^{\rho_\varphi}_{\mathcal{B}_\varphi} \varphi$. We do so by starting off from a "vacuous" triple $T = (\{\varnothing\}, \varnothing, \varnothing)$ of the same type `BE EXPLICIT` as an ILTS,[5] and adding information and structure to it in a principled way. The resulting triple, that we call T_φ, is then processed in such a way that the result of such processing does meet the requirements for being an ILTS.

Let us consider a few key cases, to give a sense of the intuition at play. We will first focus on the construction of the triple T_φ. For instance, take $\varphi = p$. In this case, as no modality occurs in φ, we can leave the labelling function untouched. Moreover, the empty base is the least base that does not support p. Hence, we don't add any rule to the atomic base we are considering.

`USE ACTUAL FORMULAE`

[5]This means, in particular, that we understand the second component of the triple to be a function. Of course, an empty function is a function nonetheless.

Now, take $\varphi = [a]\psi$. In this case, we need to add one a-successor $\mathcal{C}_\psi$ of $\mathcal{B}_\varphi$ to our labelling function that, together with the information in ρ_φ, does not support ψ.

Implications add a layer of complexity to the construction. Take $\varphi = \psi \supset \chi$. Given the semantic clause for material implications, for $\nVdash^{\rho_\varphi}_{\mathcal{B}_\varphi} \varphi$, we need $\Vdash^{\rho_\varphi}_{\mathcal{B}_\varphi} \psi$ and $\nVdash^{\rho_\varphi}_{\mathcal{B}_\varphi} \chi$. This means that our construction needs to consider also the case when we want to add information to T needed to force it to support formulae.

As an example of this, consider again $\varphi = [a]\psi \supset \chi$. Here we need to guarantee that $\Vdash^{\rho_\varphi}_{\mathcal{B}_\varphi} [a]\psi$ and $\nVdash^{\rho_\varphi}_{\mathcal{B}_\varphi} \chi$. For the former to be the case, we have to ensure that all a-successor of $\mathcal{B}_\varphi$ support ψ, hence we will need to add information accordingly.

To summarise, we need a procedure that produces a triple $T_\varphi = (\mathcal{F}_\varphi, f_\varphi, \mathcal{B}_\varphi)$, which will then used as a blueprint for building our final counter-ILTS. To do so, we specify two constructions, σ and τ. The former is a countermodel constructor, whereas the latter is a support constructor. Their purpose is to modify the bases in the triple, and how they relate to each other, in such a way that the result bear all the necessary information to then finally construct a counter-ILTS for the given formula.

The definitions of both σ and τ are given recursively on the structure of φ. `[ADD OBSERVATIONS ON IMPLICATION AND CONJUNCTION.]`

C1 $\varphi = p$.

$$\begin{aligned} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= (\mathcal{F}, f, \mathcal{B}) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= (\mathcal{F}, f, \mathcal{B} \cup \{\Rightarrow p\}) \end{aligned}$$

C2 $\varphi = \bot$.

$$\begin{aligned} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= (\mathcal{F}, f, \mathcal{B}) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= (\mathcal{F}, f, \mathcal{B} \cup \{\Rightarrow p : \forall p \in \Phi_0\}) \end{aligned}$$

C3 $\varphi = \psi_1 \wedge \psi_2$.

$$\begin{aligned} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= \sigma(\sigma((\mathcal{F}, f, \mathcal{B}), \psi_1), \psi_2) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= \tau(\tau((\mathcal{F}, f, \mathcal{B}), \psi_1), \psi_2) \end{aligned}$$

C4 $\varphi = \psi_1 \supset \psi_2$.

$$\begin{aligned} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= \sigma(\tau((\mathcal{F}, f, \mathcal{B}), \psi_1), \psi_2) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= \tau((\mathcal{F}, f, \mathcal{B}), \psi_2) \end{aligned}$$

C5 $\varphi = [\alpha]\psi$. Here we need to consider the structure of α as well.

(a) $\alpha = a$.

$$\begin{aligned} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= \sigma((\mathcal{F}, f', \mathcal{C}), \psi) \\ & \text{where } f'(\alpha') = f(\alpha') \text{ for all } \alpha' \neq \alpha, \text{ and } f'(a) = f(a) \cup \langle \mathcal{B}, \mathcal{C} \rangle. \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= \tau((\mathcal{F}, f, \mathcal{C}), \varphi), \text{ for all } \mathcal{C} \in \mathcal{F} \text{ s.t. } \langle \mathcal{B}, \mathcal{C} \rangle \in f(\alpha) \end{aligned}$$

(b) $\alpha = \psi?$.

$$\begin{aligned} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= \sigma((\mathcal{F}, f, \mathcal{C}), \chi \supset \psi) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= \tau((\mathcal{F}, f, \mathcal{C}), \chi \supset \psi) \end{aligned}$$

(c) $\alpha = \beta \cup \gamma$.

$$\begin{aligned} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) &= \sigma((\mathcal{F}, f, \mathcal{B}), [\beta]\psi \wedge [\gamma]\psi) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) &= \tau((\mathcal{F}, f, \mathcal{B}), [\beta]\psi \wedge [\gamma]\psi) \end{aligned}$$

(d) $\alpha = \beta;\gamma$.

$$\begin{array}{lcl} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \sigma((\mathcal{F}, f, \mathcal{B}), [\beta]([\gamma]\psi)) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \tau((\mathcal{F}, f, \mathcal{B}), [\beta]([\gamma]\psi)) \end{array}$$

(e) $\alpha = \beta^*$.

$$\begin{array}{lcl} \sigma((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \sigma((\mathcal{F}, f, \mathcal{B}), \psi) \\ \tau((\mathcal{F}, f, \mathcal{B}), \varphi) & = & \tau((\mathcal{F}, f, \mathcal{B}), \psi \wedge [\beta]([\beta^*]\psi)) \end{array}$$

As we said, this construction yields a triple T_φ that is yet to become a full-blown ILTS. Hence, we define a closure operator cl on triples as follows.

- $cl(\mathcal{F}_\varphi) = \uparrow \mathcal{F}_\varphi$;
- $cl(\mathcal{B}_\varphi) = \mathcal{B}_\varphi$;
- $cl(f_\varphi) = f'_\varphi$ is defined as follows. First, let $Inc(\Omega_\varphi) := \{\mathcal{B} \in \Omega_\varphi : \Vdash_{\mathcal{B}} \bot\}$, where $\Omega_\varphi = \uparrow \mathcal{F}_\varphi$[6]. Then $cl(f_\varphi)$ to be the least relation containing f_φ and closed under the following rules:

$$\frac{\langle \mathcal{B}, \mathcal{C} \rangle \in f'_\varphi(a) \qquad \mathcal{B}' \supseteq \mathcal{B} \qquad \mathcal{B}' \notin Inc(\Omega_\varphi)}{\langle \mathcal{B}', \mathcal{C} \rangle \in f'_\varphi(a)} \text{UpCons}$$

and

$$\frac{\langle \mathcal{B}, \mathcal{C} \rangle \in f'_\varphi(a) \qquad \mathcal{B}' \supseteq \mathcal{B} \qquad \mathcal{B}' \notin Inc(\Omega_\varphi)}{\langle \mathcal{B}', \mathcal{C} \rangle \in f'_\varphi(a)} \text{UpInc}$$

[6]Note that whether a base is inconsistent a matter of atomic derivability, hence the information provided by f_φ is not needed to define Inc.