Logique & Analyse **141-142** (1993), 75-104

ON THE INTERPRETATION OF INCOMPLETE EXPRESSIONS*

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1. Introduction

In this paper I compare and contrast two different interpretations of Frege's notion of an incomplete expression, namely those of Dummett and Geach. Whereas Dummett has written extensively about how he understands this aspect of Frege's philosophy of language, Geach has only presented his interpretation in a small number of brief passages. My relationship to their work is, therefore, different. In the case of Dummett I simply expound what he has written, but in the case of Geach I develop and expand his views about linguistic functions. I am happy to accept what I have called Geach's interpretation as my own if it is thought that my elaboration of his views is at variance with the spirit of his exegesis of Frege.

I broadly agree with the views expressed in Hugly's paper, "Ineffability in Frege's Logic". He argues that for Frege predicates —and other incomplete expressions— have to be thought of as being functions and in this paper I champion Geach's interpretation of incomplete expressions as linguistic functions. Hugly wrote, however, at a time when Dummett had published comparatively little on Frege and so his alternative interpretation of unsaturated expressions was not readily available to Hugly. In this paper I show that the interpretation of incomplete expressions as functions has several advantages over Dummett's account of them as patterns and that Dummett's exegesis of Frege is flawed on this issue.

A paper concerned with what a philosopher who died approximately 70 years ago thought about predicates and functional signs may seem to be merely a scholarly footnote in the history of the philosophy of language. Dummett, however, has written that

> *all* thought may be said to involve the discernment of pattern; even to recognise the truth of the rawest of observation statements requires us

* An earlier version of this paper was read to the Philosophy Society at the University of Birmingham in February 1992. I am grateful to the audience for giving me helpful and constructive criticism.

> to attend to particular features or notice particular similarities in the welter of detail before us.[1]

And the prototype for the general notion of a pattern that is used in this and similar passages is that notion of a pattern that Dummett employs in his exegesis of Frege's views on incomplete expressions. It is that interpretation that I criticise in this paper.

2. *Dummett on Incomplete Expressions*

Because of the influence that they have had on Dummett's understanding of incomplete expressions, it is worth quoting at length those passages from §30 of *Grundgesetze* in which Frege explains his two ways of making names out of names. (In *Grundgesetze* the term 'name' applies to any significant unit of the Beggriffsschrift, whether complete or incomplete, and 'proper name' applies to any complete expression.) Frege summarises the first method of formation as follows:[2]

> This formation is carried out in this way: a name fills the argument-places of another name that are fitting for it. Thus there arises
>
> [A] a proper name
>
> [1] from a proper name and a name of a first-level function of one argument, or
>
> [2] from a name of a first-level function and a name of a second-level function of one argument, or
>
> [3] from a name of a second-level function of one argument of type 2 and the name '$—\overset{f}{\smile}—\mu_{\beta}(f(\beta))$' of a third-level function;
>
> [B] the name of a first-level function of one argument

1 *LBM*, p.198. The italics are Dummett's. See also *FPM*, p. 37, where Dummett writes that all 'conceptual thought involves the apprehension of pattern'. In the interests of conciseness the following abbreviations are used for the titles of some of Dummett's works: *FPL* for the second edition of *Frege: Philosophy of Language*, *FPM* for *Frege: Philosophy of Mathematics*, *IFP* for *The Interpretation of Frege's Philosophy*, *LBM* for *The Logical Basis of Metaphysics* and *TOE* for *Truth and Other Enigmas*.

2 *Grundgesetze*, pp.46-47. The layout of this passage and the labelling of the various subdivisions are due to the translator of the opening sections of the *Grundgesetze*. Note that I use a roman font for those variables for which Frege uses a "Gothic" font.

> [1] from a proper name and a name of a first-level function of two arguments.
>
> The names so formed may be used in the same way for the formation of further names, and all names arising in this way succeed in denoting if the primitive simple names do so.

Two comments need to be made about this passage. The first is that a name of a second-level function of one argument of type 2 is an expression which yields a complete expression when applied to a name of a first-level function of one argument. The second is that although the formal system of the *Grundgesetze* is inconsistent, that does not entail that Frege's referentiality proof —given in §29— is faulty. Resnik has established that referentiality does not entail consistency ("Frege's Proof of Referentiality", p. 190).

Frege explains the second way of forming names of first-level functions as follows:

> [We] begin by forming a name in the first way, and we then exclude from it at all or some places, a proper name that is part of it (or coincides with it entirely) —but in such a way that these places remain recognizable as argument-places of type 1. (*Grundgesetze*, p. 47.)

Note that an argument-place of type 1 is one which is appropriate to admit a proper name. Dummett calls this 'the principle of the extraction of functions, using "function" in its *Begriffsschrift* sense' (*IFP*, p. 281), that is to say, as applying to something linguistic. In *FPL*, pp. 45-48, Dummett generalises this principle by explaining how an incomplete expression can be removed from a complete one.

According to Dummett one of the greatest steps forward that Frege made in the theory of meaning was his 'distinction between the two stages of sentence-formation— the formation of atomic sentences and their transformation into complex sentences' (*FPL*, p.195). Atomic 'sentences are formed out of basic constituents none of which are, or have been formed from, sentences' (*FPL*, p. 21) and Dummett's initial list of basic constituents consists of logically simple proper names, functional signs, predicates and relational signs (*FPL*, p. 23).[3] The construction of atomic sentences out of basic constituents is a rule-governed construction (*FPL*, pp. 32-33). For example, the atomic sentence 'Theaetetus flies' is formed

[3] In subsequent refinements this list is augmented by the addition of truth-functional connectives, quantifiers and a definite description operator.

out of the proper name 'Theaetetus' and the simple predicate 'flies'. The rule governing this construction is that an atomic sentence is formed when a proper name is prefixed to a simple one-place predicate. In other words, given a proper name X and a simple predicate Y, the result of applying this rule to X and Y is the atomic sentence $\ulcorner XY \urcorner$.[4] As another example, we can consider the atomic sentence 'Peter envies John', which has been constructed from the logically simple proper names 'Peter' and 'John' and the simple relational sign 'envies'. The rule governing the construction here is that given any proper names X and Y and an infix relational sign R, then $\ulcorner XRY \urcorner$ is an atomic sentence. This account of the construction of atomic sentences is derived from one clause of what Frege calls the first way of making names out of names in §30 of *Grundgesetze*.[5] In chapter 15 of *IFP* Dummett calls the inverse of this construction process *analysis*. The analysis of a sentence into its simple constituents is unique for him and in his scheme of things a person grasps the sense of an atomic sentence by seeing how it has been constructed from simple constituents and by knowing the senses of those constituents.[6]

In the second chapter of *FPL* Dummett expounds the notion of the step-by-step construction of a sentence from a given stock of atomic sentences. He says that three operations are involved. These are: (i) the operation of constructing a sentence out of one or more sentences by means of the sentential connectives; (ii) the operation of constructing a complex predicate out of a sentence by means of replacing one or more occurrences of a single proper name in that sentence with the Greek letter xi; and (iii) the operation of constructing a sentence out of a complex one-place predicate by means of replacing the Greek letter xi with a sign of generality (*FPL*, pp. 16 and 23). Concerning step (ii) Dummett says that 'the general notion of a one-place predicate' cannot be thought of 'as synthesized from its components, but as formed by omission of a proper name from a sentence' (*FPL*, pp. 22-23). Step (ii) is derived from Frege's second way of making names out of names and steps (i) and (iii) from his first way.[7] Thus, for example,

[4] I use "corners" to represent quasi-quotation.

