

Inferentialist Game Semantics (Extended Abstract)

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Abstract

Game semantics is an elegant approach to the formal semantics of reasoning and computation that grounds model-theoretic concepts of truth and validity in game-theoretic concepts that emphasize the dynamic and interactive aspects of logical reasoning. In Hyland-Ong games, plays are traces of interactions between a player and an environment and such games provide a naturally appealing semantics for computation that is derived from proof-search in logical systems. Such a semantics can be seen as providing an intensional theory of meaning for systems of logic in terms of (the computation of) proofs. In logic, an intensional theory of meaning for systems of logic is offered by proof-theoretic semantics; in particular, by base-extension semantics (B-eS), in which the model-theoretic interpretation of atomic propositions in a satisfaction relation is replaced by a validity relation which uses provability in ‘bases’ of atomic rules. We establish a fully abstract correlation between B-eS and Hyland-Ong game semantics, employing techniques similar to those used by Sandqvist to give a sound and complete B-eS for intuitionistic propositional logic. We illustrate our semantics through the example of 4x4 Sudoku.

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1 Introduction

Game semantics is an elegant approach to the formal semantics of reasoning and computation that grounds model-theoretic concepts of truth and validity in game-theoretic concepts that emphasize the dynamic and interactive aspects of logical reasoning. A formula is interpreted by a two-player game between a *Proponent*, who defends a claim, and an *Opponent*, who represents the environment. The game specifies which challenges Opponent may issue and how Proponent may meet them; a proof, or a program, is then interpreted as a *winning strategy* — a policy by which Proponent meets every challenge that Opponent can mount. The dialogical analysis of logical validity in this style goes back to Lorenzen [26, 27], for whom a formula is intuitionistically valid just in case Proponent has a winning strategy in the associated dialogue — see Felscher [14] for a careful technical development. In the semantics of computation, the dialogical idea was sharpened into a systematic approach in

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the work of Hyland and Ong [25]. The methodology has been extended several times; see, for example, Blass [4] and Abramsky et al. [1, 2], and Nickau [30].

Two features of this picture matter for the present paper. First, game semantics is *intensional*: a strategy records not merely *that* a claim is defensible but *how* it is defended, move by move. Second, the interaction it describes reads naturally as *proof-search*: Opponent’s challenges decompose a putative conclusion into sufficient sub-goals, and Proponent’s responses discharge them. Indeed, Hyland–Ong games have been used to give a semantics for proof-search in reductive logic [39] (game semantics for proof-search has also been considered in [29]). Read this way, game semantics provides an intensional theory of meaning for systems of logic in terms of (the computation of) proofs.

An alternative intensional theory of meaning has been developed under the title *proof-theoretic semantics* (P-tS). In this framework, the meaning of the logical constants is explicated not by reference to truth in models but by reference to proof-theoretic constructions. P-tS may be seen as a logical realization of *inferentialism* [5, 6, 24], as developed by Dummett [9, 10] and Prawitz [35, 36], in which the meaning of linguistic and logical constructs is determined by their use. P-tS in the sense we require here originates with Prawitz’s ‘Ideas and Results in Proof Theory’ [35] (Appendix A), in which Prawitz seeks a theory of validity for (natural deduction) *proofs* that is grounded not in terms of denotations but in a pre-logical — that is, connective-free — notion of provability in a *base* of atomic rules; see Schroeder-Heister [42] for the contrast with model-theoretic semantics. We defer the details to Section 2. Concretely, the version of P-tS we adopt here is called *base-extension semantics* and originates with Sandqvist [40, 41].

As games and P-tS offer two intensional theories of meaning it is natural to ask if any correspondence can be made between them. That is, can the concepts of game semantics be reconstructed from the primitives of P-tS, so that the games are themselves proof-theoretic in character rather than an externally imposed model? In this paper, we answer this question positively for *intuitionistic propositional logic* (IPL). We reconstruct the basic concepts of Hyland–Ong game semantics from the primitives of B-eS according to the following dictionary: a base of atomic rules is an *arena*; an argument from assumptions is a *play*; a proof is a *winning strategy*. Replacing provability-in-a-base by the existence of a winning strategy in the corresponding arena throughout Sandqvist’s clauses yields a *game-extension semantics* for IPL: our main logical result is that IPL is sound and complete for it.

Section 2 reviews Sandqvist’s P-tS for IPL [41], defining bases, derivability in a base, and the support relation; these definitions are used throughout the paper. Section 3 develops the game-theoretic apparatus from the inferentialist point of view realized by P-tS: atomic arenas, moves, plays, and (winning) strategies, with the dictionary above made precise. The point of departure is the use of Hyland–Ong games in [39] to give a semantics for proof-search in reductive logic. Section 4 gives a substantial worked example, namely 4×4 Sudoku, which shares the player-vs-environment setup of the game semantics for proof-search in [39] while being a familiar game that is not immediately about logical consequence. Section 5 establishes the key properties of the basic setup: the correspondence between derivability in a base and winning strategies in the associated arena. Section 6 then delivers the main result: NJ is sound and complete for the validity relation of the game-extension semantics. Finally, Section 7 summarizes the contribution and suggests directions for further research. Some proofs are deferred to the appendices.

Notations & Conventions. Lower-case italic letters p, q, r, s , etc. denote propositional atoms, and upper-case italic letters P, Q, R, S , etc. denote finite (possibly empty) sets thereof. It is essential that we have a denumerable set of atoms $At = \{p_1, p_2, \dots\}$. For

formulae, we use lower-case Greek letters ϕ, ψ, γ , etc., and for finite (possibly empty) sets of formulae upper-case Greek letters Γ, Δ, Σ , and so on. We write $\mathcal{B}, \mathcal{C}, \mathcal{D}$, etc. for bases, $\vdash_{\mathcal{B}}$ for derivability in a base $\mathcal{B}$, and $\Vdash_{\mathcal{B}}$ for the support relation relative to a base $\mathcal{B}$. For arenas, we use $\mathbb{A}, \mathbb{B}, \mathbb{C}$ etc., and $\Vdash'_{\mathbb{A}}$ for the support relation relative to an arena $\mathbb{A}$. We write $\vdash_{\text{NJ}}$ to denote provability/consequence in the natural deduction system NJ [15, 34] (see Figure 3 in Appendix C).

2 Background: Proof-theoretic Semantics

The version of P-tS we use in this paper is the *base-extension semantics* (B-eS) for IPL given by Sandqvist [41]. This section defines the semantics in full.

The primitive of B-eS is the idea of a *base* — that is, a set of rules over atomic formulae (i.e., propositional atoms, not $\perp$). They function as inferential models in caparison to the denotational models of Tarski/Beth/Kripke semantics. No logical constants occurs anywhere in a base; this is what makes derivability in a base a pre-logical notion, fit to ground a semantics for the connectives without circularity. The logical constants are defined inductively by a semantic judgment called *support*.

