Homological Projective Duality

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1 Preliminaries

Motivating Question: Assume that we know $D(X)$, the bounded derived category of coherent sheaves on a smooth projective variety $X \subset \mathbb{P}^n$. What can we say about $D(X_H)$ for $X_H = X \cap H$, a hyperplane section of X?

First, we need to explain what we mean by claiming to "know" $D(X)$.

Definition 1.1. *Let $\mathcal{T}$ be a triangulated category. A* semiorthogonal decomposition *of $\mathcal{T}$ is a collection $\mathcal{T}_0, \dots, \mathcal{T}_{n-1}$ of full triangulated subcategories such that*

1. *$Hom(\mathcal{T}_i, \mathcal{T}_j) = 0$ for $i > j$,*
2. *for any $F \in \mathcal{T}$ there exists a chain of morphisms $0 = F_n \to F_{n-1} \to \dots \to F_1 \to F_0 = F$ such that $Cone(F_{i+1} \to F_i) \in \mathcal{T}_i$.*

Remark 1.2. If $\mathcal{T} = D(X)$ for a smooth projective X then the categories $\mathcal{T}_i \subset D(X)$ are admissible, i.e. there exist both left and right adjoints to the inclusion functor.

Remark 1.3. Because of the first condition the chain $0 = F_n \to F_{n-1} \to \dots \to F_1 \to F_0 = F$ is unique and functorial.

The simplest triangulated category is $D(\mathbf{k})$ - the derived category of $\mathbf{k}$-vector spaces.

Definition 1.4. *An object $E \in \mathcal{T}$ is exceptional if $Hom(E, E) = \mathbf{k}$ and $Ext^i(E, E) = Hom(E, E[i]) = 0$ for $i \neq 0$.*

If E is an exceptional object then the functor $D(\mathbf{k}) \to \mathcal{T}$ defined by $V \mapsto V \otimes E$ is fully faithful.

Definition 1.5. *A sequence $E_0, \dots, E_{n-1}$ of exceptional objects is an* exceptional collection *if $Ext^k(E_i, E_j) = 0$ for $i > j$ and all k. An exceptional collection is* full *if $\langle E_0, \dots, E_{n-1} \rangle = \mathcal{T}$, where $\langle E_0, \dots, E_{n-1} \rangle$ denotes the smallest triangulated subcategory of $\mathcal{T}$ containing the objects $E_0, \dots, E_{n-1}$.*

If $E_0, \dots, E_{n-1}$ is a full exceptional collection, then we have a semiorthogonal decomposition $\mathcal{T} = \langle E_0, \dots, E_{n-1} \rangle = \langle D(\mathbf{k}), \dots, D(\mathbf{k}) \rangle$ with n components.

Example. If $X = \mathbb{P}^n$, then for example $D(X) = \langle \mathcal{O}, \dots, \mathcal{O}(n) \rangle$.

Now, we can reformulate the question we have started with.

Motivating Question: Suppose we know a semiorthogonal decomposition for $D(X)$. Can we construct a semiorthogonal decomposition for $D(X_H)$?

We need some compatibility conditions between the semiorthogonal decomposition and the projective embedding $f \colon X \hookrightarrow \mathbb{P}^n$.

Examples.

1. For $\mathrm{id} \colon X = \mathbb{P}^n \hookrightarrow \mathbb{P}^n$ and a hyperplane $H \subset \mathbb{P}^n$ we have $D(X_H) = D(H) = \langle \mathcal{O}, \dots, \mathcal{O}(n-1) \rangle$.
2. For the second Veronese embedding $\nu_2 \colon X = \mathbb{P}^n \hookrightarrow \mathbb{P}^N$, $N = \binom{n+1}{2} - 1$ and the hyperplane $H \subset \mathbb{P}^N$, the hyperplane section $X_H \cong Q^{n-1}$ is isomorphic to a quadric and we have $D(X_H) = \langle \mathcal{C}_H, \mathcal{O}, \dots, \mathcal{O}(n-2) \rangle$.

*Lecture notes taken during the workshop " Homological Projective Duality and Noncommutative Geometry" at University of Warwick, 8-13 October 2012.

Remark 1.6. Abstractly the category $\mathcal{C}_H$ does not depend on the place we put it in the decomposition; we have

$$D(Q^{n-1}) = \langle \mathcal{C}_H, \mathcal{O}, \dots, \mathcal{O}(n-2)\rangle = \left\langle \mathcal{O}, \mathcal{C}_H^1, \mathcal{O}(1), \dots, \mathcal{O}(n-2)\right\rangle = \ldots = \left\langle \mathcal{O}, \dots, \mathcal{O}(n-2), \mathcal{C}_H^{n-1}\right\rangle$$

and

$$\mathcal{C}_H \cong \mathcal{C}_H^1 \cong \ldots \cong \mathcal{C}_H^{n-1}.$$

3. More generally, for $d \le n+1$ and the d-th Veronese embedding $\nu_d \colon X = \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ we have $D(X_H) = \left\langle \mathcal{C}_H^d, \mathcal{O}, \dots, \mathcal{O}(n-d)\right\rangle$,

4. For $\nu_d \colon X = \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ and hyperplanes $H_1, \dots, H_k \subset \mathbb{P}^N$ such that $\dim X_{H_1 \dots H_k} = \dim X - k$ we have $D(X_{H_1 \dots H_k}) = \left\langle \mathcal{C}^d_{H_1 \dots H_k},\ \mathcal{O}, \dots, \mathcal{O}(n-dk)\right\rangle$.

Definition 1.7. *A* Lefschetz decomposition *of $D(X)$ with respect to $\mathcal{O}_X(1)$ is a chain of full triangulated subcategories $0 \subset \mathcal{A}_{i-1} \subset \mathcal{A}_{i-2} \subset \ldots \subset \mathcal{A}_1 \subset \mathcal{A}_0$ such that $D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1)\rangle$ is a semiorthogonal decomposition. Here,*

$$\mathcal{A}_k(k) := \{A(k) \mid A \in \mathcal{A}_k\}.$$

Examples.

1. For $X = \mathbb{P}^n$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n}(1)$ we can consider the following Lefschetz decompositions:

   - a Lefschetz decomposition of length $i = n+1$ with $\mathcal{A}_{i-1} = \ldots = \mathcal{A}_0 = \langle \mathcal{O}\rangle$,
   - a Lefschetz decomposition of length $i = n$ with $\mathcal{A}_0 = \langle \mathcal{O}, \mathcal{O}(1)\rangle$, $\mathcal{A}_1 = \ldots = \mathcal{A}_{i-1} = \langle \mathcal{O}(1)\rangle$

   and there are many other.

   In particular, we see that a Lefschetz decomposition is an additional structure on an exceptional collection.

2. For $X = \mathbb{P}^n$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n}(2)$

   - if $n+1$ is even then $i = \frac{n+1}{2}$ and $\mathcal{A}_{i-1} = \ldots = \mathcal{A}_0 = \langle \mathcal{O}, \mathcal{O}(H)\rangle$ is a Lefschetz decomposition,
   - if $n+1$ is odd then $i = \frac{n+2}{2}$ and $\mathcal{A}_{i-1} = \langle \mathcal{O}\rangle$, $\mathcal{A}_{i-2} = \ldots = \mathcal{A}_0 = \langle \mathcal{O}, \mathcal{O}(1)\rangle$ is a Lefschetz decomposition.

3. $X = Q^n$, an n-dimensional quadric, $\mathcal{O}_X(1) = \mathcal{O}_{Q^n}(1)$

   - If n is odd, then $D(X) = \langle S, \mathcal{O}, \dots, \mathcal{O}(n-1)\rangle$ for a spinor bundle S. X has a Lefschetz decomposition with $i = n$, $\mathcal{A}_0 = \langle S, \mathcal{O}\rangle$, and $\mathcal{A}_1 = \ldots = \mathcal{A}_{i-1} = \langle \mathcal{O}\rangle$.
   - If n is even, $D(X) = \langle S_+, S_-, \mathcal{O}, \dots, \mathcal{O}(n-1)\rangle$ for spinor bundles S_+ and S_-. X has a Lefschetz decomposition with $i = n$, $\mathcal{A}_0 = \langle S_+, S_-, \mathcal{O}\rangle$ and $\mathcal{A}_1 = \ldots = \mathcal{A}_{i-1} = \langle \mathcal{O}\rangle$.
   - By mutation we also get that $D(X) = \langle S_+, \mathcal{O}, S_+(1), \mathcal{O}(1), \dots, \mathcal{O}(n-1)\rangle$. Then we get a Lefschetz decomposition with $i = n$, $\mathcal{A}_0 = \mathcal{A}_1 = \langle \mathcal{O}, S_+\rangle$ and $\mathcal{A}_2 = \ldots = \mathcal{A}_{i-1} = \langle \mathcal{O}\rangle$.