[5] The clause in question is the one that Furth labels (A1) in his translation restricted to those cases in which the constituents are not sentences and the results are.

[6] Things are slightly more complicated when the list of basic constituents includes quantifiers and a definite description operator, but the 'ultimate constituents' revealed by the analysis of a sentence will always be simple (*IFP*, p. 289).

[7] See *Grundgesetze*, §30. Step (i) corresponds to clause (A1), restricted to those cases when the components and results are sentences, and step (iii) corresponds to clause (A2),

from the sentence 'Theaetetus flies' we can, by operation (ii), construct the complex predicate 'ξ flies', and from this, by operation (iii), we can construct the sentence 'Everything flies'. As a second example consider the sentence, 'Everybody envies somebody.' This could have been constructed from the atomic sentence 'Peter envies John', which itself has been constructed from simple constituents, namely 'Peter', 'envies' and 'John', according to an appropriate rule.[8] Thus, one possible constructional history for this complex sentence is:

'Peter envies John'

↓ (ii)

'Peter envies ξ'

↓ (iii)

'Peter envies somebody'

↓ (ii)

'ξ envies somebody'

↓ (iii)

'Everybody envies somebody'.

Here the labelled arrows indicate the application of one of the steps in the account of the step-by-step construction of a sentence and the label indicates which particular step is being applied.

Although Dummett only explicitly mentions these three types of sentence-forming operations, when he comes to deal with the constructional history of more complicated sentences and sentences containing more than one sign of generality he makes use of a further two operations and he also

restricted to the case when the second-level functional sign is a quantifier. The labels (A1) and (A2) are those that Furth introduces into his translation.

8 The construction process of the complex sentence could, of course, start from any other atomic sentence of the same "form" as the one actually used in the text.

substantially qualifies step (ii). The discussion of such refinements is not relevant, however, to my main concern which is to expound Dummett's account of Frege's notion of an incomplete expression.[9] The connection between his understanding of that notion and the idea of constructing sentences in stages is brought out in the following passage (*FPL*, p. 21):

> The basic idea of the step-by-step construction of sentences involves a distinction between two classes of sentences and, correspondingly, two types of expression.

The two classes of sentences are the atomic and the complex and the two types of expression are the simple and the complex. In chapter 15 of *IFP* Dummett calls simple expressions *constituents* (of the sentences in which they occur) and complex expressions he calls *components* (of the sentences in which they occur). He uses the unique analysis of an atomic sentence into its simple constituents in order to explain our ability to understand sentences that are new to us and he uses the many decompositions of a sentence (either atomic or complex) into different combinations of complex components in order to explain the validity of those inferences in which the sentence in question can figure. For example, both the arguments, 'Peter envies someone because Peter envies John' and 'someone envies John because Peter envies John', are valid and, indeed, they are both instances of the same valid form of inference, but in order to explain their validity we have to analyse 'Peter envies John' into the complex predicate 'Peter envies ξ' and the proper name 'John' in the first case and into the complex predicate 'ξ envies John' and the proper name 'Peter' in the second case. Dummett, in fact, drives a wedge between these two distinct types of analysis in that he thinks someone could have the 'firmest grasp' of the sense of a sentence (by knowing its analysis into simple constituents and knowing the senses of those constituents) and yet not be aware of any decompositions of that sentence into collections of complex components (*FPL*, p. 29). The reason —or at least one of them— why he sees these two types of analysis as being unconnected is that only on the assumption that they are distinct can he explain how deductive inference is ampliative. Unfortunately, it would be far too much of a digression for me to explore this further here.[10]

[9] These refinements are discussed in great detail in chapter 4, "Types of Analysis", of my thesis, *Frege's Theory of Functions in Application to Linguistic Structures.*

There are two aspects to Dummett's explanation of complex expressions. The first is that —unlike simple expressions— complex expressions *have to be* obtained from whole sentences. Those sentences themselves can be either atomic or complex. In the construction of complex sentences some of the building blocks can be complex expressions, but in the construction of atomic sentences only simple ingredients can be used. Simple expressions can in no way be thought of as having been formed from a whole sentence; they are used to construct atomic sentences. (Simple expressions can also be used in the construction of complex sentences. Dummett thinks, for example, that there are simple first-level quantifiers —see *FPL*, p. 48— and these are the signs of generality that are employed in step (iii) of the construction process of complex sentences.) This is the aspect of Dummett's explanation of complex expressions that has been stressed so far in this section; the second —related— aspect is that complex expressions are *features* of sentences or *patterns* that can be discerned in sentences. In *LBM*, p. 196, Dummett illustrates the idea of a pattern —and how it can be used to form a new concept— by means of the following four sentences (which he does not label):

(1) A Harvard professor was appointed president of Harvard.
(2) A Harvard professor was appointed president of Princeton.
(3) A Stanford professor was appointed president of Harvard.
(4) A Columbia professor was appointed president of Columbia.

We can discern, he says, a common pattern in (1) and (2), another in (1) and (3), and yet a third in (1) and (4). Seeing the common pattern in (1) and (4), for example, allows us to form a new concept, namely that of an internally appointed president.

On p. 31 of *FPL* Dummett says that 'complex predicates form the prototype for Frege's general notion of an "incomplete" expression.' This claim is inaccurate, but *Dummett* —at least in *FPL*— does think that predicates are prototypical incomplete expressions. Because of this what Dummett says about complex predicates can usually be generalised to apply to all incomplete expressions. Simple —that is to say, complete— and complex predicates share several properties. For example, they belong to the same syntactic category, they both refer to concepts and Frege's auxiliary notation can be used in representing them. These shared properties also extend, *mutatis mutandis*, to simple and incomplete expressions belonging to those syntactic categories which contain at least one simple expression

[10] The interested reader should study the following passages in order to find out the way in which —according to Dummett— deduction is ampliative: *IFP*, pp. 290-291, *FPM*, chapter 4, *LBM*, chapter 8 and *TOE*, chapter 17.

(because of the prototypical nature of complex predicates).[11] Because both complete and incomplete expressions have the properties just mentioned in common, they cannot be used to distinguish between them and, thus, I do not discuss them at length in this paper.