► **Definition 1** (Base rule). A 0th-level base rule is a singleton $\langle r \rangle$, where r is an atom. A 1st-level base rule is an ordered pair $\langle Q, r \rangle$, where r is an atom and Q is a finite (possibly empty) set of atoms. An $(n+1)$ th-level base rule is an ordered pair $\langle Q, r \rangle$, where r is an atom and Q is a finite set of k th-level base rules with $k \leq n$. ◀

We identify $\langle \emptyset, r \rangle$ with $\langle r \rangle$. For base rules we write

$$(P_1 \triangleright q_1), \dots, (P_n \triangleright q_n) \triangleright r$$

in place of $\langle \{ \langle P_1, q_1 \rangle, \dots, \langle P_n, q_n \rangle \}, r \rangle$. Such a rule is meant to be read as: *given derivations of q_1 through q_n , discharge the premisses P_1 through P_n and conclude r .*

► **Definition 2** (Base). A base $\mathcal{B}$ is a (possibly infinite, possibly empty) set of base rules. A base $\mathcal{C}$ extends $\mathcal{B}$, written $\mathcal{C} \supseteq \mathcal{B}$, iff every rule of $\mathcal{B}$ is a rule of $\mathcal{C}$. ◀

► **Definition 3** (Derivability in a base). Let $\mathcal{B}$ be a base. The derivability relation $\vdash_{\mathcal{B}}$, relating sets of atoms to atoms, is defined inductively by the following clauses:

- (Ref) if $p \in S$, then $S \vdash_{\mathcal{B}} p$
- (App) if $(Q_1 \triangleright q_1), \dots, (Q_n \triangleright q_n) \triangleright r$ is a rule of $\mathcal{B}$ and $S \vdash_{\mathcal{B} \cup Q_i} q_i$, for each $i = 1, \dots, n$, then $S \vdash_{\mathcal{B}} r$

For this to make sense, we identify an atom p with the 0th-level rule $\langle p \rangle$. ◀

► **Remark 4** (Restriction). Assuming an indefinite supply of fresh atoms, bases comprising at most 2nd-level rules are as expressive as bases of arbitrary level. That is, following Stafford et al. [43]: for any base $\mathcal{B}$, there is an at most 2nd-level base $\mathcal{B}'$ such that $\vdash_{\mathcal{B}} p$ iff $\vdash_{\mathcal{B}'} p$, for every atom p occurring in $\mathcal{B}$. ◀

► **Definition 5** (Support in a Base). Let $\mathcal{B}$ be a base. The support relation $\Vdash_{\mathcal{B}}$, relating finite (possibly empty) sets of formulae to formulae, is defined inductively by the clauses of Figure 1, in which the base case (At) is the derivability relation of Definition 3, and in (Inf) $\Delta \neq \emptyset$. We say that Γ supports ϕ , written $\Gamma \Vdash \phi$, iff $\Gamma \Vdash_{\mathcal{B}} \phi$ for every base $\mathcal{B}$. ◀

$\Vdash_{\mathcal{B}} p$	iff	$\vdash_{\mathcal{B}} p$	(At)
$\Vdash_{\mathcal{B}} \phi \rightarrow \psi$	iff	$\phi \Vdash_{\mathcal{B}} \psi$	($\rightarrow$)
$\Vdash_{\mathcal{B}} \phi \wedge \psi$	iff	$\Vdash_{\mathcal{B}} \phi$ and $\Vdash_{\mathcal{B}} \psi$	($\wedge$)
$\Vdash_{\mathcal{B}} \perp$	iff	$\Vdash_{\mathcal{B}} p$, for every atom p	($\perp$)
$\Vdash_{\mathcal{B}} \phi \vee \psi$	iff	for all $\mathcal{C} \supseteq \mathcal{B}$ and every atom p , if $\phi \Vdash_{\mathcal{C}} p$ and $\psi \Vdash_{\mathcal{C}} p$, then $\Vdash_{\mathcal{C}} p$	($\vee$)
$\Delta \Vdash_{\mathcal{B}} \phi$	iff	for all $\mathcal{C} \supseteq \mathcal{B}$, if $\Vdash_{\mathcal{C}} \psi$, for all $\psi \in \Delta$, then $\Vdash_{\mathcal{C}} \phi$	(Inf)

■ **Figure 1** Sandqvist’s B-eS for IPL validity [41]

At first glance, the semantics seems to follow Kripke/Beth semantics in a proof-theoretic key. This is misleading: note the form of the clause for disjunction. The efficacy of this choice is discussed extensively in [41] and elsewhere (e.g. [23]). The salient point is that using the definition

$$\Vdash_{\mathcal{B}} \phi \vee \psi \quad \text{iff} \quad \Vdash_{\mathcal{B}} \phi \text{ or } \Vdash_{\mathcal{B}} \psi$$

does *not* give IPL [32, 31, 33, 43].

► **Theorem 6** (Sandqvist [41]). *Let Γ be finite. Then $\Gamma \vdash_{\text{NJ}} \phi$ iff $\Gamma \Vdash \phi$.* ◀

As Sandqvist [41] uses only second-level bases, this follows from Remark 4. Sandqvist [40] and Makinson [28] have considered also B-eS for classical logic. Gheorghiu [16] has developed B-eS uniformly for classical and intuitionistic first-order predicate logic, and Gheorghiu and Pym have treated second-order logic [17]. There is by now a substantial body of work, too extensive to survey here, on B-eS for modal and substructural logics (e.g., [8, 11, 12, 20, 22, 7]).

3 Game-theoretic Semantics

Following Hyland and Ong [25], the picture to have in mind is of a dialogue between two players, Proponent and Opponent. A play is a sequence of moves conducted within the discipline of an *arena*, which determines the legal actions. Different choices of arena and of the rules of play determine different games. Pym and Ritter [39] gave a version that they showed to be identical to proof-search in intuitionistic logic: a winning strategy — a function determining the moves Proponent should play in order to guarantee a win — corresponds to a proof, and the sequence of moves it generates corresponds to the construction of that proof.

The games of this section take the Pym–Ritter games as their point of departure, but the setup cannot simply be inherited. Our target is not merely the construction of proofs, but the proof-theoretic semantics of the previous section. Accordingly, the apparatus must be modified, and this is delicate: slightly different setups may result in quite different games. The definitions that follow effect this calibration; that they succeed is the content of Section 5.

► **Definition 7** (Atomic arena). *An atomic arena is a forest — that is, a set — of rooted trees of atoms. We use $\mathbb{A}, \mathbb{B}, \mathbb{C}, \dots$ to denote arenas.* ◀

Let $\mathbb{A}$ and $\mathbb{B}$ be arenas. We say that $\mathbb{B}$ *extends* $\mathbb{A}$, written $\mathbb{B} \supseteq \mathbb{A}$, iff the set of trees of $\mathbb{A}$ is a subset of the set of trees of $\mathbb{B}$.

In an arena $\mathbb{A}$, the root nodes are *P-nodes*, their children are *O-nodes*, and the children of *O-nodes* are again *P-nodes*. The polarity of nodes records which player is responsible for the corresponding claims: root nodes are conclusions that Proponent may seek to establish, their children are premisses that Opponent may challenge, and so on, alternating with depth.

We turn to the dynamic part of the dictionary: moves and plays. As in Hyland–Ong games [25], moves are of two kinds — questions, which demand justification, and answers, which supply or concede it — and each player may both question and answer, according to the polarity of the node at which the move is made.

► **Definition 8 (Move).** *A move m is an action performed by one of the players together with a datum, which is either an atom or a node of the arena. Each move is either a question or an answer. We call a move by Proponent a P-move and a move by Opponent an O-move. ◀*

The language of proof-search supplies useful intuitions for moves. The trees of an arena $\mathbb{A}$ are the rules available to Proponent for answering Opponent’s questions. A question by Opponent is a demand for proof: we write $O?p$ to denote Opponent asking ‘What is your justification for p ?’. A question by Proponent, correspondingly, invokes a rule: $P?q$ — where q is a node of $\mathbb{A}$ — asks ‘May I use the rule that concludes q ?’. Eventually, Opponent has no questions available and must answer instead: $O!r$, where r is an atom corresponding to an earlier question by Proponent, denotes Opponent conceding ‘Very well, I grant r ’. Proponent may use such concessions to answer Opponent’s questions: $P!s$, where s is an earlier question by Opponent, denotes Proponent declaring ‘So s is justified’. This intuition drives the following definition:

► **Definition 9 (Play).** *A play for an arena $\mathbb{A}$ is a finite sequence of moves $m_0, m_1, \dots, m_n$ satisfying the following conditions.*

- (i) *The move m_0 is an O-question, called the initial question.*
- (ii) *There is an index $I \geq 0$ such that $m_1, \dots, m_I$ are all O-answers, called the initial assumptions (if $I = 0$ there are none).*
- (iii) *After the initial question and the initial assumptions, moves alternate between Opponent and Proponent.*
- (iv) *Every root question — that is, every question whose node is a root of the forest of $\mathbb{A}$ — is a P-question, and its justifying question is the initial question.*
- (v) *For each question m_j with $j > I$ that is not a root question, there is a question m_k with $k < j$ whose node is the immediate predecessor, in $\mathbb{A}$, of the node of m_j ; we call m_k the justifying question for m_j .*
- (vi) *For each answer m_i with $i > I$, there is a question m_k with $k < i$ at the same node of $\mathbb{A}$. If m_j is the justifying question for m_k , we call m_j the justifying question for m_i .*
- (vii) *Each question is answered at most once.*
- (viii) *The initial question is answered only once every non-initial question has been answered.*
- (ix) *For each P-answer m_i there is an O-answer m_j with the same label and $j < i$.*
- (x) *The initial question is answered only if there is an O-answer to a root vertex with the same label as the initial question, or an initial assumption with the same label as the initial question.*

The datum of the initial question and of the initial assumptions is an atom; the datum of every other move is a node of the arena. The label of a move is its atom, or the atom labelling its node. ◀

A strategy is Proponent’s policy for continuing a play: it dictates Proponent’s responses to whatever Opponent has done so far, and it must respond whenever the rules of play permit Opponent nothing further to say.

