11th Economics Summer Seminars

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Generalized Method of Moments (GMM)

- 2sls is an instrumental variable approach which is encompassed by GMM
  - GMM allows overidentification (MM does not)
- Enhanced efficiency of GMM

GMM

- Unlike ML the calculation of the estimator for very large data sets is quite straightforward
  - No calculation of determinant or eigenvalues of *W*, unlike the maximum likelihood (ML) estimating procedure
- Consistent estimates of the spatial autocorrelation parameter(s). The resulting estimates of β and σ rely on the large sample properties of the feasible generalized-least squares (FGLS) estimator

GMM

- In 2sls the overidentification is handled by the projection matrix so that

$$W\hat{y} = Q(Q'Q)^{-1}Q'Wy = P_H Wy$$

  – This reduces the *q* instruments to the *G* needed as instruments for *G* endogenous vars

- In GMM each instrument is used, with a weighting applied to increase the efficiency of the estimator

Generalized Method of Moments (GMM)

- 2sls/GMM presents some problems
- Difficulty of finding valid instruments
- Relies on asymptotics, possible small samples may induce bias
- Weak instruments may mean 2sls/GMM worse than OLS

GMM

$y = X\beta + \varepsilon \qquad \varepsilon \sim iid(0,\Omega)$

y, ε are N x 1, X is N x k, β is k x 1

instruments Q is N x q $\quad q \geq k$

moments $\mathrm{m}_i(\beta) = Q_i'\varepsilon_i = Q_i'(y_i - X_i\beta) = 0 \qquad i = 1, \ldots, N$

$\mathrm{m}_i(\beta)$ is q x 1

averaging over N

$$\overline{m}(\beta) = \frac{1}{N}\sum_{i=1}^{N} Q_i'(y_i - X_i\beta)$$

GMM chooses $\hat{\beta}$ that solves $\overline{m}(\beta) = 0$

We choose $\hat{\beta}$ so that all q elements of are as close to 0 as possible using the function

$$J(\hat{\beta}_{GMM}) = N\overline{m}(\hat{\beta}_{GMM})'R\overline{m}(\hat{\beta}_{GMM})$$

R is a q x q symmetrical weighting matrix chosen so that the elements of $J(\hat{\beta}_{GMM})$ are as close to 0 as possible

GMM

This leads to $\hat{\beta}_{GMM} = (X'Q\tilde{W}Q'X)^{-1}X'Q\tilde{W}Q'y$

$\tilde{W} = (N^{-1}[Q'\hat{\varepsilon}\hat{\varepsilon}'Q])^{-1}$ $=(N^{-1}[Q'\Omega Q])^{-1}$ in which $\hat{\varepsilon}$ are the 2sls residuals and Ω is the variance-covariance matrix of the error process. If the errors are iid, $\tilde{W} = (\sigma_{\varepsilon}^{2}I)^{-1}$ and so is proportional to the identity matrix I.

In which case $\hat{\beta}_{GMM} = \hat{\beta}_{2sls} = (X'P_H X)^{-1}X'P_H y$

GMM for the spatial errors model

$y = X\beta + \varepsilon$

$\varepsilon = \lambda W\varepsilon + u \quad u \sim iid(0, \sigma^2 I)$

Step 1 : obtain consistent estimates of ε

$u = \varepsilon - \lambda W\varepsilon = \varepsilon - \lambda\bar{\varepsilon}$

Multiplying by W

$\bar{u} = Wu = \bar{\varepsilon} - \lambda\bar{\bar{\varepsilon}}$

squaring, summing and and dividing by N gives

$$(1) N^{-1}\sum_{i=1}^{N}(\varepsilon_i - \lambda\bar{\varepsilon}_i)(\varepsilon_i - \lambda\bar{\varepsilon}_i) = N^{-1}\sum_i \varepsilon_i^2 + \lambda^2 N^{-1}\sum_i \bar{\varepsilon}_i^2 - 2\lambda N^{-1}\sum_i \varepsilon_i\bar{\varepsilon}_i = N^{-1}\sum_i u_i^2$$

$$(2) N^{-1}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \lambda\bar{\bar{\varepsilon}}_i)(\bar{\varepsilon}_i - \lambda\bar{\bar{\varepsilon}}_i) = N^{-1}\sum_i \bar{\varepsilon}_i^2 + \lambda^2 N^{-1}\sum_i \bar{\bar{\varepsilon}}_i^2 - 2\lambda N^{-1}\sum_i \bar{\varepsilon}_i\bar{\bar{\varepsilon}} = N^{-1}\sum_i \bar{u}_i^2$$