To conclude this section I will briefly summarise Dummett's understanding of unsaturated expressions. For him Fregean incomplete expressions have two roles to play and these are that they are needed to explain the construction of complex sentences and also the validity of some inferences. They are features of or patterns that we impose on or discern in complete expressions and they are obtained by the removal, omission or extraction of an expression from a complete expression.[12] They are *not* parts of the expressions in which they are discerned (for example, *FPL*, p. 31). In the case of a complex or incomplete predicate he also says that the (valency) slots it has are 'integral to its very being' (*FPL*, p. 33), whereas this is not true of a simple or complete predicate. I consider the meaning of this remark to be exhausted by the other things that Dummett says about simple and complex predicates.

3. Geach on Incomplete Expressions

Consider the following three singular terms:[13]

'2.1^3+1',
'2.4^3+4' and
'2.5^3+5'.

[11] Throughout chapter 3 —see, for example, p. 49— of *FPL* it is assumed that appropriate simple and incomplete expressions belong to the same syntactic category —assuming that at least one simple expression of that category exists. On p. 319 of *IFP* Dummett says that both simple and complex predicates stand for concepts. As well as allowing the use of Frege's auxiliary notation to express simple predicates Dummett also allows it for relational expressions. For example, on p. 24 of *FPL* he says that 'ξ took ζ to task' is a simple, that is to say, complete, relational expression. The Greek letters xi and zeta I call Frege's *auxiliary notation*.

[12] Illustrative examples of the claims made in the text can be found in the following passages, though they are not restricted to them: 'features', *FPL*, p. 31; 'impose', *IFP*, p. 291; 'discern', *IFP*, p. 280; 'removal', *FPL*, p. 47; 'omission', *FPL*, p. 16; and 'extraction', *IFP*, pp. 280ff.

[13] This way of introducing an incomplete expression is based on a method used by Frege. See, for example, "Function and Concept", pp. 5-6, and §§1-2 of *Grundgesetze*. Geach employs a similar method on p. 142 of "Frege".

There is a common element in these numerical designations which Frege calls an *incomplete* or *unsaturated expression* and represents by means of the notation '$2.\xi^3+\xi$'. One way of explaining how this is to be understood is to begin by observing that those displayed expressions are obtained in a uniform way from the numerals '1', '4' and '5', respectively. There are a good many more similar complex expressions and it would be impossible to list them all. However, what it is possible to do is to give a recipe for constructing all such similar numerical designations. The recipe goes as follows: to construct a complex designation of a number, that belongs with those expressions, from a given numeral or complex numerical designation, first append the full stop to the numeral '2', then append the given expression to this combination of signs, then append the numeral '3' as a superscript, then the plus sign, and finally append the given numeral to the combination of signs just formed. Following Geach I call such a recipe a *linguistic function*.[14] The expressions which are the ingredients of the recipe are called the *arguments* of the linguistic function and the results of applying the recipe to those ingredients are the *values* of the linguistic function. In the above example the arguments are numerals or complex numerical designations and the values are complex designations for numbers. The recipe defining the function states that for any numerical designation X taken as argument, the value of this function is the linguistic expression $\ulcorner 2.X^3+X \urcorner$.

Under the usual conventions for using quotation marks '$2.\xi^3+\xi$' should be understood as being a linguistic expression made up of an initial numeral '2', followed by a full stop, followed by the Greek letter xi to which is appended as a superscript the numeral '3', followed by the plus sign and terminated by another occurrence of the Greek letter xi. But to so understand the expression '$2.\xi^3+\xi$' is to treat it as a complete expression and Frege is at pains to show that functional signs are incomplete and unsaturated. So, in this case we cannot understand the quotation marks in the usual way. Following Geach I have explained the incomplete expression '$2.\xi^3+\xi$' as being that linguistic function which, out of an arbitrary numerical designation X, makes the mathematical expression $\ulcorner 2.X^3+X \urcorner$. From now on I shall always understand single quotation marks around expressions containing occurrences of the Greek letter xi in this way. If ever I want to refer to the complete linguistic expression rather than the linguistic function I shall use double quotation marks, thus "$2.\xi^3+\xi$".

[14] Geach discusses linguistic functions in his papers "Frege" (pp. 143-144), "Saying and Showing in Frege and Wittgenstein" (p. 61) and "Names and Identity" (pp. 139ff.).

The functional sign '$2.\xi^3+\xi$' refers to or stands for the numerical function $2.\xi^3+\xi$. The functional sign or incomplete expression is a linguistic function which yields numerical designations when applied to numerical designations and the numerical function returns numbers when applied to numbers. It will be useful in what follows to have a succinct notation to express these facts. Mathematicians use the notation '$f:\alpha \rightarrow \beta$' to express the fact that the function f takes arguments of type α and returns values of type β.[15] Using 'J' to refer to the type which includes all the integers, and using the more accurate Fregean notation for functions, the fact that $2.\xi^3+\xi$ is a function from numbers to numbers can more concisely be represented as:

(5) $2.\xi^3+\xi: J \rightarrow J.$

Similarly, the fact that '$2.\xi^3+\xi$' is a linguistic function from numerical designations to numerical designations is represented as:

(6) '$2.\xi^3+\xi$'$: N \rightarrow N,$

where 'N' refers to the category of singular terms, which includes numerals and complex numerical designations.

There are also linguistic functions whose values are not singular numerical terms, but are propositions.[16] Just as in the case of the complex numerical designations displayed above, we can see that the propositions:

'7 = 2 + 5',
'9 = 2 + 5' and
'3 = 2 + 5',

are constructed in a uniform way from the numerals '7', '9' and '3', respectively. The recipe for their formation goes as follows: given any numerical singular term X, to obtain a proposition that belongs with these three displayed propositions, append the expression ' = 2 + 5' to X. Frege would write such an incomplete expression as 'ξ = 2 + 5'. As in the case of functional signs the quotation marks here must be understood in a

[15] By using this notation I do not wish to suggest that the function f should be understood as being a particular subset of the Cartesian product of x and y such that if (d,r) and (d,s) are both elements of this subset, then $r = s$. Such a subset is a Fregean object and, thus, cannot be a Fregean function.

[16] I use the term 'proposition' as does Geach for something linguistic.

special way. '$\xi = 2 + 5$' does not refer to an expression whose first symbol is the Greek letter xi, followed by an equals sign, which is followed by the numeral '2', to which is appended the plus sign, followed by the numeral '5'. It is rather to be understood as that linguistic function which for any numerical singular term *X* taken as argument returns as value the expression *X* @ ' = 2 + 5'.[17] If we let '*P*' denote the category of propositions, then the category of this unsaturated expression is $N \rightarrow P$.

So far I have only considered functions of a single argument, but functions of two arguments are common in mathematics. Such functions are 'doubly in need of completion' as Frege says (*Grundgesetze*, p.8; where this phrase occurs in italics). He uses the two Greek letters xi and zeta to mark these argument-places, as in '$\xi + \zeta$'. There are a number of ways of construing this. We can think of it as that linguistic function which for any two numerical singular terms *X* and *Y* taken as arguments has the value ⌜X + Y⌝. Thinking of it in this way we have a linguistic function which yields a complex numerical designation when simultaneously applied to two numerical designations in a particular order. This is represented in the following way:

(7) '$\xi + \zeta$': $(N \times N) \rightarrow N$.