► **Definition 10 (Strategy).** *A strategy for an arena $\mathbb{A}$ is a function σ mapping each play $\pi = (m_0, \dots, m_k)$ whose last move is an O-move to a (possibly empty) sequence of moves $\sigma(\pi) = (m_{k+1}, \dots, m_n)$, subject to the following conditions:*

- (coherence) *the concatenation $\pi \cdot \sigma(\pi)$ is a play;*
- (responsiveness) *if π contains no unanswered P-questions that could be answered by Opponent in the next move, then $\sigma(\pi)$ is non-empty.*

A strategy σ is winning for an atom r from a set of assumptions Q if it satisfies the following further condition:

- (winning) *every maximal play that opens with the initial question $O?r$ followed by the initial assumptions $O!q$, for $q \in Q$, and in which Proponent plays according to σ , contains a P-answer to the initial question.*

We write $Q \blacktriangleright_{\mathbb{A}} r$ to express that there is a winning strategy for r from Q in $\mathbb{A}$. ◀

To conclude this section, we observe the following whose proof is deferred to Appendix A.

► **Lemma 11 (Monotonicity).** *If $Q \blacktriangleright_{\mathbb{A}} r$ and $\mathbb{B} \supseteq \mathbb{A}$, then $Q \blacktriangleright_{\mathbb{B}} r$.* ◀

In this paper, we show that our notion of game exactly captures derivability in a base, the foundational concept of Sandqvist’s B-eS. This characterization then allows us to explicate the defining clauses of the semantics, including the clause for disjunction. Before proceeding, however, we pause to develop intuitions for these definitions through an extended example that is not immediately about logical consequence: 4×4 Sudoku.

4 Example: 4×4 Sudoku

As discussed, the concept of game developed here is modelled on proof-search. Despite the dialogical presentation, therefore, the better analogy is with single-player games of complete information: not chess or Go, but solitaire or the fourteen-fifteen puzzle. The Opponent pursues no agenda of its own but stands for the environment, granting or withholding whatever the rules permit. In this section, we formalise 4×4 Sudoku.

The rules of 4×4 Sudoku are those of regular Sudoku [13], but with a reduced board. A *state* of play is a board in which some cells, possibly none, contain numbers from 1 to 4. We represent states as propositions: let $States \subseteq At$ be the set of all possible 4×4 Sudoku states, legal and illegal alike. From these atoms we build an arena — which we call the *Sudoku arena*, denoted $\mathbb{S}$ — that models the relation between states.

In $\mathbb{S}$, trees have at most depth two. Each root vertex has at most one child. The root of each tree is labelled with a legal game state. The child of the root vertex is labelled with one of the possible game states accessible from the label of the root vertex. If the root vertex accesses no other game state, it has no child. Given a game state p with accessible game states $q_1, \dots, q_n$ ($n \geq 1$), there will be n trees with the root vertex labelled p :

$$\begin{array}{ccccc} p & p & & p \\ | & | & \dots & | \\ q_1 & q_2 & & q_n \end{array}$$

If p is a legal game state with no accessible game states from it, then $\mathbb{S}$ contains a single-vertex tree labelled p . For example, consider the game state p :

1	2	3	4
3	4	1	2
4	1	2	3
2	3	4	

There will be four trees with the root labelled p :

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Note that among the child vertices of p , only q_1 is a legal game state. Therefore, there will be no tree with a root vertex labelled q_2 , q_3 , or q_4 . Since q_1 is an ending game state, there will be a tree with just one node labelled q_1 . Therefore, Proponent has a winning strategy only if she chooses the tree $q_1 \Rightarrow p$:

$$O?p, P?p, O?q_1, P?q_1, O!q_1, P!q_1, O!p, P!p$$

We consider another game, in which Opponent asks for:

1	2	3	4
3	4	1	2
4	1	2	3
		4	

There is a winning strategy for Proponent:

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$$O?p, P?p, O?q, P?q, O?r, P?r, O?s, P?s, O!s, P!s, O!r, P!r, O!q, P!q, O!p, P!p$$

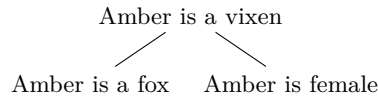
5 Faithfulness and Adequacy

This section establishes that the game-theoretic apparatus of Section 3 is an exact counterpart of the proof-theoretic apparatus of Section 2. Two grades of correspondence must be distinguished:

- *Adequacy*. This is extensional equivalence: the game interpretation is adequate for derivability in a base when an atom is derivable from given assumptions iff Proponent has a winning strategy for it in the corresponding arena. It settles the yes/no question of derivability and nothing more.
- *Faithfulness*. This operates one level down, at the level of the objects themselves rather than their mere existence: it concerns whether derivations and strategies correspond, not merely whether they coexist.

What we state and prove in this section is adequacy: Theorem 14 shows that derivability in a base yields winning strategies in the corresponding arena, Theorem 15 establishes the converse, and Corollary 16 records the biconditional. The proofs, however, deliver more than the statements record: both are compositional translations, under which plays track the construction of proofs in a base. We return to this point in Remark 17.

The correspondence rests on a simple observation: the data of a base is exactly the data of an arena. An atomic rule, read game-theoretically, is a tree: its conclusion is a node at which the rule may be invoked, its premisses are the challenges to which invoking it exposes the invoker, and its discharged hypotheses are the concessions available for meeting those challenges. Consider, for example, the rule *from* ‘Amber is a fox’ and ‘Amber is female’, *infer* ‘Amber is a vixen’, which determines the arena



relating the three logic-free sentences. A dialogue in this arena proceeds as an argument would: Opponent opens by asking whether Amber is a vixen and granting whatever assumptions are in play — say, that Amber is a fox and that Amber is female; Proponent invokes the rule by playing a question at the root; Opponent challenges the premisses by questioning the child nodes; Proponent answers each challenge, as she may, since Opponent has conceded the corresponding facts; and, every premiss being met, Opponent concedes the root, whereupon Proponent answers the initial question and the argument is won. A strategy that wins in this way against every line of challenge is nothing other than a derivation in the base.

The comparison proceeds according to the following dictionary, which the results of this section prove exact at the level of consequence:

<i>level</i>	<i>proof-theoretic concept</i>	<i>game-theoretic concept</i>
data	base	arena
dynamics	proof-search	play
consequence	derivability	winning strategy

We begin by showing that bases and arenas are isomorphic as data structures. The isomorphism is witnessed by a pair of translations, defined by structural recursion, that convert atomic trees to base rules and back.