4. For any X we have a stupid decomposition with $i = 1$ and $\mathcal{A}_0 = D(X)$.

Proposition 1.8. *Let $D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1)\rangle$ be a Lefschetz decomposition, $H \in \Gamma(X, \mathcal{O}_X(1))$ and $i \colon X_H = \{H = 0\} \hookrightarrow X$. Note that X_H does not have to be smooth. Then*

1. *The left derived functor $i^*|_{\mathcal{A}_k} \colon \mathcal{A}_k \to D(X_H)$ is fully faithful for $k \ge 1$. (It is not fully faithful on $\langle \mathcal{A}_k, \mathcal{A}_{k+1}(1)\rangle$.)*

2. *$i^*(\mathcal{A}_1(1)), \dots, i^*(\mathcal{A}_{i-1}(i-1))$ are semiorthogonal in $D(X_H)$.*

3. *$D(X_H) = \langle \mathcal{C}_H, i^*(\mathcal{A}_1(1)), \dots, i^*(\mathcal{A}_{i-1}(i-1))\rangle$.*

Proof. 1. Projection formula gives

$$\mathrm{Hom}(i^*F, i^*G) = \mathrm{Hom}(F, i_*i^*G) = \mathrm{Hom}(F, G \otimes i_*\mathcal{O}_{X_H}).$$

The short exact sequence on X

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to i_*\mathcal{O}_{X_H} \to 0$$

gives after tensoring with G

$$0 \to G(-1) \to G \to G \otimes i_* \mathcal{O}_{X_H} \to 0.$$

We get a long exact sequence

$$\ldots \to \mathrm{Hom}(F, G(-1)) \to \mathrm{Hom}(F, G) \to \mathrm{Hom}(F, G \otimes i_* \mathcal{O}_{X_H}) \to \ldots$$

As $F \in \mathcal{A}_k(k)$ and $G(-1) \in \mathcal{A}_k(k-1) \subset \mathcal{A}_{k-1}(k-1)$, we know that $\mathrm{Hom}(F, G(-1)) = 0$. Hence we get an isomorphism $\mathrm{Hom}(F, G) \cong \mathrm{Hom}(i^* F, i^* G)$.

2. For $k > l \geq 1$, $F \in \mathcal{A}_k(k)$ and $G \in \mathcal{A}_l(l)$ the same argument as above proves semiorthogonality.

3. We need to show that $i^*(\mathcal{A}_i(i))$ is an admissible subcategory of $D(X_H)$. We know that $\mathcal{A}_i(i)$ is an admissible subcategory of $D(X)$. It follows that $\mathcal{A}_i(i)$ is saturated and any fully faithful embedding is admissible.

□

Some properties of Lefschetz decomposition

Given a Lefschetz decomposition $\langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_{i-1}(i-1) \rangle$ we have:

- $\langle \mathcal{A}_0, \mathcal{A}_0(1), \ldots, \mathcal{A}_0(r) \rangle = \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_r(r) \rangle = \langle \mathcal{A}_{r+1}(r+1), \ldots, \mathcal{A}_{i-1}(i-1) \rangle^{\perp}$.

  The last equality is by the semiorthogonality of the Lefschetz decomposition. The first equality follows from it and from the obvious inclusions

$$\langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_r(r) \rangle \subset \langle \mathcal{A}_0, \mathcal{A}_0(1), \ldots, \mathcal{A}_0(r) \rangle,$$

$$\langle \mathcal{A}_0, \mathcal{A}_0(1), \ldots, \mathcal{A}_0(r) \rangle \subset \langle \mathcal{A}_{r+1}(r+1), \ldots, \mathcal{A}_{i-1}(i-1) \rangle^{\perp}.$$

- A Lefschetz decomposition can be reconstructed from its $\mathcal{A}_0$ via the recurrent formula

$$\mathcal{A}_r = {}^{\perp} \mathcal{A}_0(-r) \cap \mathcal{A}_{r-1}.$$

  Question: Find nice sufficient conditions on $\mathcal{A}_0$ which would imply that $\mathcal{A}_0$ extends to a Lefschetz decomposition. The subtlety here is in showing admissibility.

- There is a natural partial order on the set of Lefschetz decompositions, $\mathcal{A}_\bullet \leq \mathcal{A}'_\bullet$ if $\mathcal{A}_0 \subset \mathcal{A}'_0$.

Definition 1.9. *A Lefschetz decomposition is* minimal *if it is minimal with respect to the above partial order.*

Question: Is it true that a minimal Lefschetz decomposition always exists?
Can a decreasing sequence of admissible subcategories be infinite?
If $\mathcal{A}_\bullet < \mathcal{A}'_\bullet$ then $\mathcal{A}_0 \subsetneq \mathcal{A}'_0$ and

$$\left\langle \mathcal{A}'_1(1), \ldots, \mathcal{A}'_{j-1}(j-1) \right\rangle = {}^{\perp}\mathcal{A}'_0 \subsetneq {}^{\perp}\mathcal{A}_0 = \langle \mathcal{A}_1(1), \ldots, \mathcal{A}_{i-1}(i-1) \rangle.$$

Question: How can one prove a Lefschetz decomposition to be minimal?

Definition 1.10. *A Lefschetz decomposition is* rectangular *if* $\mathcal{A}_0 = \mathcal{A}_1 = \ldots = \mathcal{A}_{i-1}$.

Examples:

- For $X = \mathbb{P}^n$, an exceptional collection $\langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$ gives a rectangular Lefschetz decomposition with respect to $\mathcal{O}(d)$ if and only if $n+1$ is divisible by d.

- For $X = \mathrm{Gr}(2,5)$ and the Plücker embedding $i : X \hookrightarrow \mathbb{P}^9$ the Kapranov's exceptional collection gives a Lefschetz decomposition

$$D(X) = \langle \mathcal{A}_0, \ldots, \mathcal{A}_3(3) \rangle$$

  with respect to $i^* \mathcal{O}(1)$. Here

$$\mathcal{A}_0 = \left\langle \mathcal{O}, \mathcal{U}^*, S^2(\mathcal{U}^*), S^3(\mathcal{U}^*) \right\rangle, \quad \mathcal{A}_1 = \left\langle \mathcal{O}, \mathcal{U}^*, S^2(\mathcal{U}^*) \right\rangle, \quad \mathcal{A}_2 = \langle \mathcal{O}, \mathcal{U}^* \rangle, \quad \mathcal{A}_3 = \langle \mathcal{O} \rangle$$

  and $\mathcal{U}$ is the tautological vector bundle on $\mathrm{Gr}(2,5)$.

There is also another, rectangular Lefschetz decomposition with respect to the same line bundle

$$D(X) = \langle \mathcal{A}'_0, \dots, \mathcal{A}'_4(4) \rangle$$

with $\mathcal{A}'_0 = \dots = \mathcal{A}'_4 = \langle \mathcal{O}, \mathcal{U}^* \rangle$. This Lefschetz decomposition gives by Proposition 1.8 eight exceptional objects on a hyperplane section of X; if the section is generic these objects generate $D(X_H)$. Note that Kapranov's decomposition gives only six exceptional objects on X_H.