Also

$$(3) N^{-1}\sum_{i=1}^{N}(\varepsilon_i - \lambda\bar{\varepsilon}_i)(\bar{\varepsilon}_i - \lambda\bar{\bar{\varepsilon}}_i) = N^{-1}\sum_i \varepsilon_i\bar{\varepsilon}_i + \lambda^2 N^{-1}\sum_i \bar{\varepsilon}_i\bar{\bar{\varepsilon}}_i - \lambda\left(N^{-1}\sum_i \varepsilon_i\bar{\bar{\varepsilon}}_i + N^{-1}\sum_i \bar{\varepsilon}_i^2\right) = N^{-1}\sum_i u_i\bar{u}_i$$

GMM for the spatial errors model

$$(1) N^{-1}\sum_{i=1}^{N}(\varepsilon_i - \lambda\bar{\varepsilon}_i)(\varepsilon_i - \lambda\bar{\varepsilon}_i) = N^{-1}\sum_{\mathrm{i}}\varepsilon_{\mathrm{i}}^2 + \lambda^2 N^{-1}\sum_{\mathrm{i}}\bar{\varepsilon}_{\mathrm{i}}^2 - 2\lambda N^{-1}\sum_{\mathrm{i}}\varepsilon_{\mathrm{i}}\bar{\varepsilon}_{\mathrm{i}} = N^{-1}\sum_{\mathrm{i}}u_{\mathrm{i}}^2$$

$$(2) N^{-1}\sum_{i=1}^{N}(\bar{\varepsilon}_i - \lambda\bar{\bar{\varepsilon}}_i)(\bar{\varepsilon}_i - \lambda\bar{\bar{\varepsilon}}_i) = N^{-1}\sum_{\mathrm{i}}\bar{\varepsilon}_{\mathrm{i}}^2 + \lambda^2 N^{-1}\sum_{\mathrm{i}}\bar{\bar{\varepsilon}}_{\mathrm{i}}^2 - 2\lambda N^{-1}\sum_{\mathrm{i}}\bar{\varepsilon}_{\mathrm{i}}\bar{\bar{\varepsilon}} = N^{-1}\sum_{\mathrm{i}}\bar{u}_{\mathrm{i}}^2$$

Also

$$(3) N^{-1}\sum_{i=1}^{N}(\varepsilon_i - \lambda\bar{\varepsilon}_i)(\bar{\varepsilon}_i - \lambda\bar{\bar{\varepsilon}}_i) = N^{-1}\sum_{i}\varepsilon_i\bar{\varepsilon}_i + \lambda^2 N^{-1}\sum_{i}\bar{\varepsilon}_i\bar{\bar{\varepsilon}}_i - \lambda\left(N^{-1}\sum_{i}\varepsilon_i\bar{\bar{\varepsilon}}_i + N^{-1}\sum_{i}\bar{\varepsilon}_i^2\right) = N^{-1}\sum_{i}u_i\bar{u}_i$$

Moments equations

$$(1) N^{-1}\sum_{\mathrm{i}}u_{\mathrm{i}}^2 \rightarrow E[\frac{1}{N}u'u] = \sigma^2$$

$$(2) N^{-1}\sum_{\mathrm{i}}\bar{u}_{\mathrm{i}}^2 \rightarrow E[\frac{1}{N}u'W'Wu] = \frac{\sigma^2}{N}tr(W'W)$$

$$(3) N^{-1}\sum_{i}u_i\bar{u}_i \rightarrow E[\frac{1}{N}u'W'u] = 0$$

GMM for the spatial errors model

These equations and moment conditions in matrix form

<u>Step 2: estimate parameters of Autoregressive error process</u>

$$g_N = G_N \theta + v_N(\lambda,\ \sigma^2)$$

$$v_N(\lambda,\ \sigma^2) = g_N - G_N \theta$$

where $v_N(\lambda,\ \sigma^2)$ is a vector of residuals, $\theta = [\lambda,\ \lambda^2, \sigma^2]'$,