► **Definition 12** (Translations). *Define* $\text{Base}(-)$ *on rooted trees of atoms and* $\text{Arena}(-)$ *on base rules by mutual structural recursion:*

$$\text{Base}((\Rightarrow p)) := \langle p \rangle \quad \text{Base}((\Psi_1, \dots, \Psi_n \Rightarrow p)) := \langle \{\text{Base}(\Psi_1), \dots, \text{Base}(\Psi_n)\}, p \rangle$$

and

$$\text{Arena}(\langle p \rangle) := (\Rightarrow p) \quad \text{Arena}(\langle \{\rho_1, \dots, \rho_n\}, p \rangle) := (\text{Arena}(\rho_1), \dots, \text{Arena}(\rho_n) \Rightarrow p)$$

where the Ψ_i are atomic trees and the ρ_i base rules. Extend both maps elementwise:

$$\text{Base}(\mathbb{A}) := \{\text{Base}(\Psi) \mid \Psi \in \mathbb{A}\} \quad \text{and} \quad \text{Arena}(\mathcal{B}) := \{\text{Arena}(\rho) \mid \rho \in \mathcal{B}\} \quad \blacktriangleleft$$

That the translations constitute an isomorphism is the content of the following lemma, whose proof is deferred to Appendix B.

► **Lemma 13.** *The functions $\text{Base}(-)$ and $\text{Arena}(-)$ are mutually inverse: for every base $\mathcal{B}$ and every arena $\mathbb{A}$, $\text{Arena}(\text{Base}(\mathbb{A})) = \mathbb{A}$ and $\text{Base}(\text{Arena}(\mathcal{B})) = \mathcal{B}$.* ◀

The translations thus identify bases with arenas at the level of data. It remains to show that the identification extends to the level of consequence: derivability in a base obtains precisely when there is a winning strategy in the corresponding arena. We treat the two directions separately, beginning with the soundness of the game interpretation.

► **Theorem 14** (Soundness for the game interpretation). *Let $\mathcal{B}$ be a base, let S be a finite set of atoms, and let p be an atom. If $S \vdash_{\mathcal{B}} p$, then $S \blacktriangleright_{\text{Arena}(\mathcal{B})} p$.*

Proof. By Definition 3, the family of relations $\vdash_{\mathcal{B}}$, indexed by bases $\mathcal{B}$, is the least family closed under (Ref) and (App). It therefore suffices to show that the family of relations $\blacktriangleright_{\text{Arena}(\mathcal{B})}$ is closed under the corresponding conditions:

- (Ref) if $p \in S$, then $S \blacktriangleright_{\text{Arena}(\mathcal{B})} p$;
- (App) if $(Q_1 \triangleright q_1), \dots, (Q_n \triangleright q_n) \triangleright r$ is a rule of $\mathcal{B}$ and $S \blacktriangleright_{\text{Arena}(\mathcal{B} \cup Q_i)} q_i$,
for each $i = 1, \dots, n$, then $S \blacktriangleright_{\text{Arena}(\mathcal{B})} r$.

The theorem then follows by induction on the definition of $S \vdash_{\mathcal{B}} p$.

Case (Ref). Suppose $p \in S$. By clauses (i) and (ii) of Definition 9, every play for p from S opens with the initial question $O?p$ followed by the initial assumptions $O!s_1, \dots, O!s_k$, where $s_1, \dots, s_k$ enumerates S ; since $p \in S$, the opening contains the initial assumption $O!p$. Let σ be the strategy that responds to this opening with $P!p$. The move is legitimate: clause (ix) is met by the initial assumption $O!p$; clause (x) permits answering the initial question, again by that initial assumption; and clause (viii) is satisfied vacuously, the play containing no non-initial question. The resulting play

$$O?p, O!s_1, \dots, O!s_k, P!p$$

is maximal — the initial question has been answered, and by clause (vii) no question may be answered twice — and it is the unique maximal play in which Proponent follows σ . It contains a P-answer to the initial question, so σ is winning and $S \blacktriangleright_{\text{Arena}(\mathcal{B})} p$.

Case (App). Suppose $\rho = (Q_1 \triangleright q_1), \dots, (Q_n \triangleright q_n) \triangleright r$ is a rule of $\mathcal{B}$ and, for each $i = 1, \dots, n$, let σ_i be a winning strategy witnessing $S \blacktriangleright_{\text{Arena}(\mathcal{B} \cup Q_i)} q_i$. We construct a winning strategy σ witnessing $S \blacktriangleright_{\text{Arena}(\mathcal{B})} r$. We first record the shape of the arenas involved. The arena $\text{Arena}(\{\rho\})$ is a single tree Ψ whose root is labelled r , whose children are labelled $q_1, \dots, q_n$, and in which the subtrees below q_i are exactly the trees $\text{Arena}(Q_i)$. Observe that Ψ is an element of $\text{Arena}(\mathcal{B})$ and $\text{Arena}(\mathcal{B} \cup Q_i) = \text{Arena}(\mathcal{B}) \cup \text{Arena}(Q_i)$.

By clauses (i) and (ii) of Definition 9, every play for r from S opens $O?r, O!s_1, \dots, O!s_k$, with $s_1, \dots, s_k$ enumerating S . The strategy σ extends this opening with the root question $P?r$ at the root of Ψ , which clause (iv) licenses: its node is a root of the forest $\text{Arena}(\mathcal{B})$, and its justifying question is the initial question. Opponent now has two kinds of move available: by clause (v), Opponent may challenge a premiss, playing $O?q_i$ at a child of the root of Ψ , justified by $P?r$; by clause (vi), Opponent may instead concede, answering $P?r$

with $O!r$. (If $n = 0$, conceding is Opponent's only option.) These remain Opponent's only options throughout, alongside the moves arising inside the sub-plays described below.

Suppose Opponent challenges with $O?q_i$. Then σ continues by playing according to σ_i , under the following identification: regard $O?q_i$ as the initial question, and the initial assumptions $O!s_1, \dots, O!s_k$ of the ambient play as the initial assumptions, of a simulated play in the arena $\text{Arena}(\mathcal{B} \cup Q_i)$. Moves of σ_i at nodes of $\text{Arena}(\mathcal{B})$ carry over unchanged, since $\text{Arena}(\mathcal{B})$ is a sub-forest of both arenas. Moves of σ_i at nodes of $\text{Arena}(Q_i)$ are transported to the corresponding nodes of Ψ below q_i . The only clause of Definition 9 sensitive to this relocation is the justification of questions at the roots of $\text{Arena}(Q_i)$: in the simulated play these are root questions, justified by the initial question $O?q_i$ via clause (iv), while in the ambient play the corresponding nodes are children of q_i in Ψ , so the same questions are justified by $O?q_i$ via clause (v). In either case the justifying question is $O?q_i$, and every other justification is preserved. Clause (x) constrains only the answer to the ambient initial question, so it imposes nothing on the sub-plays; clauses (vi), (vii), and (ix) hold in the ambient play because they hold, move for move, in the simulated one.

Opponent may interleave the challenges $O?q_1, \dots, O?q_n$ and the sub-plays they open. No interference results: justification in Definition 9 relates *occurrences* of moves, so each non-initial move of the ambient play other than $P?r$ and $O!r$ descends, by tracing justifying questions and the responses of σ , from a unique challenge $O?q_i$, and σ answers each O-move according to the sub-strategy σ_i of the challenge from which it descends. The sub-plays therefore proceed independently, each a play of σ_i under the identification above. Since σ_i is winning, every maximal continuation of the sub-play opened by $O?q_i$ contains a P-answer to its simulated initial question (i.e., the move $P!q_i$) and, since σ_i respects clause (viii) in the simulated play, every question of that sub-play has been answered by the time $P!q_i$ is played.

Finally, suppose Opponent concedes $O!r$ (whether immediately after $P?r$ or after any number of challenges). Once every non-initial question of the play has been answered, σ answers the initial question with $P!r$. The move is legitimate: clause (ix) is met by $O!r$; clause (x) is met because $O!r$ is an O-answer at a root vertex whose label coincides with that of the initial question; and clause (viii) holds by the hypothesis that no non-initial question stands unanswered.