Lemma 1.11. *Assume that $\langle \mathcal{A}_0, \dots, \mathcal{A}_{i-1}(i-1) \rangle$ is a rectangular Lefschetz decomposition and $\omega_X = \mathcal{O}_X(-i)$. Then $\mathcal{A}_\bullet$ is minimal.*

Proof. Suppose, contrary to our claim, that $\mathcal{A}'_\bullet < \mathcal{A}_\bullet$. Then $\mathcal{A}'_0 \subsetneq \mathcal{A}_0$ and we have

$$\left\langle \mathcal{A}'_0, \mathcal{A}'_1(1), \dots, \mathcal{A}'_{i-1}(i-1) \right\rangle \subsetneq \langle \mathcal{A}_0, \mathcal{A}_0(1), \dots, \mathcal{A}_0(i-1) \rangle$$

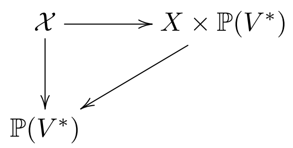
since both the LHS and the RHS are semiorthogonal decompositions and $\mathcal{A}'_r(r) \subset \mathcal{A}'{}_0(r) \subsetneq \mathcal{A}_0(r)$.

Since the RHS is the whole of $D(X)$, we conclude that $\mathcal{A}'_i$ is non-trivial. Let F be a non-zero element of $\mathcal{A}'_i$. We have by Serre duality

$$\operatorname{Hom}^\bullet(F, F) \simeq \operatorname{Hom}^\bullet(F, F(-i)[\dim X]) = \operatorname{Hom}^\bullet(F(i), F)\,[\dim X] \tag{1}$$

and the RHS of (1) is zero since $\operatorname{Hom}^\bullet(\mathcal{A}'_i(i), \mathcal{A}'_0) = 0$ by the semiorthogonality of $\mathcal{A}'_\bullet$. But this is impossible, as the LHS of (1) must contain the identity endomorphism of F. □

For an embedding $f\colon X \to \mathbb{P}(V)$ and a Lefschetz decomposition with respect to $\mathcal{O}_X(1) = f^*(\mathcal{O}_{\mathbb{P}(V)}(1))$ we get a category $\mathcal{C}_H$ in $D(X_H)$ for every hyperplane $H \in \mathbb{P}(V^*)$. Hence we get a family of categories $\{\mathcal{C}_H\}_{H\in\mathbb{P}(V^*)}$. To understand how these categories fit together we look at the universal hyperplane section $\mathcal{X} = \{(x, H) \in X \times \mathbb{P}(V^*) \mid x \in X_H\}$ of X. $\mathcal{X}$ fits into a diagram



By the Küneth formula we have

$$\operatorname{Hom}_{X\times\mathbb{P}(V^*)}(F_1 \boxtimes G_1, F_2 \boxtimes G_2) = \operatorname{Hom}_X(F_1, F_2) \otimes \operatorname{Hom}_{\mathbb{P}(V^*)}(G_1, G_2)$$

and, therefore, we obtain a Lefschetz decomposition

$$D(X \times \mathbb{P}(V^*)) = \langle \mathcal{A}_0 \boxtimes D(\mathbb{P}(V^*)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes D(\mathbb{P}(V^*)) \rangle\,,$$

where

$$\mathcal{A}_k \boxtimes D(\mathbb{P}(V^*)) := \{A \boxtimes F \mid A \in \mathcal{A}_k,\ F \in D(\mathbb{P}(V^*)\} \hookrightarrow D(X \times \mathbb{P}(V^*)).$$

It is a Lefschetz decomposition with respect to $\mathcal{O}_X(1) \boxtimes L$ for any line bundle L on $\mathbb{P}(V^*)$.

$\mathcal{X} \subset X \times \mathbb{P}(V^*)$ is a divisor whose structure sheaf $\mathcal{O}_{\mathcal{X}}$ fits into a short exact sequence

$$0 \to \mathcal{O}_X(-1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(-1) \to \mathcal{O}_{X\times\mathbb{P}(V^*)} \to \mathcal{O}_{\mathcal{X}} \to 0.$$

Arguing as in the proof of Prop. 1.8 we obtain a semiorthogonal decomposition

$$D(\mathcal{X}) = \langle \mathcal{C}, \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^*)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes D(\mathbb{P}(V^*)) \rangle\,.$$

The category $\mathcal{C}$ is the total family of $\{\mathcal{C}_H\}$ in the sense detailed below.

Base change for semiorthogonal decompositions

Definition 1.12. *For $X \xrightarrow{p} S$ a subcategory $\mathcal{T} \subset D(X)$ is S-*linear *is for any $F \in D^{perf}(S)$ we have an inclusion $\mathcal{T} \otimes p^*F \subset \mathcal{T}$.*

Remark 1.13. A pullback of $F \in D(S)$ can be unbounded; if p is flat then we can replace $D^{\mathrm{perf}}(S)$ by $D(S)$ in the definition above.

Definition 1.14. *Morphisms $p\colon X \to S$ and $f\colon S' \to S$ are* Tor-independent *if for any $x \in X$ and $s' \in S'$ such that $p(x) = s = f(s')$ the groups $Tor_{>0}^{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, \mathcal{O}_{S',s'})$ are zero.*

Remark 1.15. If either p or f is flat then p and f are Tor-independent. If p and f are closed embeddings than they are Tor-independent if and only if they are transversal.

Let $X' = X \times_S S'$ and consider the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\widetilde{f}} & X \\ {\scriptstyle \widetilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S. \end{array}$$

Theorem 1.16. *Let $D(X) = \langle \mathcal{T}_0, \ldots, \mathcal{T}_{n-1} \rangle$ be an S-linear semiorthogonal decomposition. If p and f are Tor-independent, there exists an S'-linear semiorthogonal decomposition $D(X') = \left\langle \mathcal{T}'_0, \ldots, \mathcal{T}'_{n-1} \right\rangle$ such that $\widetilde{f}^*(\mathcal{T}_i^{perf}) \subset \mathcal{T}'_i$. Here, $\mathcal{T}_i^{perf} = \mathcal{T}_i \cap D^{perf}(X)$. In fact, $\mathcal{T}'_i$ is the completion of $\mathcal{T}_i^{perf} \boxtimes D^{perf}(S')$ with respect to certain homotopy colimits.*

We have the following picture

$$\begin{array}{ccccc} X_H & \xrightarrow{\iota} & \mathcal{X} & \longrightarrow & X \times \mathbb{P}(V^*) \\ \downarrow & & \downarrow & & \\ \mathrm{Spec}(k) & \longrightarrow & \mathbb{P}(V^*) & & \end{array}$$

As the subcategories $\mathcal{A}_i(i) \boxtimes D(\mathbb{P}(V^*))$ are $D(\mathbb{P}(V^*))$-linear, the category

$$\mathcal{C} =^{\perp} \langle \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^*)), \mathcal{A}_2(2) \boxtimes D(\mathbb{P}(V^*)), \ldots, \mathcal{A}_{i-1}(i-1) \boxtimes D(\mathbb{P}(V^*)) \rangle$$

is also $D(\mathbb{P}(V^*))$-linear. The map $\mathcal{X} \to \mathbb{P}(V^*)$ is flat and hence by the base change $\mathcal{C}_H$ is the completion of $\iota^*\left(\mathcal{C}^{perf}\right)$ with respect to certain homotopy colimits.

2 Main theorem

Let $\gamma\colon \mathcal{C} = \mathrm{Tot}\{\mathcal{C}_H\}_{H \in \mathbb{P}(V^*)} \to D(\mathcal{X})$ be the natural inclusion and let $\gamma^* : D(\mathcal{X}) \to \mathcal{C}$ be its left adjoint. Let π be the projective bundle map $\mathcal{X} \xrightarrow{\pi} X$.

Definition 2.1. *A* Homological Projective Dual *of*

$$(X, f\colon X \to \mathbb{P}(V), \mathcal{A}_\bullet),$$

where $\mathcal{A}_\bullet = \mathcal{A}_{i-1} \subset \ldots \subset \mathcal{A}_0$ is a Lefschetz decomposition of $D(X)$ with respect to $f^*(\mathcal{O}(1))$, is

$$(Y, g\colon Y \to \mathbb{P}(V^*), \mathcal{B}_\bullet),$$

where $\mathcal{B}_\bullet = \mathcal{B}_{j-1} \subset \ldots \subset \mathcal{B}_0$ is a Lefschetz decomposition of $D(Y)$ with respect to $g^*(\mathcal{O}(1))$, such that there exists $\mathcal{E} \in D(Y \times_{\mathbb{P}(V^*)} \mathcal{X})$ inducing a $\mathbb{P}(V^*)$ -linear equivalence $\Phi_\mathcal{E}\colon D(Y) \to \mathcal{C}$ and $\gamma^*\pi^*(\mathcal{A}_0) = \Phi_\mathcal{E}(\mathcal{B}_0)$.