The use of the word 'simultaneously' here is not meant to suggest that we are dealing with any sort of temporal phenomenon. The word is used metaphorically and the point of so doing is to deny that either of the arguments of this function is prior to the other and this is how the category-notation '$(x \times y) \rightarrow z$' is to be understood. Neither the argument of category x nor that of category y can be present without the other. Any attempt to leave one of the argument-places of such a linguistic function unfilled results in a meaningless expression. From a certain perspective it looks as if we are dealing here with a function of *one* argument. The one argument, however, is a complex object, namely, an ordered pair of singular terms.

There is another way in which to think of two-place linguistic functions. Consider the incomplete expressions 'ξ+2', 'ξ+3' and 'ξ+7'. These are built up in a uniform way. In order to see what this amounts to, let us spell out just what these three incomplete expressions are. They are, respectively:

[17] The sign '@' is used to represent concatenation.

(8) That linguistic function which for any numerical singular term X taken as argument has the value $\ulcorner X+2 \urcorner$.

(9) That linguistic function which for any numerical singular term X taken as argument has the value $\ulcorner X+3 \urcorner$.

(10) That linguistic function which for any numerical singular term X taken as argument has the value $\ulcorner X+7 \urcorner$.

The two-place function in question is such that when it is applied to the numerals '2', '3' and '7', respectively, it yields the above linguistic functions as values. It can be spelled out in this way:

(11) That function which for any numerical singular term Y taken as argument has for its value that linguistic function which for any numerical singular term X taken as argument has the value $\ulcorner X+Y \urcorner$.

This unsaturated expression can also be written as '$\xi+\zeta$' and we can represent its category as:

(12) '$\xi+\zeta$' $: N \to (N \to N)$.

It is also possible to construe the functional sign for addition in a third way to give another function of category $N \to (N \to N)$.. This time we take the arguments in a different order, beginning from the incomplete expressions '$3+\zeta$', '$7+\zeta$' and '$9+\zeta$', for example. This is also represented as '$\xi+\zeta$'.

The universal quantifier used in first-order logic is a second-level linguistic function. It can be recognized in each of these propositions:

'$(\forall x)x<2+x$',
'$(\forall x)x^2 \geq x$' and
'$(\forall x)x \neq x$'.

These propositions are obtained in a uniform way from the incomplete expressions '$\xi<2+\xi$', '$\xi^2 \geq \xi$' and '$\xi \neq \xi$', respectively. This can be spelled out more explicitly as follows:

(13) '$(\forall x)\varphi(x)$' is that linguistic function which for any given first-level linguistic function of category $N \to P$ taken as argument yields as value that expression which is formed by concatenating '$(\forall x)$' to

the result of applying that first-level linguistic function to the variable 'x'.

There is a slight difficulty here in that —by analogy with '$2.\xi^3 + \xi$'— an incomplete predicate like '$\xi < 2 + \xi$' is that linguistic function which for any numerical singular term X taken as argument yields the expression $\ulcorner X < 2 + X \urcorner$ as its value. The difficulty is that the arguments of this linguistic function are numerical singular terms, whereas in spelling out '$(\forall x)\varphi(x)$' such a function is applied to a variable (construed as something linguistic). This is, however, easy to remedy. We just extend the account of predicates to allow variables and pseudo-terms to be their arguments. (A *pseudo-term* is like a singular term except that it can contain variables as well as numerals.[18]) Thus, the incomplete expression '$\xi < 2 + \xi$' is now to be understood as that linguistic function which for any numerical pseudo-term X taken as argument yields the expression $\ulcorner X < 2 + X \urcorner$ as its value.[19] Similarly, the existential quantifier can be ex-plained as follows:

> (14) '$(\exists x)\varphi(x)$' is that linguistic function which for any given first-level linguistic function of category $N \to P$ taken as argument yields as value that expression which is formed by concatenating '$(\exists x)$' to the result of applying that first-level linguistic function to the variable 'x'.

An example of a third-level linguistic function would be one which mapped such second-level functions to propositions. Adapting Frege's notation (*Grundgesetze*, p. 41) we can represent the universal quantifier of category $((N \to P) \to P) \to P$ by means of the incomplete expression '$(\forall f)(\mu_\beta)f(\beta)$'. One's first thought on spelling out what this is runs as follows:

> (15) '$(\forall f)(\mu_\beta)f(\beta)$' is that linguistic function which for any linguistic function of category $(N \to P) \to P$ taken as argument yields as value

[18] Dummett also needs to make use of pseudo-terms in explaining higher-level incomplete expressions. See *FPL*, pp. 46-48.

[19] I realise that talking about "extending" the account that I gave of linguistic functions, if not interpreted charitably, opens me up to a criticism analogous to Frege's criticism of piecemeal definition. In order to simplify my discussion earlier I decided not to introduce linguistic functions as here given right from the outset. If I was being more rigorous, I would have done so.

> the expression consisting of an initial sign '$(\forall f)$' followed by the value of the second-level linguistic function for the argument 'f'.

The problem with this is right at the end, namely the phrase 'the argument "f" '. The arguments of second-level functions of category $(N \to P) \to P$ are of category $N \to P$. Thus, we have to construct a "dummy" first-level linguistic function to fit the bill. A suitable one is '$f\xi$', namely, that linguistic function which for any singular term or appropriate variable X taken as argument yields $\ulcorner fX \urcorner$. Hence, putting all these things together, we have:

> (16) '$(\forall f)(\mu_\beta)f(\beta)$' is that linguistic function which for any given linguistic function of category $(N \to P) \to P$ taken as argument yields as value the expression consisting of an initial sign '$(\forall f)$' followed by the value of the given function for the argument which is that first-level linguistic function which for any singular term or suitable variable X taken as argument yields $\ulcorner fX \urcorner$.

This is not very easy to take in. It may help to realise that the value of this linguistic function for the argument '$(\exists x)\phi(x)$' is '$(\forall f)(\exists x)f(x)$'. A further refinement of this account of higher-level linguistic functions is needed to avoid the clash of bound variables, but as it is of only technical interest it will not be presented here.[20].