It remains to check that σ is winning. Let π be a maximal play in which Proponent follows σ ; recall that plays are finite sequences (Definition 9). The question $P?r$ must be answered in π : while it stands unanswered, clause (vi) provides Opponent with the legal move $O!r$ whenever it is Opponent's turn, and responsiveness (Definition 10) ensures that the play cannot halt on Proponent's turn; hence a play in which $P?r$ is unanswered is not maximal. By the same reasoning, every challenge $O?q_i$ that Opponent raises is answered with $P!q_i$, since σ_i is winning, and every question internal to a sub-play is answered before the corresponding $P!q_i$, as observed above. Thus every non-initial question of π is answered, $O!r$ occurs in π , and σ concludes with $P!r$. Every maximal play in which Proponent follows σ therefore contains a P-answer to the initial question, so σ is winning and $S \blacktriangleright_{\text{Arena}(\mathcal{B})} r$. ◀

► **Theorem 15** (Completeness for the game interpretation). *Let $\mathbb{A}$ be an arena, let P be a finite set of atoms, and let r be an atom. If $P \blacktriangleright_{\mathbb{A}} r$, then $P \vdash_{\text{Base}(\mathbb{A})} r$.*

Proof. By clauses (i) and (ii) of Definition 9, every play opens with the initial question and the initial assumptions

$$\pi_0 := O?r, O!p_1, \dots, O!p_k$$

where $p_1, \dots, p_k$ enumerates P . Let σ be a winning strategy witnessing $P \blacktriangleright_{\mathbb{A}} r$. Opponent is *obstinate* in a play if, whenever he can ask a question not previously asked, he does so. Let

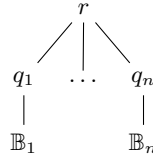
ℓ be the minimal length of the maximal plays in which Opponent is obstinate and Proponent plays according to σ . We proceed by induction on ℓ .

Base case: $\ell = 1$. In this case, there is only one move after the opening π_0 . As σ is winning, that move must be $P!r$. By rule (ix) of Definition 9, a P-answer $P!r$ must be preceded by an O-answer with the same label; since no move other than the opening and $P!r$ occurs, that O-answer is one of the initial assumptions $O!p_1, \dots, O!p_k$. Hence $r \in P$, and $P \vdash_{\text{Base}(\mathbb{A})} r$ by (Ref) of Definition 3.

Inductive step: $\ell > 1$. Consider an arbitrary play π for r in $\mathbb{A}$ with assumptions P . By clause (ix), a P-answer $P!r$ must be preceded by an O-answer with the same label. If $O!r$ is in the initial segment, then $r \in P$ and $P \vdash_{\text{Base}(\mathbb{A})} r$ by (Ref). So suppose $r \notin P$. By clause (x), the P-answer $P!r$ is licensed by an O-answer $O!r$ at a root vertex of $\mathbb{A}$. By clause (vi), that O-answer responds to an earlier P-question at the same vertex; its vertex being a root of the forest, this is a root question. Thus π has the form

$$\pi_0, \dots, P?r, \dots, O!r, P!r,$$

and there is a tree $\Psi \in \mathbb{A}$ whose root is labelled r and at which Proponent has played $P?r$:



Let $\rho := \text{Base}(\Psi)$; by Definition 12, ρ is a rule of $\text{Base}(\mathbb{A})$. We distinguish two cases.

Case $n = 0$. Then Ψ is the single vertex labelled r , so ρ is the premiss-free rule $\triangleright r$. Using (App), with its premiss condition vacuously met, yields $P \vdash_{\text{Base}(\mathbb{A})} r$.

Case $n \geq 1$. We claim that $P \blacktriangleright_{\mathbb{A} \cup \mathbb{B}_i} q_i$ for each i . To this end, we construct a winning strategy σ_i witnessing this from σ . Let θ be any play for q_i from P in $\mathbb{A} \cup \mathbb{B}_i$, and let θ' be obtained from θ by relocating every move at a node of $\mathbb{B}_i$ to the corresponding node in the copy of $\mathbb{B}_i$ sitting below q_i in Ψ (moves at nodes of $\mathbb{A}$ are left unchanged). We check that

$$\pi_i := \pi_0, P?r, O?q_i, \theta'$$

is a legal play for r from P in $\mathbb{A}$. The only clause of Definition 9 sensitive to the relocation is the justification of questions at the root q_i of $\mathbb{B}_i$: in θ such a question is a root question justified by the initial question $O?q_i$ (clause (iv)), whereas in π_i the corresponding node is a child of r in Ψ , so the same question is justified by $O?q_i$ via clause (v). In either case the justifying question is $O?q_i$, and every other justification is preserved; clauses (vi)–(x) transfer move for move. Hence π_i is a play for r in $\mathbb{A}$ assuming P . We now define σ_i by relocating σ 's responses backwards: set $\sigma_i(\theta)$ to be the sequence obtained from $\sigma(\pi_i)$ by returning every move at a node of Ψ below q_i to the corresponding node of $\mathbb{B}_i$ (and leaving moves at nodes of $\mathbb{A}$ unchanged). This is well defined, since π_i ends with an O-move exactly when θ does, and σ is defined on such plays; coherence and responsiveness of σ_i follow from those of σ under the relocation, which is a bijection on the relevant nodes. Since σ is a winning strategy for r from P in $\mathbb{A}$, every maximal play in which Proponent follows σ_i contains the P-answer $P!q_i$; hence σ_i is winning. By construction, every maximal obstinate play for q_i from P in $\mathbb{A} \cup \mathbb{B}_i$ in which Proponent follows σ_i embeds as a proper subsequence of a maximal obstinate play

for r from P in $\mathbb{A}$ in which Proponent follows σ ; hence its length is strictly smaller than ℓ , and the induction hypothesis yields $P \vdash_{\text{Base}(\mathbb{A} \cup \mathbb{B}_i)} q_i$.

By the conversion between bases and arenas, $\text{Base}(\mathbb{A} \cup \mathbb{B}_i) = \text{Base}(\mathbb{A}) \cup \text{Base}(\mathbb{B}_i) = \text{Base}(\mathbb{A}) \cup \{\text{Base}(\mathbb{B}_i) \triangleright q_i\}$. Now $\text{Base}(\mathbb{B}_i) \triangleright q_i$ is precisely the i -th premiss pair of the rule ρ , so writing $Q_i := \text{Base}(\mathbb{B}_i)$ for the rules discharged by that premiss we have $P \vdash_{\text{Base}(\mathbb{A}) \cup Q_i} q_i$ for each $i = 1, \dots, n$. Since ρ is a rule of $\text{Base}(\mathbb{A})$, clause (App) of Definition 3 yields $P \vdash_{\text{Base}(\mathbb{A})} r$, as required. $\blacktriangleleft$

Combining the two theorems with Lemma 13, we obtain the adequacy of the game interpretation.

► **Corollary 16 (Adequacy).** *Let S be a finite set of atoms and let p be an atom. For every base $\mathcal{B}$, $S \vdash_{\mathcal{B}} p$ iff $S \blacktriangleright_{\text{Arena}(\mathcal{B})} p$ and, equivalently, for every arena $\mathbb{A}$, $S \blacktriangleright_{\mathbb{A}} p$ iff $S \vdash_{\text{Base}(\mathbb{A})} p$.*

► **Remark 17 (Faithfulness).** Corollary 16 is extensional: it asserts the coincidence of two relations, not a correspondence between their witnesses. The proofs, however, are constructive and compositional. The strategy exhibited in the proof of Theorem 14 is defined by recursion on the derivation: each application of (App) becomes a root question together with sub-strategies meeting Opponent's challenges to the premisses of the rule invoked, and each appeal to (Ref) becomes an immediate answer from the initial assumptions. Conversely, the proof of Theorem 15 reads a derivation off a winning strategy. In this sense, plays track the construction of proofs in a base. Elevating this observation to a full-and-faithful correspondence — fullness: every winning strategy is the interpretation of some derivation; faithfulness: distinct derivations, up to a suitable notion of identity, are interpreted by distinct strategies — would give an atomic analogue of the full completeness theorems of game semantics [1, 25]. Doing so requires fixing notions of identity for derivations and for strategies, and we leave it to future work; see Section 7. $\blacktriangleleft$

Corollary 16 discharges the dictionary with which this section opened: bases and arenas are the same structures and, at the level of consequence, derivability and the existence of winning strategies coincide. In particular, provability-in-a-base may be replaced by the existence of a winning strategy in the corresponding arena without loss. Section 6 makes exactly this replacement throughout Sandqvist's clauses, yielding the game-extension semantics for IPL.