Theorem 2.2. *Let $(Y, g, \mathcal{B}_\bullet)$ be a Homological Projective Dual of $(X, f, \mathcal{A}_\bullet)$. Choose $L \subset V^*$ of dimension r and put $X_L = X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$, $Y_L = Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$, where $L^\perp = Ker(V \to L^*)$. If*

$$dim X_L = dim X - r, \qquad dim Y_L = dim Y - N + r \tag{2}$$

(where $N = dim V$) then

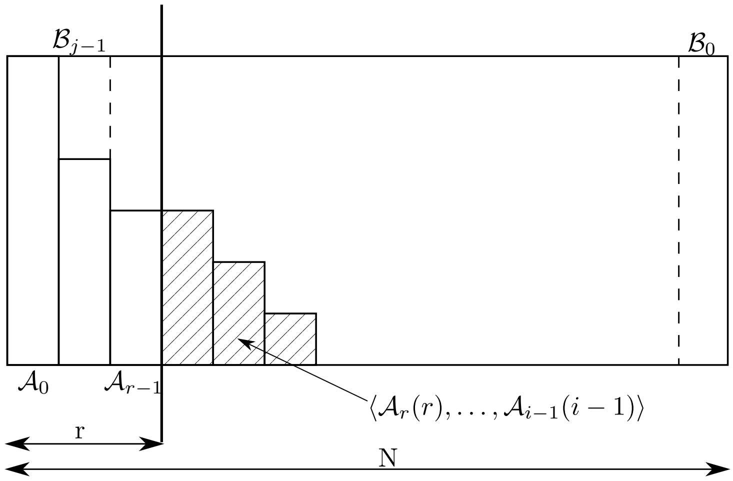
$$D(X_L) = \langle \mathcal{C}_L, \mathcal{A}_r(r), \ldots, \mathcal{A}_{i-1}(i-1) \rangle$$
$$D(Y_L) = \langle \mathcal{C}_L, \mathcal{B}_{N-r}(N-r), \ldots, \mathcal{B}_{j-1}(j-1) \rangle .$$

Remark 2.3. If X_L and Y_L do not have expected dimensions then we do not have Tor-independence and have to consider a derived fiber product.

It is possible to show that

$$j = N - 1 - \max\{k \mid \mathcal{A}_k = \mathcal{A}_0\}.$$

Lefschetz decompositions $\mathcal{A}_\bullet$ and $\mathcal{B}_\bullet$ fit into the following picture.

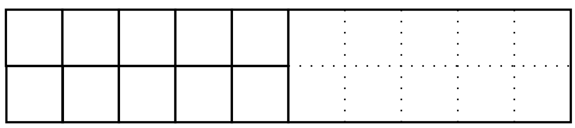


In particular, if the decomposition $\mathcal{A}_\bullet$ is rectangular then either $\mathcal{C}_L = D(X_L)$ or $\mathcal{C}_L = D(Y_L)$.

Example. Let X be $\mathrm{Gr}(2, W) \subset \mathbb{P}(\Lambda^2 W)$ embedded via the Plücker embedding and let $\dim W = 5$. Recall that $D(X)$ has a rectangular Lefschetz decomposition

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_4(4)\rangle$$

with each $\mathcal{A}_i = \langle \mathcal{O}, \mathcal{U}^* \rangle$ where $\mathcal{U}$ is the tautological bundle on X. We have $N = 10$ and the picture for $\mathcal{A}_\bullet$ is



The homological projective dual of X is $Y = \mathrm{Gr}(2, W^*) \subset \mathbb{P}(\Lambda^2 W^*)$. We have $\dim X = \dim Y = 6$.

- For $r = 1$ the condition (2) forces Y_L to be empty. Hence $\mathcal{C}_L = 0$ and $D(X_L) = \langle \mathcal{A}_1(1), \dots, \mathcal{A}_4(4)\rangle$ is a full exceptional collection consisting of eight vector bundles.
- For $r = 2$ and general L we have again $Y_L = \emptyset$, $\mathcal{C}_L = 0$ and $D(X_L) = \langle \mathcal{A}_2(2), \dots, \mathcal{A}_4(4)\rangle$ is a full exceptional collection of six vector bundles.
- For $r = 3$ an general L we have $Y_L = \emptyset$, $\mathcal{C}_L = 0$ and X_L, a Fano threefold of index two and degree five, has a semiorthogonal decomposition $D(X_L) = \langle \mathcal{A}_3(3), \mathcal{A}_4(4)\rangle$.
- For $r = 4$ and general L the variety Y_L is a union of five points, $\mathcal{C}_L = D(Y_L)$ and X_L has a semiorthogonal decomposition $D(X_L) = \langle D(Y_L), \mathcal{A}_4(4)\rangle$. If Y_L is smooth, then $D(Y_L)$ is generated by five exceptional objects and X_L is a del Pezzo of degree 5.
- For $r = 5$ and general L both X_L and Y_L are elliptic curves and $D(X_L) = D(Y_L)$.
- For $r = 6, \dots 10$ the situation is symmetric.

Lemma 2.4. *Let $(Y, g, \mathcal{B}_\bullet)$ be a homological projective dual of $(X, f, \mathcal{A}_\bullet)$. Then the set Crit$(g)$ of critical values of g is the classical projective dual of X*

$$X^\vee = \{H \in \mathbb{P}(V^*) \mid X_H \text{ is singular }\}.$$

Indeed, by definition there is a semiorthogonal decomposition

$$D(\mathcal{X}) = \langle D(Y), \mathcal{A}_1(1) \boxtimes D(\mathbb{P}(V^*)), \mathcal{A}_2(2) \boxtimes D(\mathbb{P}(V^*)), \dots, \mathcal{A}_{i-1}(i-1) \boxtimes D(\mathbb{P}(V^*))\rangle\,.$$

By base change we get that

$$D(X_H) = \langle D(Y_H), \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1)\rangle\,.$$

Recall also that a projective scheme Z is smooth if and only if $D(Z)$ is Ext-finite. The categories $\mathcal{A}_i(i)$ are subcategories of $D(X)$ and therefore Ext-finite. Using this fact and the above semiorthogonal decomposition one can show that X_H is smooth if and only if Y_H is . Then for a hyperplane $H \in \mathbb{P}(V^*)$ we've

$$H \in \mathrm{Crit}(g) \Leftrightarrow Y_H \text{ is not smooth } \Leftrightarrow X_H \text{ is not smooth } \Leftrightarrow H \in X^\vee.$$

Remark 2.5. If $(Y, g, \mathcal{B}_\bullet)$ is an HPD of $(X, f, \mathcal{A}_\bullet)$ then Y is smooth (uses the fact that $\mathcal{X}$ is smooth).

For a Lefschetz decomposition

$$D(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_{i-1}(i-1)\rangle$$

denote by $\mathfrak{a}_k$ the category $\mathcal{A}_{k+1}^\perp \cap \mathcal{A}_k$, called the *k-th primitive category of the Lefschetz decomposition of* $D(X)$. Then $\mathcal{A}_k = \langle \mathfrak{a}_k, \mathcal{A}_{k+1}\rangle = \langle \mathfrak{a}_k, \dots, \mathfrak{a}_{i-1}\rangle$. In particular

$$\mathcal{A}_0 = \langle \mathfrak{a}_0, \dots, \mathfrak{a}_{i-1}\rangle\,.$$

Denote by $\alpha_0^* : D(X) \to \mathcal{A}_0$ the left adjoint to the inclusion functor. Then $\alpha_0^*(\mathcal{A}_k(k)) = 0$ and $\alpha_0^*(\mathfrak{a}_k(l)) = 0$ for $l \leq k$.