Having presented linguistic functions of first-, second- and third-level it should be clear how even higher-level functions could be introduced. As I have explained the method of construction, I will not give any more examples of its use. It is important to realise, however, that Frege's second way of forming expressions is available to someone who accepts the interpretation of incomplete expressions as linguistic functions presented in this section. Thus, there is no difficulty in combining a relational expression like '$\xi = \zeta + 1$' with a universal and an existential quantifier to yield the proposition '$(\forall x)(\exists y)x = y + 1$'. The process of construction is easy to describe. First, we note that the relational expression '$\xi = \zeta + 1$' can be construed as that linguistic function which for any numerical pseudo-term X returns as its value that linguistic function which for any numerical pseudo-term Y returns as its value the expression $\ulcorner X = Y + 1 \urcorner$. The value of '$\xi = \zeta + 1$' for the argument '7', say, is that linguistic function which for any numerical pseudo-term Y returns as its value the expression $\ulcorner 7 = Y + 1 \urcorner$. The existential quantifier '$(\exists y)\phi(y)$' can then be applied to this lin-

[20] The interested reader may consult the first chapter of my thesis *Frege's Theory of Functions in Application to Linguistic Structures.*

guistic function, namely '$7 = \zeta + 1$', to yield '$(\exists y)7 = y + 1$'. From this by means of Frege's second way of forming expressions we can form the linguistic function '$(\exists y)\xi = y + 1$' and then the universal quantifier can be applied to this to yield '$(\forall x)(\exists y)x = y + 1$'.

In expounding Dummett's interpretation of incomplete expressions above I said that for him an unsaturated expression *has to be* obtained by removing an expression from another expression. By contrast, in Geach's exegesis an unsaturated expression *can be* obtained from another expression but *does not have to be*. This is more fully explained in my paper, "On the Sense of Unsaturated Expressions", pp. 72-73.

4. *Comparison*

Higher-level Incomplete Expressions

One of the consequences of Dummett's interpretation of Frege's notion of an incomplete expression is that it is impossible to form an incomplete expression which is such that when some of its argument-places are filled up the result is still an incomplete expression (though belonging to a different category). As he himself states (*FPL*, p. 40):

It is important to observe that Frege does not allow for second-level functional expressions which yield, when their argument-places are filled, first-level function expressions. In other words, to abandon for a moment the practice we have so far adhered to, of speaking wholly in terms of different types of expression, rather than the kinds of entity for which they stand, Frege did not recognize the existence of functionals (second-level functions) whose values were themselves functions.

Both of the claims that Dummett makes here are false. Frege did think that there were two-place predicates and two-place functional signs and he explicitly construed them as higher-level expressions which yield one-place incomplete expressions when one of their argument-places are filled up. For example, in §30 of *Grundgesetze* he writes that 'the name of a first-level function of one argument' arises 'from a proper name and a name of a first-level [*sic*] function of two arguments.' (Frege calls it 'a name of a *first-level* function of two arguments' because its *arguments* are complete expressions: I would call it 'higher-level' because its *value* is an incomplete expression.) This means of forming 'the name of a first-level function of one argument' is, in fact, part of his account of the first way to form a name, that is to say, it is part of the formation rules for the Begriffsschrift of the *Grundgesetze*.

And on the ontological level Frege did recognise the existence of functionals whose values are themselves functions. In §4 of *Grundgesetze* he writes:

> Hitherto I have spoken only of functions of a single argument; but we can easily pass on to *functions of two arguments.* These are *doubly in need of completion*, in the sense that a function of one argument is obtained once a completion by means of one argument has been effected. Only by means of yet another completion do we attain an object, and this is then called the *value* of the function for the two arguments.

As an example he considers the function $(\xi+\zeta)^2+\zeta$ which yields the function $(\xi+1)^2+1$ for the argument 1. That is to say, he is understanding the function $(\xi+\zeta)^2+\zeta$ as belonging to the type $J \to (J \to J)$.

On the linguistic level we would have that the incomplete expression '$(\xi+\zeta)^2+\zeta$' yields the incomplete expression '$(\xi+1)^2+1$' when it is applied to the numeral '1'. There is no difficulty accommodating such incomplete expressions on Geach's account of the matter. '$(\xi+\zeta)^2+\zeta$' is that linguistic function which when given a singular term Y as argument yields as its value that linguistic function which when given X as its argument returns $\ulcorner (X+Y)^2+Y \urcorner$. This, when applied to the numeral '1', yields that linguistic function which when given X as its argument returns $\ulcorner (X+1)^2+1 \urcorner$, that is to say, '$(\xi+1)^2+1$'.

Dummett also lays down the general principle 'that an incomplete expression may never be considered as derived from another incomplete expression by the removal of some constituent expression: we have always to start with a complete expression' (*FPL*, p. 40). Frege, however, does derive some incomplete expressions from other incomplete expressions. In "Function and Concept", p. 27, he says that '3 > 2' can be split up into '3' and '$x > 2$' —where the letter ex plays the same role as the letter xi more commonly plays in his writings— and he goes on to say:

> We can further split up the 'unsaturated' part '$x > 2$' in the same way, into '2' and
>
> $$x > y,$$
>
> where 'y' enables us to recognize the empty place previously filled up by '2'. In
>
> $$x > y$$
>
> we have a function with two arguments, one indicated by 'x' and the other by 'y' ...

As already mentioned, if we understand incomplete expressions as linguistic functions, then there is no problem about interpreting doubly incomplete expressions.

In *IFP*, p. 286, Dummett acknowledges that Frege did —in §30 of *Grundgesetze*— allow an expression for a one-place function to be constructed from an expression for a two-place function and a singular term. He goes on to say that it 'goes against Frege's own principles' and also that it is 'quite redundant in his stipulations'. There are, however, two distinct ways in which this method of construction is redundant. In the sense intended by Dummett it means that the formal language generated by Frege's formation rules *including* this one is exactly the same as that generated by Frege's formation rules *excluding* this one. That is undeniably true, but this method of construction is also redundant in a second sense which is that it can be derived from Frege's other formation rules. We start from an expression for a two-place function, say '$\xi + \zeta$' and then we form a complete expression by saturating both argument-places to give us, say, '2 + 3' and then we remove the numeral '2' to give us the incomplete expression 'ξ+ 3'. The procedure that allows us to form 'ξ+ 3' from '$\xi + \zeta$' by saturating the ζ-argument-place with '3' can then be defined as the sequence of operations just given. That is to say, the operation of forming 'ξ+ 3' from '$\xi + \zeta$' and '3' is the composition of the operations of forming '2 + 3' from '$\xi + \zeta$' and '2' and '3' with the operation of forming 'ξ+ 3' of '2 + 3', by removing '2' from it, where the composition of two operations Γ_1 and Γ_2 simply means that you first do Γ_1 and then you do Γ_2. Similarly, the operation of combining a quantifier with a relational expression to form a predicate can be constructed out of Frege's formation rules. Note that here I am talking about constructing (derived) formation rules out of other formation rules by the method of sequential composition and in order to do that I have to mention the expressions involved in the construction process. This can be compared to the operation of constructing derived rules of inference out of primitive inference rules by the method, say, of identifying the conclusion of one rule with one of the premises of another and in doing that the propositions or formulas involved in the rules have to be mentioned although it is not the derivation of propositions that is at issue.