6 Logical Metatheory

We have thus far established that, at the atomic level, the proof-theoretic and the game-theoretic readings based on proof-search coincide: derivability in a base is exactly the existence of a winning strategy in the corresponding arena (Corollary 16). The correspondence is, so far, limited to *atomic* formulae. The question of this section is whether the games extend from atoms to formulae: What is it for Proponent to *win* a formula ϕ in an arena $\mathbb{A}$, and does the resulting notion of validity characterize IPL?

There are some intuitive choices of definitions. For example, Proponent wins a conjunction $\phi \wedge \psi$ in an arena $\mathbb{A}$ just in case she wins each conjunct ϕ and ψ in $\mathbb{A}$. This captures the intensional characterization of a conjunction as two separate obligations. Similarly, Proponent wins an implication $\phi \rightarrow \psi$ in $\mathbb{A}$ just in case in any arena $\mathbb{B}$ extending $\mathbb{A}$ in which she wins ϕ , she also wins ψ . These heuristic accounts are unproblematic.

Disjunction requires more care, and it is here that the proof-search motivation does real work. A naïve approach would be that Proponent wins the disjunction $\phi \vee \psi$ in $\mathbb{A}$ just in case she wins either ϕ or ψ in $\mathbb{A}$. This condition is sufficient: if Proponent wins a disjunct,

$\Vdash'_{\mathbb{A}} p$	iff	$\blacktriangleright_{\mathbb{A}} p$	(At)
$\Vdash'_{\mathbb{A}} \phi \rightarrow \psi$	iff	$\phi \Vdash'_{\mathbb{A}} \psi$	($\rightarrow$)
$\Vdash'_{\mathbb{A}} \phi \wedge \psi$	iff	$\Vdash'_{\mathbb{A}} \phi$ and $\Vdash'_{\mathbb{A}} \psi$	($\wedge$)
$\Vdash'_{\mathbb{A}} \perp$	iff	$\Vdash'_{\mathbb{A}} p$, for every atom $p \in At$	($\perp$)
$\Vdash'_{\mathbb{A}} \phi \vee \psi$	iff	for all $\mathbb{B} \supseteq \mathbb{A}$ and every atom $p \in At$, if $\phi \Vdash'_{\mathbb{B}} p$ and $\psi \Vdash'_{\mathbb{B}} p$, then $\Vdash'_{\mathbb{B}} p$	($\vee$)
$\Delta \Vdash'_{\mathbb{A}} \phi$	iff	for all $\mathbb{B} \supseteq \mathbb{A}$, if $\Vdash'_{\mathbb{B}} \psi$ for all $\psi \in \Delta$, then $\Vdash'_{\mathbb{B}} \phi$	(Inf)

■ **Figure 2** Support in an Arena

then she wins the disjunction. But it is not necessary: that Proponent wins a disjunction does not yield a winning strategy for either disjunct. The lottery example by Gheorghiu and Gu [19] makes this clear: in playing a lottery one knows that *some* ticket will win, but does not know *which* ticket will win. So what is the appropriate treatment for disjunction?

The proof-search reading helps motivate an answer. A winning strategy is a policy for *continuing the search for a proof*; accordingly, to hold a disjunction is not to hold a disjunct but to know how every search that could be completed from ϕ , and could be completed from ψ , may be completed outright. That is, the disjunction is explicated by the *continuations* it serves: whatever (atomic) goal p is winnable from ϕ and winnable from ψ is winnable simpliciter. This is a continuation-passing-style reading of $\vee$ — compare the interpretation of classical disjunction via the $\lambda\mu\nu$ -calculus and continuation models [37, 39]. This is precisely the game-theoretic counterpart of Sandqvist’s second-order clause. Indeed, the reading can be made exact: Gheorghiu [18] shows that support in a fixed base coincides with proof-search in a second-order hereditary Harrop logic program under a CPS encoding of the connectives, with $\phi \vee \psi$ encoded as $\forall X ((\phi \rightarrow X) \rightarrow (\psi \rightarrow X) \rightarrow X)$. The quantification over extensions $\mathbb{B} \supseteq \mathbb{A}$ reflects that search may enlarge the arena — Proponent must be able to continue however the available rules grow — and the restriction to atomic continuations keeps the clause pre-logical.

► **Definition 18** (Support in an Arena). *Let $\mathbb{A}$ be an arena. The support relation $\Vdash'_{\mathbb{A}}$, relating finite (possibly empty) sets of formulae to formulae, is defined inductively by the clauses of Figure 2, in which, in (Inf), $\Delta \neq \emptyset$. We say that Γ supports ϕ , written $\Gamma \Vdash' \phi$, iff $\Gamma \Vdash'_{\mathbb{A}} \phi$, for every arena $\mathbb{A}$.* ◀

Since the clauses of Figure 2 are those of Figure 1 with provability-in-a-base replaced, at the base case, by the existence of a winning strategy, the results of Section 5 transfer the two relations into one another. We call this semantics *game-extension semantics* (G-eS).

► **Corollary 19.** *Let Γ be a finite set of formulae, let ϕ be a formula, let $\mathbb{A}$ be an arena, and let $\mathcal{B}$ be a base. Then*

$$\Gamma \Vdash'_{\mathbb{A}} \phi \quad \text{iff} \quad \Gamma \Vdash_{\text{Base}(\mathbb{A})} \phi,$$

and

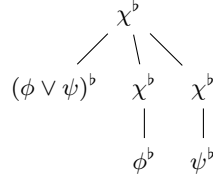
$$\Gamma \Vdash'_{\text{Arena}(\mathcal{B})} \phi \quad \text{iff} \quad \Gamma \Vdash_{\mathcal{B}} \phi.$$

Proof. By induction on ϕ . The base case is Corollary 16; the inductive cases follow from the clause-by-clause match of Figure 2 with Figure 1, together with Lemma 13, which carries the quantification over extensions across the translations. ◀

Combining Corollary 19 with Theorem 6 yields the main logical result:

► **Corollary 20.** *Let Γ be finite. Then $\Gamma \vdash_{\text{NJ}} \phi$ iff $\Gamma \Vdash' \phi$.* ◀

Although Corollary 20 follows by transfer, both soundness and completeness admit direct proofs, obtained by adapting Sandqvist’s arguments [41] to the game-theoretic setting; these are given in Appendix C and remain constructive and elementary. Soundness is standard: each rule of NJ preserves support in an arena. Completeness proceeds by constructing a *simulation arena* $\mathbb{N}$ in which each subformula ξ of the formulae in question is represented by a fresh atom ξ^b , and support and proof-search are simulated simultaneously. It is worth pausing on the trees that $\mathbb{N}$ allots to a disjunction, for they exhibit the continuation-passing reading concretely: alongside the trees encoding the introductions, $\phi^b \Rightarrow (\phi \vee \psi)^b$ and $\psi^b \Rightarrow (\phi \vee \psi)^b$, the arena contains, for each atomic continuation χ^b , the tree



Read game-theoretically: Proponent answers a question at χ^b by invoking the disjunction together with two sub-searches, one continuing from ϕ^b and one from ψ^b . A winning strategy for a disjunction is thus used exactly as the clause $(\vee)$ says it may be: not by producing a disjunct, but by supplying, for either eventuality, a continuation of the search [23]. That the completeness argument goes through with this clause — and fails for the naïve one (cf. [31, 33, 23]) — is the technical expression of the same point.