Lemma 2.6. $\alpha_0^*(\mathfrak{a}_k(k+1)) \to \mathcal{A}_0$ *is fully faithful and*

$$\mathcal{A}_0 = \langle \alpha_0^*(\mathfrak{a}_0(1)), \alpha_0^*(\mathfrak{a}_1(2)), \dots \alpha_0^*(\mathfrak{a}_{i-1}(i)))\rangle\,.$$

The decomposition in the above lemma is the *dual primitive decomposition* of $\mathcal{A}_0$.

Recall the composition of functors $D(X) \xrightarrow{\pi^*} D(\mathcal{X}) \xrightarrow{\gamma^*} \mathcal{C}$.

Lemma 2.7. *The composition* $\gamma^* \circ \pi^*$ *is fully faithful on* $\mathcal{A}_0$ *and we denote its image by* $\mathcal{C}_0 = \gamma^*\pi^*(\mathcal{A}_0)$.

The category $\mathcal{C}_0 \subset \mathcal{C}$ is admissible because $\mathcal{A}_0$ is saturated. For any $k > 0$ define category $\mathcal{C}_k$ as

$$\gamma^*\pi^*\big(\langle \alpha_0^*(\mathfrak{a}_0(1)), \alpha_0^*(\mathfrak{a}_1(2)), \dots, \alpha_0^*(\mathfrak{a}_{N-k-2}(N-k-1))\rangle\big) \subset \mathcal{C}_0.$$

Since $\mathcal{C}$ is $\mathbb{P}(V^*)$-linear the map $p : \mathcal{X} \to \mathbb{P}(V^*)$ allows to define the endofunctor $[-](1)$ on the category $\mathcal{C}$ by $F \mapsto F \otimes p^*\mathcal{O}_{\mathbb{P}(V^*)}(1)$.

Lemma 2.8. $\mathcal{C}_\bullet$ *is a Lefshetz decomposition of* $\mathcal{C}$ *with respect to the endofunctor* $[-](1)$ *defined above.*

This is the dual decomposition of $\mathcal{C} \cong D(Y)$. These categories will be used in the proof of the main theorem.

Theorem 2.9. *Let* $(X, f, \mathcal{A}_\bullet)$ *let* $g : Y \to \mathbb{P}(V^*)$ *and* $\mathcal{E} \in D(\mathcal{X} \times_{\mathbb{P}(V^*)} Y)$ *be such that*

1. $\Phi_\mathcal{E}$ *is fully faithful and factors through* $\mathcal{C}$, *i.e.* $\Phi_\mathcal{E} = D(Y) \xrightarrow{\phi_\mathcal{E}} \mathcal{C} \xrightarrow{\gamma} D(\mathcal{X})$ *for some fully faithful* $\phi_\mathcal{E}$;
2. $D(X) \xrightarrow{\pi^*} D(\mathcal{X}) \xrightarrow{\Phi_\mathcal{E}^*} D(Y)$ *is fully faithful on* $\mathcal{A}_0$;
3. $\mathcal{B}_0, \mathcal{B}_1(1), \dots, \mathcal{B}_{j-1}(j-1)$ *are semiorthogonal in* $D(Y)$, *where* $\mathcal{B}_k = \Phi_\mathcal{E}^*(\mathcal{C}_k)$.

Then Theorem 2.2 holds. In particular $(Y, g, \mathcal{B}_\bullet)$ *is a homological projective dual to* $(X, f, \mathcal{A}_\bullet)$.

There is also a relative version of Theorem 2.2. Consider a base scheme T and let $\mathcal{L} \subset V^* \otimes \mathcal{O}_T$ be a vector subbundle of rank r. If the dimension of V is N then the rank of $\mathcal{L}^\perp = \mathrm{Ker}(V \otimes \mathcal{O}_T \to \mathcal{L}^*)$ is $N - r$. Define the family of linear sections $X_\mathcal{L} = X \times_{\mathbb{P}(V)} \mathbb{P}_T(\mathcal{L}^\perp)$ and $Y_\mathcal{L} = Y \times_{\mathbb{P}(V^*)} \mathbb{P}_T(\mathcal{L})$.

Theorem 2.10. *If $(Y, g, \mathcal{B}_\bullet)$ is an HPD of $(X, f, \mathcal{A}_\bullet)$ then for any base scheme T and any $\mathcal{L} \subset V^* \otimes \mathcal{O}_T$ such that*

$$\dim(X_{\mathcal{L}}) = \dim(X) + \dim(T) - r$$

$$\dim(Y_{\mathcal{L}}) = \dim(Y) + \dim(T) - N + r$$

there exists a triangulated category $\mathcal{C}_{\mathcal{L}}$ and semiorthogonal decompositions

$$D(X_{\mathcal{L}}) = \langle \mathcal{C}_{\mathcal{L}}, \mathcal{A}_r(r) \boxtimes D(T), \ldots \mathcal{A}_{i-1}(i-1) \boxtimes D(T) \rangle ,$$

$$D(Y_{\mathcal{L}}) = \langle \mathcal{C}_{\mathcal{L}}, \mathcal{B}_{N-r}(N-r) \boxtimes D(T), \ldots, \mathcal{B}_{j-1}(j-1) \boxtimes D(T) \rangle .$$

Before sketching out the proof of the Theorem 2.10 we need to introduce the following auxilliary notion:

Definition 2.11. *A functor $\Phi \colon \mathcal{T}_1 \to \mathcal{T}_2$ is* right splitting *if*

1. $\operatorname{Ker}\Phi = \{T \in \mathcal{T}_1 \mid \Phi(T) = 0\}$ *is right admissible, i.e.* $\mathcal{T}_1 = \left\langle (\operatorname{Ker}\Phi)^\perp, \operatorname{Ker}\Phi \right\rangle$*;*
2. $\Phi|_{(\operatorname{Ker}\Phi)^\perp}$ *is fully faithful;*
3. $\operatorname{Im}\Phi$ *is right admissible in* $\mathcal{T}_2$*, i.e.* $\mathcal{T}_2 = \left\langle (\operatorname{Im}\Phi)^\perp, \operatorname{Im}\Phi \right\rangle$.

NB: If a morphism is not fully faithful then, in general, its image is not a triangulated subcategory.

Lemma 2.12. *The following are equivalent:*

1. Φ *is right splitting;*
2. *There exists a right adjoint* $\Phi^!$ *and the composition of the canonical morphism of functors* $\eta \colon \operatorname{Id} \to \Phi^!\Phi$ *with* Φ *gives an isomorphism* $\Phi_\eta \colon \Phi \to \Phi\Phi^!\Phi$ *(then also* $\varepsilon_\Phi : \Phi\Phi^!\Phi \to \Phi$ *is an isomorphism);*
3. *There exists a right adjoint* $\Phi^!$*, we have*

$$\mathcal{T}_1 = \left\langle \operatorname{Im}\Phi^!, \operatorname{Ker}\,\Phi \right\rangle, \quad \mathcal{T}_2 = \left\langle \operatorname{Ker}\Phi^!, \operatorname{Im}\Phi \right\rangle$$

   and Φ *and* $\Phi^!$ *give quasi-inverse equivalences* $\operatorname{Im}\Phi^! \simeq \operatorname{Im}\Phi$.

4. *There exists* $\mathcal{T}$ *and* $\mathcal{T}_1 \overset{i}{\leftarrow} \mathcal{T} \overset{j}{\rightarrow} \mathcal{T}_2$ *such that i is left admissible, j is right admissible and* $\Phi = j \circ i^*$.

Sketch of the proof of Theorem 2.10. Any family of r-planes in V^* pulls back from the tautological bundle over the grassmanian $\mathbf{P}_r = \operatorname{Gr}(r, V^*)$. It is therefore enough to prove the theorem for the base scheme T being $\mathbf{P}_r$ and the family $\mathcal{L}$ being the tautological bundle $\mathcal{L}_r \subset V^* \otimes \mathcal{O}_{\mathbf{P}_r}$. The case of general T and $\mathcal{L}$ is then obtained by base change.