Because the idea of a derived formation rule is not as familiar as that of a derived rule of inference, I will say something more about it here. I will illustrate the idea in the context of a categorial grammar.[21] The simplest

[21] The study of categorial grammars has mushroomed recently. Wood's book *Categorial Grammars* is a good survey of current research. The variety of grammar that I use is that developed by Potts —from some of Geach's work— at least in his early papers.

kind of categorial grammar has only a single formation rule, which Geach in "A Program for Syntax", p. 484, calls the *multiplying-out rule*, though it is really a rule schema. It states that an expression of category α is obtained by combining an expression of category $1\,\alpha\,\beta$ with one of category β. The category whose name in Potts's notation is $\ulcorner 1MN \urcorner$, where M and N are category names, is the same as that one whose name is $\ulcorner N \rightarrow M \urcorner$, using a more mathematical notation. Note that the order of the category names M and N is different in the two notations. The multiplying-out rule can be written like this:

(R) $1\alpha\beta + \beta \Rightarrow \alpha$,

with the meaning given above. (17) and (18) are two instances of (R):

(17) $1NN + N \Rightarrow N$.
(18) $1PN + N \Rightarrow P$.

From (17) and (18) we can obtain the derived formation rule:

(19) $1PN + (1NN + N) \Rightarrow P$,

which tells us that we can obtain a proposition by first combining an expression of category $1NN$ and a singular term to give us a singular term and then combining this with a predicate.

In support of his claim that the operation of forming an expression for a one-place function from an expression of a two-place function is redundant for Frege Dummett —on pp. 286-287 of *IFP*— says that when Frege lays down 'the semantic principles governing the quantifier, he concerns himself only with the truth-conditions of a quantified sentence, and not with the satisfaction-conditions of a predicate formed by attaching a quantifier to a relational expression.' Such "satisfaction-conditions" are, however, easily derivable and so it is not surprising that Frege does not concern himself with them. For example, consider the following two accounts of the referents of '$\xi > \zeta$' and '$(\forall x)\phi(x)$', respectively:

(20) '$\xi > \zeta$' refers to that numerical function which when applied to two numbers returns True if the first is greater than the second and False otherwise.

(21) '$(\forall x)\phi(x)$' refers to that function which yields True when applied to the referent of '$F\xi$' if that function returns True no matter what it is applied to and the referent of '$(\forall x)\phi(x)$' returns False if the referent of '$F\xi$' does not always return True.

From these it is possible to derive the "satisfaction-conditions" for '$(\forall x)\xi > x$'. From (20) we have that '$2 > \zeta$' refers to that function which when applied to a number returns True if that number is less than 2 and False otherwise. From this and (21) we obtain that '$(\forall x)2 > x$' refers to the True if all numbers are less than 2 and False otherwise. From '$(\forall x)2 > x$' we can form '$(\forall x)\xi > x$' by the principle of the extraction of functions and this refers to that function which returns True if it is applied to a number which has the property that it is greater than all numbers. For Dummett the principle of the extraction of functions is a linguistic operation: the "satisfaction conditions" for '$(\forall x)\xi > x$' were obtained by means of the ontological analogue of this.

Pathological Functions and Illegitimate Patterns

As I have already explained —for Geach— the Fregean incomplete expression '$2.\xi^3 + \xi$' is a recipe for constructing expressions. It is that linguistic function which for any numerical designation X taken as argument returns as value the linguistic expression $\ulcorner 2.X^3 + X \urcorner$. In this subsection I show that not all linguistic functions are unsaturated expressions and I propose a criterion which distinguishes between those linguistic functions that are incomplete expressions and those which are not. Then I go on to show that not all features of sentences —as understood by Dummett— are incomplete expressions, but in this case no natural criterion exists which draws a line between those patterns that are incomplete expressions and those that are not.

Frege extended the sense of the word 'function' in several directions. One of them was to allow certain symbols to be used in the construction of functional signs which previously no one had ever thought of allowing. For example, he let relational expressions such as the equals sign and the less-than sign be used in the construction of functional signs like '$\xi = 2$' and '$5 < \xi + 2$'. But there were limits beyond which even Frege would not go. For example, he did not allow the assertion sign to be used in the construction of functional signs, as he says in "Function and Concept", (p. 22, fn. *):

> The assertion sign [*Urtheilsstrich*] cannot be used to construct a functional sign; for it does not serve, in conjunction with other signs, to designate an object. '$\vdash\!\!-2+3=5$' does not designate anything; it asserts something.

It is possible, however, to formulate linguistic functions which make use of the assertion sign. That linguistic function which for any given numerical singular term X taken as argument yields as value the expression $\ulcorner\vdash\!\!-2+X=5\urcorner$ is a perfectly legitimate and respectable linguistic function. Although Frege would not call it a functional sign, we can express this linguistic function with the help of his xi-notation as '$\vdash\!\!-2+\xi=5$'.

Therefore, although every functional sign is a linguistic function the converse is not true. There are perfectly legitimate linguistic functions which are not incomplete expressions. I call such linguistic functions *pathological* (by analogy with the use of the word 'pathological' in mathematical analysis), because their existence makes the characterisation of unsaturated expressions more difficult than it would otherwise have been.

Implicit in the quotation from Frege just given there is a criterion for distinguishing between unsaturated expressions and pathological linguistic functions, but before making it explicit I need to explain rigorously the Fregean hierarchy of types:

> Let B be the set of basic types. It contains the type of objects (J) and that of truth-values (H).[22] Then, the set T, consisting of all the types in the Fregean hierarchy, can be defined as follows:
> (a) $B \subseteq T$.
> (b) If $\alpha, \beta \in T$, then $\alpha \rightarrow \beta \in T$.
> An entity which has type $\alpha \rightarrow \beta$ is a function whose arguments are drawn from α and whose values are drawn from β.

"Frege's" criterion of demarcation between unsaturated expressions and pathological linguistic functions can now be stated explicitly as follows: an unsaturated expression is a linguistic function that can be represented by means of Frege's xi-notation (or an extension of this notation in order to cope with functions of higher-level or greater polyadicity) and whose referent is an entity which is of a type that occurs somewhere in the Fregean hierarchy of types and which is not a basic type.

The reason why '$\vdash\!\!-2+\xi=5$' is not a functional sign for Frege is that its value for any numerical singular term taken as argument is an assertion and assertions do not refer to anything. The only difference between '$\vdash\!\!-2+3=5$' and '$-2+3=5$' is that the former has a symbolic indicator attached which indicates that we are dealing with an asserted proposition.[23] In ordinary language —and even in mathematical discourse— there

22 As is well known in his later writings Frege thought that truth-values were objects. Here it is not important whether or not we follow him in this.

are no mandatory assertoric force indicators. The same string of words can be used as an asserted proposition or an unasserted one. It is only from the context that we can say whether '2 + 3 = 5', say, in a mathematical text is to be understood as having or lacking assertoric force. There are indicators, however, in ordinary language which show whether sentences have forces other than the assertoric attaching to them. Interrogative sentences, such as 'Does Jill love Jack?', are used to ask questions and so can be said to be uttered with interrogative force.[24] This suggests the formation of linguistic functions which have interrogative sentences as their values. An example of such a linguistic function is the one which for any singular term *X* taken as argument has the value ⌜Does *X* love Jack?⌝ Quite clearly this has no reference and so could not be a functional sign for Frege, but it is a perfectly legitimate linguistic function.