We close by comparing this treatment of disjunction with that of the Pym-Ritter games from which our setup departs [39]. There, the arena for a formula containing $\psi_1 \vee \psi_2$ contains a switching vertex: Opponent’s demand for the disjunction is met by Proponent playing L or R , whereupon Opponent demands a proof of the chosen disjunct. This is the introduction rule read reductively, and it is apt when Proponent is *constructing* a proof of the disjunction; but it exacts a commitment that proof-search cannot in general afford — choose the wrong disjunct and the search fails although the goal is provable. In the Pym-Ritter framework, the remedy is a matter of *control*: backtracking, modelled by embedding the intuitionistic games in classical ones, where Proponent may return to the switch and play the other move. Our clause internalizes that remedy semantically. Rather than committing to a disjunct and reserving the right to backtrack, a winning strategy for $\phi \vee \psi$ carries continuations for both disjuncts, so no commitment is ever made that the search might need to retract. Where the switching reading gives disjunction the type of a choice, the present reading gives it the type of its eliminations — exactly the shift effected by continuation-passing-style translations, under which the control operators of classical search become intuitionistically definable [39]. Sandqvist’s clause for $\vee$, which within B-eS can appear a technical device required for completeness, thus acquires a direct computational content in G-eS: it is the type of a strategy that knows how to continue the search, however the disjunction is eventually resolved [23].

7 Discussion

The main contribution of this work is the establishment of a direct correspondence between game-theoretic and proof-theoretic notions. Lemma 13 identifies bases with arenas at the level of data and Corollary 16 identifies, at the level of consequence, winning strategies in arenas and derivability in bases. Corollary 19 equates validity in bases with validity in arenas and Corollary 20 demonstrates that the semantics developed in this paper is, in fact, a semantics for IPL. In this sense, the semantics presented in this paper follows the inferentialist project: we have shown that concepts of game semantics can be reconstructed from the primitives of P-tS.

Gheorghiu [18], showed that support in a base coincides with uniform proof-search in a second-order hereditary Harrop program. As a corollary of the correspondences established here, this result acquires an operational reading: validity in an arena is not merely characterized by winning strategies, but is executable, with strategies computed by goal-directed search against the arena's base.

The focus of this work is the development of a game semantics rooted in proof-theoretical concepts. Future directions might include an extension of the semantics presented in this paper to the cases corresponding to first- and higher-order predicate logic (cf. [16, 17]). Connections with categorical accounts of B-eS (e.g., [38]) and proof-relevant semantics should also be considered.

One might also explore different notions of play and how they give rise to different logics. Specifically, how can substructural reasoning (cf. [20, 21]) be captured by constraining the definition of play: contraction, for example, can be restricted by limiting the number of times players can move on the same node. Also, relevance can be imposed by a condition that forces the existence of a question relative to every vertex in an arena.

One might also consider how classical reasoning might be captured using the inferentialist game semantics apparatus. At minimum, classical logic can be obtained from the current game-extension semantics if the depth of trees in arenas is restricted to trees with maximum depth two and the disjunction clause is omitted (cf. [16, 40]). This paper offers an alternative way following [39]: to relax clause (ix) of Definition 9 so that for any P-answer m_i there exists an O-question m_k and an O-answer m_j such that m_i is hereditarily justified by m_k and m_j is an O-answer with the same label as m_k and $k < j < i$. Another approach would be to develop concurrent games, as in [3], to capture classical reasoning.

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A Proofs for Section 3

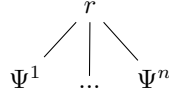
Proof of Lemma 11 (Monotonicity). Suppose $Q \blacktriangleright_{\mathbb{A}} r$ and $\mathbb{B} \supseteq \mathbb{A}$. There are two cases.

Case $r \in Q$. Since $r \in Q$, the atom r is among the initial assumptions of any play for r from Q . Hence Proponent may answer the initial question immediately, and

$$O?r, O!q_1, \dots, O!q_n, P!r \quad (q_1, \dots, q_n \in Q)$$

is a winning strategy in any arena; in particular, $Q \blacktriangleright_{\mathbb{B}} r$.

Case $r \notin Q$. By clause (x) of Definition 9, there is a tree $\Psi \in \mathbb{A}$ whose root vertex is labelled r ; let $\Psi^1, \dots, \Psi^n$ be its immediate subtrees:



By (winning), a winning strategy witnessing $Q \blacktriangleright_{\mathbb{A}} r$ begins with the initial question $O?r$ and the initial assumptions $O!q_1, \dots, O!q_n$ for $q_1, \dots, q_n \in Q$. For Proponent to answer the initial question, there must be an O-answer at the root of Ψ and hence, by clause (vi) of Definition 9, a P-question at that node. Since $\mathbb{B} \supseteq \mathbb{A}$, we have $\Psi \in \mathbb{B}$; and by Definition 7 the root of Ψ is a P-node in $\mathbb{B}$, so Proponent may question it in $\mathbb{B}$ exactly as in $\mathbb{A}$, whatever further trees $\mathbb{B}$ contains. Hence the strategy that answers the initial question $O?r$ from the initial assumptions $O!q_1, \dots, O!q_n$ in $\mathbb{A}$ answers it from the same assumptions in $\mathbb{B}$; that is, $Q \blacktriangleright_{\mathbb{B}} r$. ◀

B Proofs for Section 5

Proof of Lemma 13. We prove two claims: first, that $\text{Arena}(\text{Base}(\Psi)) = \Psi$ for every atomic tree Ψ , by induction on the depth of Ψ ; second, that $\text{Base}(\text{Arena}(\Gamma)) = \Gamma$ for every base rule Γ , by induction on the level of Γ .

First claim.

Base case. Suppose $\Psi = (q_1, \dots, q_n \Rightarrow p)$. By Definition 12, $\text{Base}(\Psi) = (q_1, \dots, q_n \triangleright p)$, and, again by Definition 12, $\text{Arena}((q_1, \dots, q_n \triangleright p)) = (q_1, \dots, q_n \Rightarrow p) = \Psi$.

Inductive step. Suppose $\Psi = (\Theta_1, \dots, \Theta_n \Rightarrow p)$, where each Θ_i (with $1 \leq i \leq n$) is an immediate subtree of p , and assume $\text{Arena}(\text{Base}(\Theta_i)) = \Theta_i$ for every Θ_i . By Definition 12, applied twice,

$$\begin{aligned} \text{Arena}(\text{Base}((\Theta_1, \dots, \Theta_n \Rightarrow p))) &= \text{Arena}((\text{Base}(\Theta_1), \dots, \text{Base}(\Theta_n) \triangleright p)) \\ &= (\text{Arena}(\text{Base}(\Theta_1)), \dots, \text{Arena}(\text{Base}(\Theta_n)) \Rightarrow p) \end{aligned}$$

By the induction hypothesis, the right-hand side is $(\Theta_1, \dots, \Theta_n \Rightarrow p) = \Psi$.

Second claim.

Base case. Suppose $\Gamma = (q_1, \dots, q_n \triangleright p)$. By Definition 12, $\text{Arena}(\Gamma) = (q_1, \dots, q_n \Rightarrow p)$, and, again by Definition 12, $\text{Base}((q_1, \dots, q_n \Rightarrow p)) = (q_1, \dots, q_n \triangleright p) = \Gamma$.

Inductive step. Suppose $\Gamma = (\Delta_1, \dots, \Delta_n \triangleright p)$, where each Δ_i (with $1 \leq i \leq n$) is a base rule of lower level, and assume $\text{Base}(\text{Arena}(\Delta_i)) = \Delta_i$ for every Δ_i . By Definition 12, applied twice,

$$\begin{aligned} \text{Base}(\text{Arena}((\Delta_1, \dots, \Delta_n \triangleright p))) &= \text{Base}((\text{Arena}(\Delta_1), \dots, \text{Arena}(\Delta_n) \Rightarrow p)) \\ &= (\text{Base}(\text{Arena}(\Delta_1)), \dots, \text{Base}(\text{Arena}(\Delta_n)) \triangleright p) \end{aligned}$$

By the induction hypothesis, the right-hand side is $(\Delta_1, \dots, \Delta_n \triangleright p) = \Gamma$. ◀

C Proofs for Section 6

That IPL is sound with respect to the semantics is shown in an entirely standard way. One shows that support in an arena is preserved by the rules of the system NJ for IPL [15, 34], as given in Figure 3.