So define the universal linear sections

$$\mathcal{X}_r = (X \times \mathbf{P}_r) \times_{\mathbb{P}(V) \times \mathbf{P}_r} \mathbb{P}_{\mathbf{P}_r}(\mathcal{L}_r^\perp),$$

$$\mathcal{Y}_r = (Y \times \mathbf{P}_r) \times_{\mathbb{P}(V^*) \times \mathbf{P}_r} \mathbb{P}_{\mathbf{P}_r}(\mathcal{L}_r).$$

Explicitly, in terms of the maps $X \xrightarrow{f} \mathbb{P}(V)$ and $Y \xrightarrow{g} \mathbb{P}(V^*)$, we have

$$\begin{aligned}\mathcal{X}_r &= \{(x, L, v) \in X \times \mathbf{P}_r \times \mathbb{P}(V) \mid v \in L^\perp,\, f(x) = v\} = \\ &= \{(x, L) \in X \times \mathbf{P}_r \mid L \subset f(x)^\perp\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{Y}_r &= \{(y, L, v) \in Y \times \mathbf{P}_r \times \mathbb{P}(V^*) \mid v \in L,\, g(y) = v\} = \\ &= \{(y, L) \in Y \times \mathbf{P}_r \mid L^\perp \subset g(y)^\perp\}.\end{aligned}$$

From this description we see that $\mathcal{X}_r \to X$ is a fiber bundle with fibers $\operatorname{Gr}(r, N-1)$ and $\mathcal{Y}_r \to Y$ is a fiber bundle with fibers $\operatorname{Gr}(N-r, N-1)$. The varieties X and Y are smooth and hence so are $\mathcal{X}_r$ and $\mathcal{Y}_r$. Finally, note that

$$\mathcal{X}_1 = \mathcal{X},\ \mathcal{Y}_1 = Y \quad \text{and} \quad \mathcal{X}_{n-1} = X,\ \mathcal{Y}_{n-1} = \mathcal{Y}.$$

Consider the commutative diagram

$$\begin{array}{ccc} & & \mathcal{X}_r \times_{\mathbf{P}_r} \mathcal{Y}_r \\ & & \downarrow{\scriptstyle \psi_r} \\ X \times Y & \longleftarrow & Q(X,Y) \\ \downarrow & & \downarrow \\ \mathbb{P}(V) \times \mathbb{P}(V^*) & \longleftarrow & Q \end{array}$$

where $Q(X,Y) = \mathcal{X} \times_{\mathbb{P}(V^*)} Y$ and $Q = \{(v,H) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \,|\, v \in H\}$ is the incidence quadric.

Let $\mathcal{E} \in D(Q(X,Y))$ be the object in the definition of the homological projective dual. Then $\mathcal{E}_r = \psi_r^* \phi_* \mathcal{E}$ is an object of $D(\mathcal{X}_r \times_{\mathbf{P}_r} \mathcal{Y}_r)$ and we write Φ_r for the corresponding functor $D(\mathcal{X}_r) \to D(\mathcal{Y}_r)$. We then show by induction on r that for all $r \geq 1$ the functor Φ_r satisfies the condition (2) of Lemma 2.12 and is therefore right splitting.

Since $\mathcal{Y}_1 = Y$ and $\mathcal{X}_1 = \mathcal{X}$, the base case of the induction ($r = 1$) follows from the definition of Y being a homological projective dual of X. To establish the inductive step we change the base to the flag variety $\mathbf{S}_r = \mathrm{Fl}(r-1, r; V^*)$. For

$$\widetilde{\mathcal{X}_{r+1}} = \mathcal{X}_{r+1} \times_{\mathbf{P}_{r+1}} \mathbf{S}_{r+2}, \qquad \widetilde{\mathcal{X}_r} = \mathcal{X}_r \times_{\mathbf{P}_r} \mathbf{S}_{r+1},$$
$$\widetilde{\mathcal{Y}_r} = \mathcal{Y}_r \times_{\mathbf{P}_r} \mathbf{S}_{r+1}, \qquad \widetilde{\mathcal{Y}_{r+1}} = \mathcal{Y}_{r+1} \times_{\mathbf{P}_{r+1}} \mathbf{S}_{r+2}$$

we have it that $\widetilde{\mathcal{X}_{r+1}}$ is a divisor in $\widetilde{\mathcal{X}_r}$ and $\widetilde{\mathcal{Y}_r}$ is a divisor in $\widetilde{\mathcal{Y}_{r+1}}$. Using this presentation we can compare Φ_{r-1}, Φ_r, $\widetilde{\Phi_{r-1}}$ and $\widetilde{\Phi_r}$. This allows us to establish the inductive step: if Φ_{r-1} is right splitting, then so is Φ_r.

Once it is established that Φ_r is right splitting, it follows that

$$D(\mathcal{X}_r) = \left\langle \operatorname{Im} \Phi_r^!, \operatorname{Ker} \Phi_r \right\rangle,$$

$$D(\mathcal{Y}_r) = \left\langle \operatorname{Ker} \Phi_r^!, \operatorname{Im} \Phi_r \right\rangle$$

with $\operatorname{Im} \Phi_r^! \simeq \operatorname{Im} \Phi_r$. We therefore set $\mathcal{C}_{\mathcal{L}_r}$ to be $\operatorname{Im} \Phi_r$ and it remains to show that

$$\operatorname{Ker} \Phi_r^! = \langle \mathcal{B}_{N-r}(N-r) \boxtimes D(\mathbf{P}_r), \ldots, \mathcal{B}_{j-1}(j-1) \boxtimes D(\mathbf{P}_r) \rangle, \tag{3}$$
$$\operatorname{Ker} \Phi_r = \langle \mathcal{A}_r(r) \boxtimes D(\mathbf{P}_r), \ldots, \mathcal{A}_{i-1}(i-1) \boxtimes D(\mathbf{P}_r) \rangle. \tag{4}$$

One can easily check that $\mathcal{A}_k(k) \boxtimes D(\mathbf{P}_r) \subset \operatorname{Ker} \Phi_r$ for $k \geq r$ and that $\mathcal{B}_k(k) \boxtimes D(\mathbf{P}_r) \subset \operatorname{Ker} \Phi_r^!$ for $k \geq N - r$. The issue is to show that the semiorthogonal collections in the RHS of (3)-(4) are full, i.e. they generate the whole of $\operatorname{Ker} \Phi_r$ and $\operatorname{Ker} \Phi_r^!$. For $\operatorname{Ker} \Phi_r^!$ this is done by an ascending induction on r and for $\operatorname{Ker} \Phi_r$ — by a descending induction on r. In both cases, the inductive step uses the base change to $\mathbf{S}_r = \mathrm{Fl}(r-1, r; V^*)$ described above. □

3 Examples

1). Take $X = \mathbb{P}(V)$ for a vector space V of dimension N, $f = \mathrm{id} : \mathbb{P}(V) \to \mathbb{P}(V)$ and Lefschetz decomposition

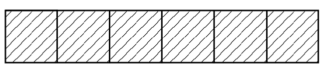
$$D(X) = \left\langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(N-1) \right\rangle$$

with $\mathcal{A}_0 = \langle \mathcal{O} \rangle$. Then $\mathcal{X} \subset X \times \mathbb{P}(V^*)$ is the incidence quadric and we have a semiorthogonal decomposition

$$D(\mathcal{X}) = \langle \mathcal{C}, \mathcal{O}(1) \boxtimes D(\mathbb{P}(V^*)), \ldots, \mathcal{O}(N-1) \boxtimes D(\mathbb{P}(V^*)) \rangle.$$

On the other hand, let $E \to \mathbb{P}(V^*)$ be the rank $N-1$ incidence vector bundle, whose fiber over $H \in \mathbb{P}(V^*)$ is $H \subset V$. Then $\mathcal{X} = \mathbb{P}_{\mathbb{P}(V^*)}(E)$ and hence, by a theorem of Orlov, $\mathcal{C} = 0$. Hence the homological projective dual to $(\mathbb{P}(V), \mathrm{Id}, \mathcal{A}_\bullet)$ is $Y = \emptyset$.