$$G_N = \begin{bmatrix} \frac{2}{N}\hat{\varepsilon}'\hat{\bar{\varepsilon}} & \frac{-1}{N}\hat{\bar{\varepsilon}}'\hat{\bar{\varepsilon}} & 1 \\ \frac{2}{N}\hat{\bar{\bar{\varepsilon}}}'\hat{\bar{\varepsilon}} & \frac{-1}{N}\hat{\bar{\bar{\varepsilon}}}'\hat{\bar{\bar{\varepsilon}}} & \frac{1}{N}tr(W'W) \\ \frac{1}{N}(\hat{\varepsilon}'\hat{\bar{\bar{\varepsilon}}} + \hat{\bar{\varepsilon}}'\hat{\bar{\varepsilon}}) & \frac{-1}{N}(\hat{\bar{\varepsilon}}'\hat{\bar{\bar{\varepsilon}}}) & 0 \end{bmatrix},\ g_N = \begin{bmatrix} \frac{1}{N}\hat{\varepsilon}'\hat{\varepsilon} \\ \frac{1}{N}\hat{\bar{\varepsilon}}'\hat{\bar{\varepsilon}} \\ \frac{1}{N}\hat{\varepsilon}'\hat{\bar{\varepsilon}} \end{bmatrix}$$

$$\hat{\varepsilon} = y - X\hat{\beta}$$

The vector θ contains the parameters to be estimated.

The GMM estimator $\hat{\theta}_{GMM}$ is the θ that minimise $v_N(\lambda,\ \sigma^2)'\ v_N(\lambda,\ \sigma^2)$

hence $\left(\hat{\lambda},\ \hat{\sigma}^2\right) = \arg\min\left\{v_N(\lambda,\ \sigma^2)'\ v_N(\lambda,\ \sigma^2)\right\}$

GMM for the spatial errors model

Given $\hat{\lambda}$ and $\hat{\sigma}^2$ do Cochrane-Orcutt transform

$$B = (I - \hat{\lambda} W)$$

$$X^* = BX$$

$$y^* = By$$

then using $\hat{X}^* = P_H X^*$

$$\hat{\beta}^* = (\hat{X}^{*\prime} \hat{X}^*)^{-1} \hat{X}^{*\prime} y^*$$

To obtain standard errors of $\hat{\beta}^*$

$$\hat{C} = \hat{\sigma}^2 \left[X^{*\prime} X^* \right]^{-1}$$

$$s.e. \hat{\beta}^* = \sqrt{diag(\hat{C})}$$

$$t = \frac{\hat{\beta}^*}{s.e. \hat{\beta}^*}$$

This does not produce $s.e. \hat{\lambda}$

However this can be obtained via Bootstrap or by the method of Kelejian-Prucha (1998) available in software sem_gmm.m

Step 3 : eliminate error dependence
Then estimate regression coefficients

GMM for the spatial errors model

- the GM estimator is both consistent and asymptotically efficient Kelejian and Prucha (1998,1999)
- Step 1: The OLS residuals are used as predictors of ε.
- Step 2: Once OLS has been estimated, the system can be solved for $\hat{\lambda}, \hat{\sigma}^2$ using non-linear least squares
- Typically done via nonlinear least-squares estimation via a modified Newton-Raphson method which is suitable for minimizing any nonlinear function. This depends on numerical differences so there is no need to specify derivatives.
- Step 3: Once a solution for the spatial parameter is found, estimates for the vector of exogenous variables and model variance can be derived using feasible generalized least squares (FGLS).