► **Theorem 21 (Soundness).** *If $\Gamma \vdash_{\text{NJ}} \phi$, then $\Gamma \Vdash' \phi$.*

$$\begin{array}{c}
\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge I \\
\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \vee I \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi} \vee I \\
\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \supset \psi} \supset I \\
\frac{\Gamma \vdash \perp}{\Gamma \vdash \phi} \perp E \\
\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} \wedge E \quad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi} \wedge E \\
\frac{\Gamma \vdash \phi \vee \psi \quad \Gamma, \phi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi} \vee E \\
\frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi \supset \psi}{\Gamma \vdash \psi} \supset E
\end{array}$$

■ **Figure 3** Natural Deduction System NJ for IPL (eliding $\top$)

The more interesting direction is completeness. It too is articulated through NJ. The proof of completeness is similar to that for the corresponding result in by Sandqvist [41]. We construct a ‘simulation’ arena $\mathbb{N}$ by ‘flattening’, $(-)^b$, which is defined in the proof of completeness, below. In this arena, we simultaneously simulate support and proof-search and show that they coincide.

► **Definition 22** (Simulation arena). *We define the simulation arena $\mathbb{N}$ associated with a set of formulae Θ . Let Ξ be the set containing all elements of Θ and their subformulae. Then associate with each element ξ of Ξ a distinct atomic counterpart ξ^b such that*

■ $(\phi \wedge \psi)^b$:

$$\begin{array}{ccc}
(\phi \wedge \psi)^b & \phi^b & \psi^b \\
/ \quad \backslash & | & | \\
\phi^b & \psi^b & (\phi \wedge \psi)^b
\end{array}$$

■ $\phi \rightarrow \psi$

$$\begin{array}{ccc}
(\phi \rightarrow \psi)^b & & \\
| & & \\
\psi^b & & \psi^b \\
| & / \quad \backslash & \\
\phi^b & (\phi \rightarrow \psi)^b & \phi^b
\end{array}$$

■ $\phi \vee \psi$

$$\begin{array}{ccc}
& \chi^b & \\
/ \quad | \quad \backslash & & \\
(\phi \vee \psi)^b & \chi^b & \chi^b \\
| & | & | \\
\phi^b & \psi^b & (\phi \vee \psi)^b \\
& & | \\
& & \psi^b
\end{array}$$

■ $\perp$

$$\begin{array}{c} \phi^b \\ | \\ \perp^b \end{array}$$

This defines the simulation arena $\mathbb{N}$ associated with Θ . ◀

Atoms in $\mathbb{N}$ simulate the behaviour of the subformulae in ϕ .

► **Theorem 23** (Completeness). *If $\Xi \Vdash' \zeta$, then $\Xi \vdash_{\text{NJ}} \zeta$.*

Proof. Assume $\Xi \Vdash' \zeta$. Let Γ be the set containing all members of $\Xi \cup \{\zeta\}$ and their subsentences. With every non-atomic $\gamma \in \Gamma$ associate an atom $\gamma^b \notin \Gamma$ in such a way that $\gamma_1^b \neq \gamma_2^b$ whenever $\gamma_1 \neq \gamma_2$. Also for every atom $g \in \Gamma$, set $g^b = g$. Conversely, with every atom p associate a sentence $p^\sharp$ in such a way that $(\gamma^b)^\sharp = \gamma$ for every $\gamma \in \Gamma$; when p is not in the range of b , set $p^\sharp = p$. For any $\Phi^b = \phi^b | \phi \in \Phi$ and $P^\sharp = p^\sharp | p \in P$.

To prove completeness, Sandqvist, in [41], constructs a simulation base $\mathcal{N}$ (Figure 4) with the following properties:

$$\begin{array}{c} \frac{\phi^b \quad \psi^b}{(\phi \wedge \psi)^b} \wedge I \qquad \frac{\frac{\perp^b}{\phi^b} \perp E \quad (\phi \wedge \psi)^b}{\phi^b} \wedge E \quad \frac{(\phi \wedge \psi)^b}{\psi^b} \wedge E \\[10pt] \frac{\phi^b}{(\phi \vee \psi)^b} \vee I \quad \frac{\psi^b}{(\phi \vee \psi)^b} \vee I \qquad \frac{(\phi \vee \psi)^b \quad \frac{[\phi^b] \quad [\psi^b]}{\chi^b} \vee E}{\chi^b} \vee E \\[10pt] \frac{[\phi^b] \quad \psi^b}{(\phi \supset \psi)^b} \supset I \qquad \frac{\phi^b \quad (\phi \supset \psi)^b}{\psi^b} \supset E \end{array}$$

■ **Figure 4** Sandqvist's simulation base $\mathcal{N}$ [41]

(†) for every $\gamma \in \Gamma$ and every $\mathcal{B} \supseteq \mathcal{N}$, $\Vdash_{\mathcal{B}} \gamma^b$ iff $\Vdash_{\mathcal{B}} \gamma$

(‡) for any P and q , if $P \Vdash_{\mathcal{N}} q$, then $P^\sharp \vdash_{\text{NJ}} q^\sharp$

From these two properties, Sandqvist gets to the desired conclusion by the following argument: from the hypothesis that $\Xi \Vdash' \zeta$ it follows that $\Xi^b \Vdash_{\mathcal{N}} \zeta^b$ — for if $\mathcal{B} \supseteq \mathcal{N}$ and $\Vdash_{\mathcal{B}} \xi^b$ for every $\xi^b \in \Xi^b$, then by (†), $\Vdash_{\mathcal{B}} \xi$ for every $\xi \in \Xi$, whence $\Vdash_{\mathcal{B}} \zeta$ since $\Xi \Vdash' \zeta$, whence $\Vdash_{\mathcal{B}} \zeta^b$ by (†) again; but, then, because $T \Vdash_{\mathcal{B}} q$ iff $T \Vdash_{\mathcal{B}} q$ (Theorem 3.1 of [41]), it follows that $\Xi^b \Vdash_{\mathcal{N}} \zeta^b$, whence by (‡), $(\Xi^b)^\sharp \vdash_{\text{NJ}} (\zeta^b)^\sharp$, which is just to say that $\Xi \vdash_{\text{NJ}} \zeta$, as desired.

Notice that **Base** ($\mathbb{N}$) is exactly Sandqvist's simulation base $\mathcal{N}$ and, conversly, **Arena** ($\mathcal{N}$) = $\mathbb{N}$. From this fact, together with corollary 16 it follows that:

$$S \Vdash_{\mathcal{N}} p \quad \text{iff} \quad S \blacktriangleright_{\text{Arena}(\mathcal{N})} p,$$

and, equivalently,

$$S \blacktriangleright_{\mathbb{N}} p \quad \text{iff} \quad S \Vdash_{\text{Base}(\mathbb{N})} p.$$

which is just to say,

$$S \blacktriangleright_{\mathbb{N}} p \quad \text{iff} \quad S \Vdash_{\mathcal{N}} p.$$

Also, for every extension $\mathbb{B} \supseteq \mathbb{N}$ it follows that $S \blacktriangleright_{\mathbb{B}} p$ iff $S \Vdash_{\text{Base}(\mathbb{B})} p$ and, conversely for every extension $\mathcal{B} \supseteq \mathcal{N}$, $S \Vdash_{\mathcal{B}} p$ iff $S \blacktriangleright_{\text{Arena}(\mathcal{B})} p$. Hence, by corollary 16 together with Theorem 19 it follows that:

(†*) for every $\gamma \in \Gamma$ and every $\mathbb{B} \supseteq \mathbb{N}$, $\Vdash'_{\mathbb{B}} \gamma^b$ iff $\Vdash'_{\mathbb{B}} \gamma$

(‡*) for any P and q , if $P \blacktriangleright_{\mathbb{N}} q$, then $P^{\sharp} \vdash_{\text{NJ}} q^{\sharp}$

The same argument as in [41] can then be used to conclude that $\Xi^b \blacktriangleright_{\mathcal{N}} \zeta^b$ and, consequently, by (‡*), $(\Xi^b)^{\sharp} \vdash_{\text{NJ}} (\zeta^b)^{\sharp}$, which is just to say $\Xi \vdash_{\text{NJ}} \zeta$. ◀