The picture is (case $N = 6$):



1′). Consider the same situation as before but with the Lefschetz decomposition

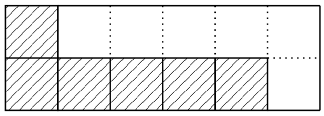
$$D(X) = \big\langle \langle \mathcal{O}, \mathcal{O}(1)\rangle, \mathcal{O}(2), \dots, \mathcal{O}(N-1) \big\rangle$$

with $\mathcal{A}_0 = \langle \mathcal{O}, \mathcal{O}(1)\rangle$. Then

$$D(\mathcal{X}) = \big\langle \mathcal{C}, \mathcal{O}(2) \boxtimes D(\mathbb{P}(V^*)), \dots, \mathcal{O}(N-1) \boxtimes D(\mathbb{P}(V^*)) \big\rangle$$

and it follows from the previous example that $\mathcal{C} = \mathcal{O}(1) \boxtimes D(\mathbb{P}(V^*)$. Therefore $\mathcal{C}$ is equivalent to $D(\mathbb{P}(V^*))$, so the homological projective dual is $Y = \mathbb{P}(V^*)$ with the Lefschetz decomposition defined by $\mathcal{B}_0 = \langle \mathcal{O}, \mathcal{O}(1)\rangle$.

The picture is (case $N = 6$):



1″). Consider the same situation as before but with the Lefschetz decomposition

$$D(X) = \big\langle \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle, \mathcal{O}(3), \dots, \mathcal{O}(N-1) \big\rangle$$

with $\mathcal{A}_0 = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$. Then arguing as before we get $\mathcal{C} = \langle \mathcal{O}(1) \boxtimes D(\mathbb{P}(V^*)), \mathcal{O}(2) \boxtimes D(\mathbb{P}(V^*))\rangle$ and $\mathcal{C}_H = \langle \mathcal{O}, \mathcal{O}(1)\rangle \subset D(\mathbb{P}^{N-2})$ is a "noncommutative projective space". There is no geometrical homological projective dual Y, but instead we can consider $\mathcal{C}$ itself, a fibration in noncommutative projective spaces, to be the "noncommutative homological projective dual" of $(\mathbb{P}(V), \mathrm{Id}, \mathcal{A}_\bullet)$.

For the next few examples, we need to consider a version of the homological projective duality where we consider X relative to some base S. Namely, let X and Y be algebraic varieties over a base scheme S with X globally smooth over the base field k. Suppose we have projective maps $f\colon X \to S \times \mathbb{P}(V)$, $g\colon Y \to S \times \mathbb{P}(V^*)$ and S-linear Lefschetz decompositions $\mathcal{A}_\bullet$ and $\mathcal{B}_\bullet$ of $D(X)$ and $D(Y)$, respectively. Then $\langle Y, g, \mathcal{B}_\bullet\rangle$ is a *homological projective dual of* $\langle X, f, \mathcal{A}_\bullet\rangle$ *relative to* S if there exists $\mathcal{E} \in D(Y \times_{S\times\mathbb{P}(V^*)} \mathcal{X})$ inducing a $S \times \mathbb{P}(V^*)$-linear equivalence $\Phi_{\mathcal{E}} : D(Y) \to \mathcal{C}$ such that $\gamma^*\pi^*(\mathcal{A}_0) = \Phi_{\mathcal{E}}(\mathcal{B}_0)$.

The Theorems 2.2 and 2.10 can be shown to hold in this relative setting. Note that we've only asked X to be smooth over the base field k. So the individual fibers of X over S might be singular. Indeed, we can then obtain by base change the Theorems 2.2 and 2.10 for these singular fibers.

2). Let S be a smooth, not necessarily projective variety. Let E be a rank k vector bundle on S and $X = \mathbb{P}_S(E)$. Assume that $f\colon X \hookrightarrow S \times \mathbb{P}(V)$ is linear on fibers, i.e. it is defined by some $\psi\colon E \hookrightarrow \mathcal{O}_S \otimes V$. Let $\phi\colon \mathcal{O}_S \otimes V^* \to E^*$ be the map dual to ψ. Note that $f^*\mathcal{O}_{S\times\mathbb{P}(V)}(1) = \mathcal{O}_{X/S}(1)$.

By a theorem of Orlov X has a Lefschetz decomposition

$$D(X) = \big\langle D(S), D(S) \otimes \mathcal{O}_{X/S}(1), \dots, D(S) \otimes \mathcal{O}_{X/S}(k-1) \big\rangle$$

with $\mathcal{A}_0 = D(S)$. The universal hyperplane section of X fits into the diagram

$$\begin{array}{ccccc} \mathcal{X} & \subset & \mathbb{P}_S(E) \times \mathbb{P}(V^*) & = & \mathbb{P}_{S\times\mathbb{P}(V^*)}(E) \\ \downarrow & \searrow & & \swarrow{\scriptstyle \mathbb{P}^{k-1}} & \\ \mathbb{P}(V^*) & & S \times \mathbb{P}(V^*) & & \end{array}$$

For any point $(s, H) \in S \times \mathbb{P}(V^*)$ the fiber $\mathcal{X}_{s,H}$ in $\mathbb{P}(E_s)$ consists of those lines of E_s which are mapped by $\psi_s\colon E_s \to V$ to the hyperplane $H \subset V$. In other words, it is precisely the vanishing set in $\mathbb{P}(E_s)$ of $\phi_s(H) \subset E_s^*$. Thus there are two possibities: if $\psi_s(H)$ is a line in E_s^*, then $\mathcal{X}_{s,H}$ is the corresponding hyperplane in $\mathbb{P}(E_s)$. On the other hand, if $\phi_s(H) = 0$ then $\mathcal{X}_{s,H}$ is the whole of $\mathbb{P}(E_s)$.

Let $E^\perp$ be the kernel of ϕ. For any $s \in S$, restricting to s the short exact sequence

$$0 \to E^\perp \hookrightarrow \mathcal{O}_S \otimes V^* \xrightarrow{\phi} E^* \to 0$$

we deduce that $\phi_s(H) = 0 \Leftrightarrow H \subset E_s^\perp$. So the locus of the points in $S \times \mathbb{P}(V^*) \simeq \mathbb{P}_S(\mathcal{O}_S \otimes V^*)$ where $\mathcal{X}_{s,H}$ is the whole of $\mathbb{P}(E_s)$ is precisely $\mathbb{P}_S(E^\perp)$. Thus, the picture is

$$\begin{array}{ccc} \mathcal{X} & \hookleftarrow & \mathbb{P}_S(E) \times_S \mathbb{P}_S(E^\perp) \\ {\scriptstyle \text{generically } \mathbb{P}^{k-2}} \downarrow & & \downarrow {\scriptstyle \mathbb{P}^{k-1}} \\ S \times \mathbb{P}(V^*) & \hookleftarrow & \mathbb{P}_S(E^\perp) \end{array}$$

In particular, for $k = 2$ the map $\mathcal{X} \to S \times \mathbb{P}(V^*)$ is simply the blow-up of $\mathbb{P}_S(E^\perp)$.

We have the semiorthogonal decomposition

$$D(\mathcal{X}) = \big\langle \mathcal{C}, D(S \times \mathbb{P}(V^*)) \otimes \mathcal{O}(1), \ldots D(S \times \mathbb{P}(V^*)) \otimes \mathcal{O}(k-1) \big\rangle.$$

Away from $\mathbb{P}_S(E^\perp)$, the variety $\mathcal{X}$ is an $\mathbb{P}^{k-2}$-fiber bundle, so the semiorthogonal collection

$$\big\langle D(S \times \mathbb{P}(V^*)) \otimes \mathcal{O}(1), \ldots D(S \times \mathbb{P}(V^*)) \otimes \mathcal{O}(k-1) \big\rangle$$

generates everything there. In other words, the restriction of $\mathcal{C}$ away from $\mathbb{P}_S(E^\perp)$ is 0. On the other hand, over $\mathbb{P}_S(E^\perp)$ we've a $\mathbb{P}^{k-1}$-fiber bundle, so the restriction of $\mathcal{C}$ to $\mathbb{P}_S(E^\perp)$ is $D(\mathbb{P}_S(E^\perp))$. It can be shown that, indeed, $\mathcal{C} = D(\mathbb{P}_S(E^\perp))$ and $Y = \mathbb{P}_S(E^\perp)$.