GMM for the spatial errors model

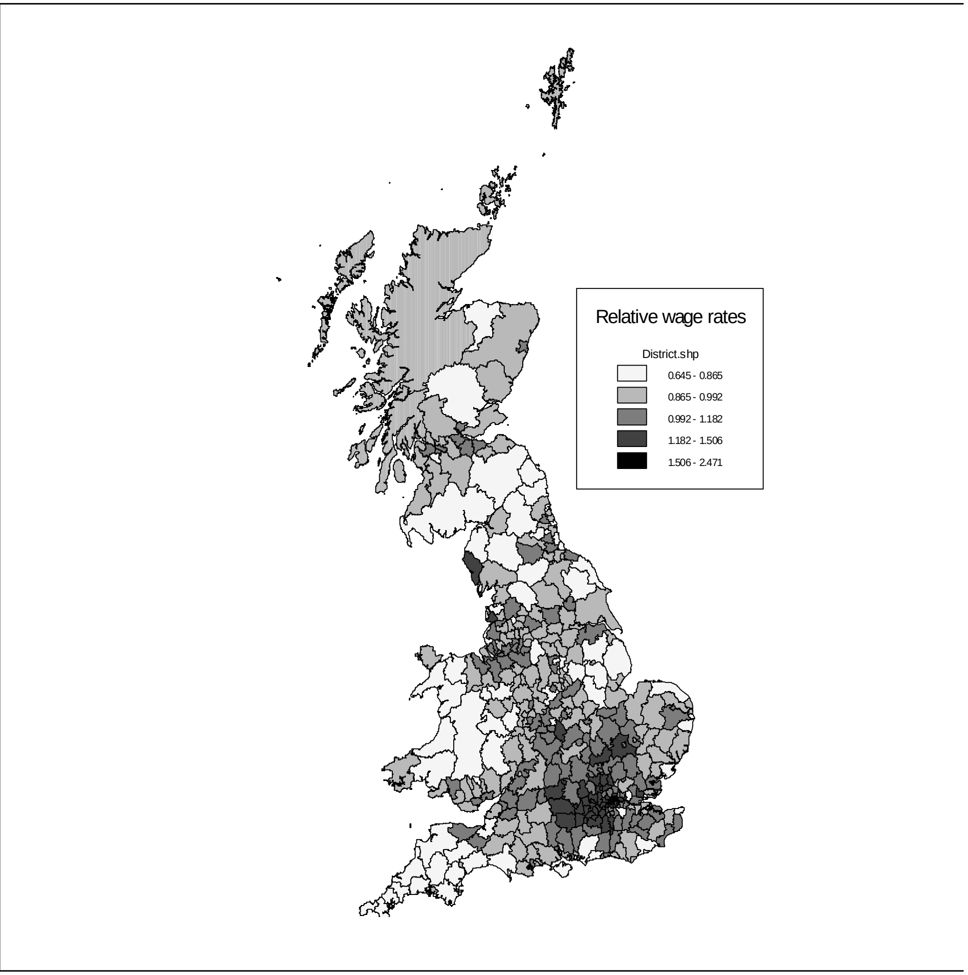

Fig. 2. Wage rates (relative to the mean)