It is also possible to construct linguistic functions whose values are derivations.[25] For example, there is that linguistic function whose value for any singular term *X* taken as argument is the derivation ⌜*X* loves Jill, therefore *X* is crazy⌝. This can be represented by means of Frege's xi-notation as 'ξ loves Jill, therefore ξ is crazy'.

All of the examples of pathological linguistic functions mentioned so far fail to satisfy the natural criterion because they do not refer to anything. It is easy to construct lots more examples of such functions whose values are sentences which have indicators that relate to the force with which they are conventionally uttered, but there is little point in doing so. There is another class of pathological linguistic functions which, although they cannot be expressed by means of Frege's auxiliary notation, do have a referent of the right type. One example of this class is that linguistic function which maps a person's name onto his father's name. This function takes, for example, the arguments 'Isaac', 'Jacob' and 'Judah' onto the values 'Abraham', 'Isaac' and 'Jacob', respectively. It is impossible to represent this function by means of Frege's xi-notation, but its referent is of the correct kind. The referent of this linguistic function is that ontological function which maps a person onto his father. Thus, its type is $J \rightarrow J$.

23 It is a fairly common mistake to think that the assertion sign is the combination of symbols '⊢—'. The assertion sign is simply the vertical bar. In the *Begriffsschrift* Frege called the horizontal the content-stroke, but in *Grundgesetze* he simply referred to it as the horizontal.

24 There are also other ways of indicating interrogative force in ordinary language which I shall ignore here. For example, accompanied by suitable prosodic and paralinguistic modulations the sentence 'Jill loves Jack' can be used to ask a question.

25 A more natural word than 'derivation' here would be 'argument', but to use this here would be confusing because I also talk of the argument of a function.

Another example of this class of pathological linguistic functions is that function which for any proper name X taken as argument has as value the name of the father of the bearer of X concatenated with the word 'loves' concatenated with the name of the mother of the bearer of X.[26] For example, this function yields the value 'Abraham loves Sarah' for the argument 'Isaac'. It is again impossible to represent this by means of Frege's xi-notation. The referent of this linguistic function is that ontological function which maps a person onto the truth-value the True if his father loves his mother and onto the False otherwise.[27]

In the context of the interpretation of incomplete expressions as linguistic functions the existence of pathological linguistic functions poses no real problem as there is a natural criterion for isolating them. In the context of Dummett's interpretation of incomplete expressions the existence of pathological patterns poses a very serious problem. Dummett interprets Frege's incomplete expressions as patterns or features of sentences (*FPL*, p. 31):

> There is no part in common to the sentences 'Brutus killed Brutus' and 'Cassius killed Cassius' which is not also a part of the sentence 'Brutus killed Caesar': yet the predicate 'ξ killed ξ' is said to occur in the first two and not in the third. Such a complex predicate is, rather, to be regarded as a *feature* in common to the two sentences, the feature, namely, that in both the simple relational expression '... killed ...' occurs with the same names in both of its argument-places.

By analogy with the notion of a pathological linguistic function it is possible to construct pathological patterns or to discern, say, a common pathological feature in two interrogative sentences. For example, there is no part in common to the interrogative sentences 'Did Brutus kill Brutus?' and 'Did Cassius kill Cassius?' which is not also a part of the interrogative sentence 'Did Brutus kill Caesar?', but the first two have a common *feature*, namely, that the expression 'Did ... kill ...?' occurs in both of them with the same proper name in both of its argument-places. Dummett does not raise —let alone try to answer— the question of how the pattern or feature, *qua* pattern or feature, 'ξ killed ξ' differs from the pathological

26 If there exists a person who has more than one name, then this recipe will not be a function. To be certain that we are dealing here with a function we could insist that if a person has more than one name, then we choose the one which is earliest alphabetically.

27 More examples of pathological linguistic functions can be found in chapter 2 of my thesis, *Frege's Theory of Functions in Application to Linguistic Structures*, where I also discuss the roots of the idea in the works of Husserl and Wittgenstein.

pattern or feature 'Did ξ kill ξ?' He does not ask what property the first pattern possesses that the second lacks in virtue of which the first is and the second is not an incomplete expression. Similarly, a pathological pattern can be discerned in each of the following inferences:

'Peter envies someone because Peter envies John.'
'Jason envies someone because Jason envies John.'
'Maxine envies someone because Maxine envies John.'

The pattern that each of these three indicative sentences exhibits can be expressed by the notation, 'ξ envies someone because ξ envies John.' And again, the following sentences exhibit a common pattern:[28]

'Lord George Bentinck knew that Maccabeus ran in the Derby.'
'Lord George Bentinck knew that Running Rein ran in the Derby.'
'Lord George Bentinck knew that Red Rum ran in the Derby.'

The common pattern here is 'Lord George Bentinck knew that ξ ran in the Derby' but this is not an unsaturated expression. As a final example we can consider these sentences:

'The man who killed Jack's mother snores.'
'One of the people that served Jack's mother snores.'
'At least one of the policemen responsible for the arrest of Jack's mother snores.'

Here again a pattern can be discerned in each sentence, namely that they all end with the words 'Jack's mother snores'. These three sentences do indeed exhibit a common pattern even though the words 'Jack's mother snores' are neither a grammatical nor a logical unit of any of them.

Someone might try to devise a criterion of demarcation in this case by analogy with what I have called the natural criterion of demarcation. Because Dummett does not allow us to remove an expression from an unsaturated expression, the criterion would be that a pattern or a feature of a sentence is an incomplete expression only if it is obtained when an expression —either complete or incomplete— with a referent is removed or omitted from either a singular term that has a referent or from a sentence that is either true or false. In this case a further argument can be brought to bear against Dummett's interpretation of unsaturated expressions as

[28] This example is based on one used by Geach for a different purpose. Geach's example can be found in Lewis (ed.), *Peter Geach*, p. 281.

patterns and this makes use of the parallelism that exists in Frege's philosophy of language between the linguistic realm, the realm of sense and the ontological realm. There are unsaturated entities in each of these three realms and Frege talks about them in very similar terms. Dummett, however, interprets incomplete expressions and their senses as patterns while interpreting their referents as functions. My argument proceeds in two stages. In the first part I show that there is no reason to think that 'incomplete' and related terms mean different things when used of expressions, their senses and their referents. In the second part I show that incomplete entities in the ontological realm cannot be thought of as patterns or features.

There is no textual justification in Frege's writings to support a different interpretation of the notion of incompleteness as it occurs in each of the three realms. Frege, for example, uses the same terminology when speaking of incomplete entities in the linguistic and ontological reals. On p. 6 of "Function and Concept" he writes:

I am concerned to show that the argument does not belong with a function, but goes together with the function to make up a complete whole; for a function by itself must be called incomplete, in need of supplementation, or "unsaturated." And in this respect functions differ fundamentally from numbers.