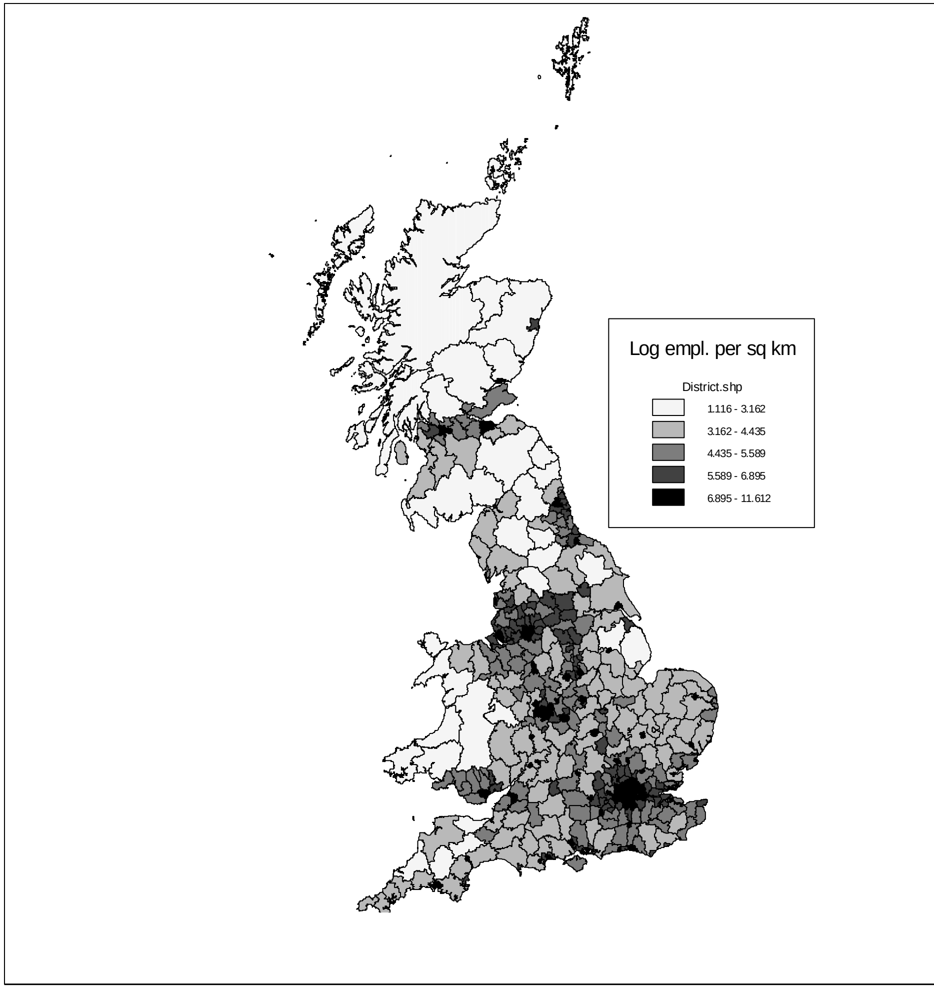

Fig. 3. Employment density

GMM for the spatial errors model

ML estimates

$$y = X\beta + \varepsilon$$

$$\varepsilon = \lambda W \varepsilon + u$$

$$u \sim N(0, \sigma^2)$$

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Variable	Coefficient	Asymptot t-stat	z-probability
const	5.653089	167.098250	0.000000
emp. density	0.022036	5.003729	0.000001
lambda	0.748979	21.960853	0.000000

GMM estimates

$$y = X\beta + \varepsilon$$

$$\varepsilon = \lambda W \varepsilon + u$$

$$u \sim iid(0, \sigma^2)$$

Constant = 5.60714   emp. density = 0.0316163

t constant=191.208 t emp. density = 6.76165

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lambda estimate (AR error process) = 0.556276

lambda t ratio = 13.2035

GMM/GS2SLS for the SARAR model

spatial error	$y = X\beta + \varepsilon$	$\varepsilon = \lambda M\varepsilon + u \quad u \sim iid(0, \sigma^2 I)$
spatial lag	$y = \rho Wy + X\beta + \varepsilon$	$\varepsilon \sim iid(0, \sigma^2 I)$
SARAR	$y = \rho Wy + X\beta + \varepsilon$	$\varepsilon = \lambda M\varepsilon + u \quad u \sim iid(0, \sigma^2 I)$

SARAR combines spatially lagged dependent variable with spatially
Autocorrelated error term, also possible *M* = *W*

ML very challenging with large samples, requires distributional assumptions
Consistent non-ML estimation available for all three models, i.e. GMM/2sls

Kelejian and Prucha (1998) show that the estimator is asymptotically normal
and consistent. It is not based on an assumption of a normal error process.
The only alternative is ML, which is probably not feasible with large samples

GMM/GS2SLS for the SARAR model

SARAR $y = \rho Wy + X\beta + \varepsilon \quad \varepsilon = \lambda M\varepsilon + u \quad u \sim iid(0, \sigma^2 I)$

estimation of wage equation for UK

	Wy	constant	employment density
	$\hat{\rho} = 0.270002$	$\hat{\beta}_1 = 2.90051$	$\hat{\beta}_2 = 0.0346384$
t ratios	6.99939	18.6668	6.6591

$\hat{\lambda} = 0.281005$

t ratio 14.5595

$\hat{\sigma}^2 = 0.0190167$

$R^2 = 0.335855$

$\bar{R}^2 = 0.332576$

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A GMM estimator for a spatial model with Moving Average errors

- GMM estimator for spatial regression models with moving average errors
- spatial lag
- Bootstrap inference
- application

Fingleton B (2008) ‘A Generalized Method of Moments estimator for a spatial model with Moving Average errors, with application to real estate prices’ Empirical Economics 34 35-57

A GMM estimator for a spatial model with Moving Average errors

SARAR $y = \rho Wy + X\beta + \varepsilon \qquad \varepsilon = \lambda M\varepsilon + u \qquad u \sim iid(0, \sigma^2 I)$

AR errors $\varepsilon = (I - \lambda M)^{-1} u$

MA errors $\varepsilon = (I - \lambda M) u$

SARAR $y = (I - \rho W)^{-1} X\beta + (I - \rho W)^{-1}(I - \lambda M)^{-1} u$

SARMA $y = (I - \rho W)^{-1} X\beta + (I - \rho W)^{-1}(I - \lambda M) u$

Figure 1: Shock effects with AR errors

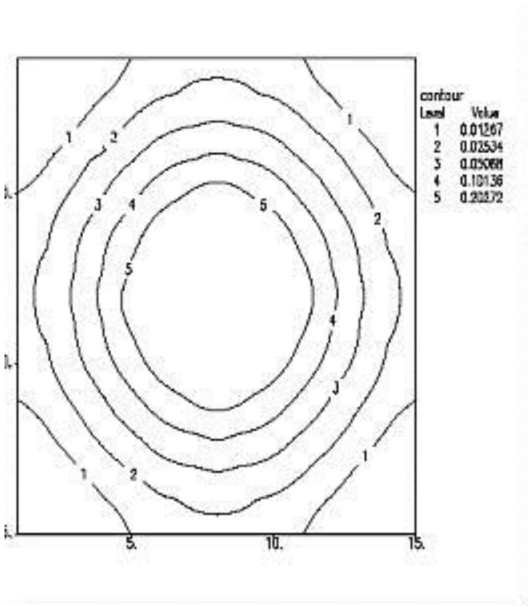


Figure 2 : Shock effects under SARAR

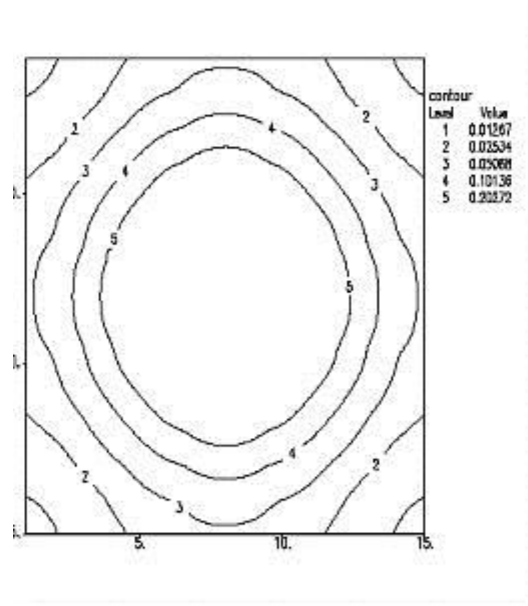


Figure 3: Shock effects with MA errors

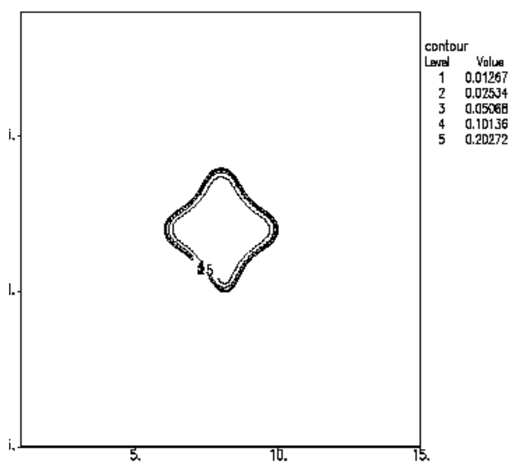
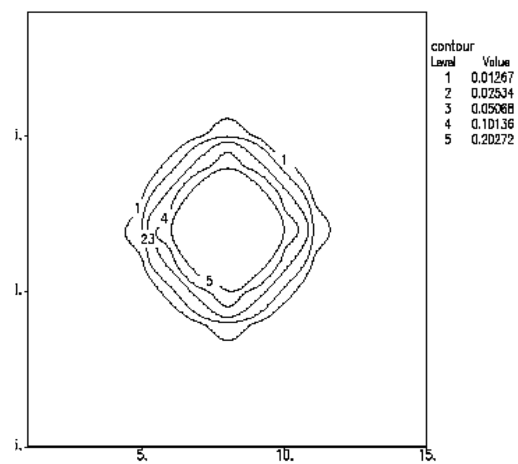