And a few pages later on p. 17 he writes:

Statements ... can be imagined to be split up into two parts; one complete in itself, and the other in need of supplementation, or "unsaturated."

Furthermore, Frege uses the same terminology to talk about the way in which incomplete and complete entities in each realm combine together (if the argument-places of the one are fitting for the other) and he also talks about the unsaturatedness of an ontological function *answering to* the unsaturatedness of the functional sign whose referent it is and the unsaturatedness of the sense of an incomplete expression *corresponding to* the unsaturatedness of that expression:

> The peculiarity of functional signs, which we here called 'unsaturatedness', naturally has something answering to it in the functions themselves. They too may be called 'unsaturated' and in this way

> we mark them out as fundamentally different from numbers ("What is a Function?", p. 665).[29]

Similarly, in the case of the senses of functional signs Frege writes:

> If we call the parts of the sentence that show gaps unsaturated and the other parts complete, then we can think of a sentence as arising from saturating an unsaturated part with a complete part ... To the unsaturated part of the sentence there corresponds an unsaturated part of the thought and to the complete part of the sentence a complete part of the thought, and we can also speak here of saturating the unsaturated part of the thought with a complete part ... Each of the sentence-parts
>
> "1 is greater than 2" and "1^2 is greater than 2"
>
> can also be seen as put together out of the proper name "1" and an unsaturated part. The corresponding holds for the related thoughts.[30]

And again, in "On Concept and Object" he writes that:

> the sense of the phrase 'the number 2' does not hold together with that of the expression 'the concept *prime number*' without a link. We apply such a link in the sentence 'the number 2 falls under the concept *prime number*'; it is contained in the words 'falls under' which need to be completed in two ways —by a subject and an accusative—; and only because their sense is thus 'unsaturated' are they capable of serving as a link. (P. 205.)

These passage show that Frege used the same terminology of functional signs, their senses and their referents. He applies, for example, the expressions 'unsaturated' and 'incomplete' to the functional entities on all three levels. Moreover, they show not only that, but also that Frege conceived of the combination of an unsaturated item with a saturated one on each level in the same terms. He would not have done this if he had radically different ideas about the modes of combination involved on each level.

29 See also "On Schoenflies: *Die logischen Paradoxien der Mengenlehre*", pp. 191-192.

30 "A Brief Survey of My Logical Doctrines", pp. 217-218. In the first sentence of this quotation the translation actually has '... saturating a saturated part with a complete part ...' This is an incorrect rendering of the German and I have altered it in the text.

Because Frege refers to both expressions and their referents as being incomplete and unsaturated one would expect that if someone interprets unsaturated expressions as patterns or features, then he is committed to interpreting their referents as patterns or features. But although there is some plausibility in saying that 'ξ killed ξ' is a feature of the sentence 'Brutus killed Brutus', there is no plausibility in saying that the concept ξ *killed* ξ is a feature of both the True and the False. (A defender of this position would be committed to saying that the concept ξ *killed* ξ was a feature of the True when that concept was applied to a person who actually did kill himself, but that it was a feature of the False when applied to a person who did not kill himself.) Concepts cannot be both functions from objects to truth-values and also features or patterns. This is because —for Dummett— the collection of patterns is disjoint from the collection of functions. This comes out most clearly in his discussion of the senses of incomplete expressions (*LBM*, p. 196; see also *FPM*, p. 37):

> The extraction of the predicate from the sentence depends upon recognising that the sentence displays a pattern in common with certain other sentences; a grasp of the sense of that predicate constitutes a grasp of a pattern in common between the thought expressed by the sentence and other thoughts.

Thus, as well as thinking that an incomplete expression is a pattern, Dummett also thinks that the sense of an incomplete expression is a pattern; but Dummett forcefully argues against the interpretation of the sense of an incomplete expression as being a function in the realm of sense.[31] If we assume that Dummett is being consistent in this area of his interpretation of Frege, then he must accept that nothing can be both a pattern and a function.

There is yet a further consideration that can be brought against the different interpretations of incompleteness that Dummett gives depending on whether we are talking about incomplete expressions or their incomplete referents. Dummett is correct in pointing out that we do not have any ontological intuition which allows us to classify entities into objects or functions independently of our ability to distinguish between the expressions that refer to them. We know that an entity is an object because it is the referent of a proper name (or singular term) and proper names behave in a certain way and we know that an entity is a concept, say, because it is the referent of a predicate. What is mysterious in Dummett's account is how we could come to know that an entity is a function on the basis of our

[31] For details of this see my paper, "On the Sense of Unsaturated Expressions".

knowledge that it is the referent of a pattern. Functions have different properties from patterns and the way a pattern combines with an expression to produce another expression is different from the way in which a function yields a value by being applied to an argument. As we have no ontological intuition into the behaviour of functions, the only way in which we can ascertain their behaviour and properties is through the behaviour and properties of those expressions whose referents are functions. If Dummett's interpretation of incomplete expressions as patterns was correct, then we would have no reason to think that the referent of an incomplete expression was a function. Under Geach's interpretation, by contrast, there is a uniform account given of incomplete expressions, their senses and their referents. Incomplete expressions are linguistic functions, their senses are functions in the realm of sense and their referents are ontological functions.[32]

It may just as well be mentioned here that no ontological intuition is involved in what I have called the natural criterion of demarcation because the fact that some linguistic expression has a reference is something that can be ascertained by how that expression behaves in inferences. In chapter 4 of *FPL* Dummett shows how the category of proper names can be individuated by looking at how certain expressions behave in some inferences and the category of propositions is the category of those indicative sentences that can figure as a premise or a conclusion in a deductive argument. (This is needed to exclude such indicative sentences as 'Jack loves someone because Jack loves Jill' from the category of propositions.)

5. *Conclusion*

In this paper I have considered the interpretations of Frege's notion of an incomplete expression put forward by Dummett and Geach and I have argued that Geach's interpretation is to be preferred to Dummett's. One of the issues involved in this debate concerns the reference of an incomplete expression. In discussing predicates and functional signs Frege is always careful to consider whether or not these expressions have a reference and what, in fact, they might refer to. Both Geach and Dummett offer interpretations which lose sight of the referent that an incomplete expression should have. It is possible to restore this connection in the case of Geach's interpretation, but it cannot be done satisfactorily in the case of

[32] In my paper, "On the Sense of Unsaturated Expressions", I defend the Church-Geach interpretation of the sense of an incomplete expression against Dummett's criticisms.

Dummett's. There are additional flaws in Dummett's exegesis, but this one I consider to be most serious. It leaves us with no adequate way of demarcating between pathological and non-pathological patterns. I would just like to emphasise that it is as part of his *exegesis* of Frege that I criticise Dummett's notion of a pattern. The notion does not mesh well with Frege's general account of the functioning of language. I do think that the notion of a pattern has an important role to play in the philosophy of language —and in philosophy in general— but in order to do so it needs to be incorporated into a non-Fregean theory of meaning. To explore that, however, is beyond the scope of this paper.

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