Figure 4: Shock effects under SARMA



ESTIMATING THE SARMA MODEL VIA GMM

- feasible Generalized Spatial two stage Least Squares (GS2SLS) estimator for the parameters of the SARAR model : Kelejian and Prucha (1998)
- GS2SLS for SARMA model

ESTIMATING THE SARMA MODEL VIA GMM

- 3 stages, but key differences
  - – new moments
  - – different Cochrane-Orcutt type transformations – require matrix inversion
  - – IV
    - because of spatial lag

New moments : MA errors

MA errors

SARMA $y = \rho Wy + X\beta + \varepsilon \qquad \varepsilon = u - \lambda Mu \quad u \sim iid(0, \sigma^2 I)$

MA errors $\varepsilon = (I - \lambda M)u$

$\varepsilon = u - \lambda Mu = u - \lambda \bar{u}$

Multiplying by M

$M\varepsilon = \bar{\varepsilon} = \bar{u} - \lambda \bar{\bar{u}}$

then squaring, summing and and dividing by N etc

AR errors

SARAR $y = \rho Wy + X\beta + \varepsilon \qquad \varepsilon = \lambda M\varepsilon + u \quad u \sim iid(0, \sigma^2 I)$

$\varepsilon = \lambda M\varepsilon + u$

$u = \varepsilon - \lambda M\varepsilon = \varepsilon - \lambda \bar{\varepsilon}$

Multiplying by M

$Mu = \bar{u} = \bar{\varepsilon} - \lambda \bar{\bar{\varepsilon}}$

then squaring, summing and and dividing by N etc

MA errors

$$\varepsilon = u - \lambda \bar{u}$$

$$\bar{\varepsilon} = \bar{u} - \lambda \bar{\bar{u}}$$

$$n^{-1}\sum (u-\lambda\bar{u})(u-\lambda\bar{u}) = n^{-1}\left(\sum u^2 + \lambda^2\sum \bar{u}^2 - 2\lambda\sum u\bar{u}\right) = n^{-1}\sum \varepsilon^2$$

$$n^{-1}\sum (\bar{u}-\lambda\bar{\bar{u}})(\bar{u}-\lambda\bar{\bar{u}}) = n^{-1}(\sum \bar{u}^2 + \lambda^2\sum \bar{\bar{u}}^2 - 2\lambda\sum \bar{u}\bar{\bar{u}}) = n^{-1}\sum \bar{\varepsilon}^2$$

$$n^{-1}\sum (u-\lambda\bar{u})(\bar{u}-\lambda\bar{\bar{u}}) = n^{-1}\left(\sum u\bar{u} + \lambda^2\sum \bar{u}\bar{\bar{u}} - \lambda\sum \bar{u}^2 - \lambda\sum u\bar{\bar{u}}\right) = n^{-1}\sum \varepsilon\bar{\varepsilon}$$

$$E(\sum u^2) = \sigma^2$$

$$E(\sum \bar{u}^2) = \sigma^2 Tr(M'M)$$

$$E(\sum \bar{\bar{u}}^2) = \sigma^2 Tr(M'MM'M)$$

$$E(\sum u\bar{u}) = 0$$

$$E(\sum u\bar{\bar{u}}) = \sigma^2 Tr(MM)$$

$$E(\sum \bar{u}\bar{\bar{u}}) = \sigma^2 Tr(M'MM)$$

$$n^{-1}(\sigma^2 + \lambda^2\sigma^2 Tr(M'M) - 2\lambda 0) = n^{-1}E(\sum \varepsilon^2)$$

$$n^{-1}(\sigma^2 Tr(M'M) + \lambda^2\sigma^2 Tr(M'MM'M) - 2\lambda\sigma^2 Tr(M'MM)) = n^{-1}E(\sum \bar{\varepsilon}^2)$$

$$n^{-1}(0 + \lambda^2\sigma^2 Tr(M'MM) - \lambda\sigma^2 Tr(M'M) - \lambda\sigma^2 Tr(MM)) = n^{-1}E(\sum \varepsilon\bar{\varepsilon})$$

MA errors

$$n^{-1}(\sigma^2+\lambda^2\sigma^2 Tr(M'M)-2\lambda 0)=n^{-1}E(\sum\varepsilon^2)$$

$$n^{-1}(\sigma^2 Tr(M'M)+\lambda^2\sigma^2 Tr(M'MM'M)-2\lambda\sigma^2 Tr(M'MM))=n^{-1}E(\sum\bar{\varepsilon}^2)$$

$$n^{-1}(0+\lambda^2\sigma^2 Tr(M'MM)-\lambda\sigma^2 Tr(M'M)-\lambda\sigma^2 Tr(MM))=n^{-1}E(\sum\varepsilon\bar{\varepsilon})$$

$$G\begin{bmatrix}\sigma^2 & \lambda^2\sigma^2 & -\lambda\sigma^2 & -\lambda\sigma^2\end{bmatrix}'-g=\zeta(\lambda\quad\sigma^2)$$

$$t_1=Tr(W'W)\ t_2=Tr(W'WW'W)\ t_3=Tr(W'WW)\ t_4=Tr(WW)$$

$$G=\begin{bmatrix}1 & \frac{t_1}{n} & 0 & 0\\ \frac{t_1}{n} & \frac{t_2}{n} & \frac{2t_3}{n} & 0\\ 0 & \frac{t_3}{n} & \frac{t_1}{n} & \frac{t_4}{n}\end{bmatrix}\begin{bmatrix}\sigma^2\\ \lambda^2\sigma^2\\ -\lambda\sigma^2\\ -\lambda\sigma^2\end{bmatrix}-\begin{bmatrix}n^{-1}E(\sum\varepsilon^2)\rightarrow\frac{\hat{\varepsilon}'\hat{\varepsilon}}{n}\\ n^{-1}E(\sum\bar{\varepsilon}^2)\rightarrow\frac{\hat{\bar{\varepsilon}}\hat{\bar{\varepsilon}}}{n}\\ n^{-1}E(\sum\varepsilon\bar{\varepsilon})\rightarrow\frac{\hat{\varepsilon}'\hat{\bar{\varepsilon}}}{n}\end{bmatrix}=\zeta(\lambda\quad\sigma^2)$$

$\zeta(\lambda\quad\sigma^2)$ is a vector of residuals and the nonlinear leastsquares estimators are

$$(\hat{\lambda},\hat{\sigma}^2)=\arg\min\{\zeta(\lambda\ \sigma^2)'\zeta(\lambda\ \sigma^2)\}$$

GMM for the spatial errors model

Given $\hat{\lambda}$ and $\hat{\sigma}^2$ do Cochrane-Orcutt transform

$$B = (I - \hat{\lambda} M)$$

$$X^* = B^{-1} X$$

$$y^* = B^{-1} y$$

$$P_H = Q(Q'Q)^{-1} Q'$$

then using $\hat{X}^* = P_H X^*$

$$\hat{\beta}^* = (\hat{X}^{*\prime} \hat{X}^*)^{-1} \hat{X}^{*\prime} y^*$$

To obtain standard errors of $\hat{\beta}^*$

$$\hat{C} = \hat{\sigma}^2 \left[X^{*\prime} X^* \right]^{-1}$$

$$s.e.\hat{\beta}^* = \sqrt{diag(\hat{C})}$$

$$t = \frac{\hat{\beta}^*}{s.e.\hat{\beta}^*}$$

This does not produce $s.e.\hat{\lambda}$

However this can be obtained via Bootstrap

Step 3 : eliminate error dependence
Then estimate regression coefficients

GMM/GS2SLS for the SARMA model

SARMA $y = \rho Wy + X\beta + \varepsilon$ $\varepsilon = u - \lambda Mu$ $u \sim iid(0, \sigma^2 I)$

estimation of wage equation for UK

	Wy	constant	employment density
	$\hat{\rho} = 0.242277$	$\hat{\beta}_1 = 4.20119$	$\hat{\beta}_2 = 0.0333281$
t ratios	6.77136	20.8969	6.39442

$\hat{\lambda} = -0.429009$

$\hat{\sigma}^2 = 0.0197793$

$R^2 = 0.309222$

$\bar{R}^2 = 0.30581$

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GMM/GS2SLS for the SARMA model

For
1. Monte Carlo demonstration
2. Application to real estate

See

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GMM/GS2SLS for the SARMA model : Issues

- choice of appropriate instruments
- most efficient optimisation method
- small sample properties of the estimator
- evidence presented here does suggest that there is scope for the practical implementation via GMM of SARMA models