2′). Suppose we have a inclusion $f\colon S \hookrightarrow \mathbb{P}(V)$. Set the vector bundle E in the previous example to be $f^*\mathcal{O}_{\mathbb{P}(V^*)}(-1)$, then $X = \mathbb{P}_S(E) = S$. The Orlov's Lefschetz decomposition is the stupid Lefschetz decomposition $\mathcal{A}_0 = D(X)$. We've

$$E^\perp = \mathrm{Ker}(V^* \otimes \mathcal{O}_X \to \mathcal{O}_X(1)) = \Omega_{\mathbb{P}(V)}(1)|_X$$

and by the previous example $\mathcal{X} \to S \times \mathbb{P}(V^*)$ is 0 outside $\mathbb{P}_S(E^\perp)$, while over $\mathbb{P}_S(E^\perp)$ it is an isomorphism. So $\mathcal{X} = \mathbb{P}_S(E^\perp)$ is a homological projective dual of X over itself.

2″). Given vector spaces A and B we can consider $X = \mathbb{P}(A) \times \mathbb{P}(B)$ over $S = \mathbb{P}(A)$. Write X as $\mathbb{P}_{\mathbb{P}(A)}(E)$ for $B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1)$. Take, as usual, the Lefschetz decomposition with $\mathcal{A}_0 = D(\mathbb{P}(A))$ with respect to $\mathcal{O}_{X/\mathbb{P}(A)}(1)$.

The embedding

$$\psi\colon B \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \hookrightarrow B \otimes A \otimes \mathcal{O}_{\mathbb{P}(A)}$$

defines the embedding

$$f\colon \mathbb{P}(B) \times \mathbb{P}(A) \hookrightarrow \mathbb{P}(B \otimes A) \times \mathbb{P}(A).$$

We have

$$E^\perp = \mathrm{Ker}(B^* \otimes A^* \otimes \mathcal{O}_{\mathbb{P}(A)} \to B^* \otimes \mathcal{O}_{\mathbb{P}(A)}(1)) = B^* \otimes \Omega_{\mathbb{P}(A)}(1)$$

and so $Y = \mathbb{P}_{\mathbb{P}_A}(B^* \otimes \Omega_{\mathbb{P}(A)}(1))$ is a homological projective dual of $\langle X, f, \mathcal{A}_\bullet \rangle$.

If A is two dimensional then $\Omega_{\mathbb{P}(A)}(1)$ is a line bundle and $Y = \mathbb{P}(A) \times \mathbb{P}(B^*) \cong \mathbb{P}(A^*) \times \mathbb{P}(B^*)$.

3). For $X = \mathbb{P}(W)$ let f be $\nu_2\colon \mathbb{P}(W) \to \mathbb{P}(S^2W)$, the second Veronese embedding, and take the Lefschetz decomposition with $\mathcal{A}_0 = \langle \mathcal{O}, \mathcal{O}(1) \rangle$. Then $\mathcal{X}$ is the universal quadric which fits into the diagram

$$\begin{array}{ccccc} X_Q & \subset & \mathcal{X} & \subset & \mathbb{P}(W) \times \mathbb{P}(S^2W^*) \\ \downarrow & & \downarrow & & \\ Q & \in & \mathbb{P}(S^2W^*) & & \end{array}$$

We get the semiorthogonal decomposition

$$D(\mathcal{X}) = \big\langle \mathcal{C}, D(\mathbb{P}(S^2W^*)) \otimes \mathcal{O}(2), \ldots, D(\mathbb{P}(S^2W^*)) \otimes \mathcal{O}(n-1) \big\rangle.$$

By a result of Kapranov, if Q is a smooth quadric then $\mathcal{C}_Q = \langle S \rangle$ or $\mathcal{C}_Q = \langle S_+, S_- \rangle$. To understand what happens for a singular Q we recall the definition of a Clifford algebra. Given a quadratic form $q \in S^2W^*$ we can define

$$\mathrm{Cl}(W, q) = T(W)/(w \otimes w' + w' \otimes w = 2\,\langle w, w' \rangle\, 1),$$

where $T(W)$ is the free tensor algebra on W and $\langle w, w'\rangle$ is the symmetric bilinear form $\frac{1}{2}(q(w+w') - q(w) - q(w'))$. Algebra $\mathrm{Cl}(W,q)$ is $\Lambda^\bullet W$ as a vector space but it has a different multiplication. Because the defining relation is not homogeneous Cl is not $\mathbb{Z}$ -graded. However, it has $\mathbb{Z}/2$ grading, $\mathrm{Cl} = \mathrm{Cl}^0 \oplus \mathrm{Cl}^1$; Cl^0 is a subalgebra and Cl^1 is a Cl^0-submodule with a bilinear map $\mathrm{Cl}^1 \otimes \mathrm{Cl}^1 \to \mathrm{Cl}^0$.

Then for any $Q \in \mathbb{P}(S^2W^*)$ we've $\mathcal{C}_Q \cong D(\mathrm{Cl}^0(W,q)$ for any quadratic form $q \in S^2W^*$ which defines Q. If the bilinear form associated to q is non-degenerate then $D(\mathrm{Cl}^0)$ is a matrix algebra or a product of two matrix algebras. It follows that $\mathcal{C} \cong D(\mathbb{P}(S^2W^*), \underline{\mathrm{Cl}^0})$, so the HPD of X is a noncommutative variety $Y = (\mathbb{P}(S^2W^*), \underline{\mathrm{Cl}^0})$ for a sheaf of Clifford algebras $\underline{\mathrm{Cl}^0}$.

3′). For the third Veronese embedding $\nu_3\colon \mathbb{P}(W) \to \mathbb{P}(S^3W)$ the result depends on $n = \dim W$. For $n = 3$ we get the stupid decomposition. For $n = 4$ the dual is $Y \to \mathbb{P}(S^3W^*)$ with a generic fiber being a finite set of points. For $n = 6$ we get a fibration in noncommutative K3 surfaces.

For the final example, let $X = \mathrm{Gr}(2,W)$ and let $f\colon X \to \mathbb{P}(\Lambda^2W)$ be the Plücker embedding. We assume that $\mathrm{char}(\mathbf{k}) = 0$ and denote by $\mathcal{U}$ the tautological vector bundle on X.

Kapranov constructed a full exceptional collection on X with $\binom{n}{2}$ elements, $\{\Sigma^\alpha\mathcal{U}^*\}$, where α is Young diagram that fits into a rectangle 2 cells tall and $n-2$ cells wide. The order on the collection is the same as the inclusion order on Young diagrams.

Adding a column on the left to a diagram α is twists $\Sigma^\alpha\mathcal{U}^*$ by $\mathcal{O}(1)$. Hence the Kapranov's collection is

$$\langle \mathcal{O}, \mathcal{U}^*, \dots, S^{n-2}(\mathcal{U}^*), \mathcal{O}(1), \mathcal{U}^*(1), \dots, S^{n-3}(\mathcal{U}^*)(1), \dots, \mathcal{O}(n-2)\rangle.$$

So X has a Lefschetz decomposition:

$$\begin{aligned} \mathcal{A}_0 &= \langle \mathcal{O}, \mathcal{U}^*, \dots, S^{n-3}(\mathcal{U}^*), S^{n-2}(\mathcal{U}^*)\rangle, \\ \mathcal{A}_1 &= \langle \mathcal{O}, \mathcal{U}^*, \dots, S^{n-3}(\mathcal{U}^*)\rangle \\ &\dots \\ \mathcal{A}_{n-2} &= \langle \mathcal{O}\rangle \end{aligned}$$

In particular, $|\mathcal{A}_0| = n-1$, $|\mathcal{A}_1| = n-2$, etc.

There is also a smaller Lefchetz decomposition with $\mathcal{A}_0 = \left\langle \mathcal{O}, \mathcal{U}^*, \dots, S^{\lfloor \frac{n}{2}\rfloor - 1}(\mathcal{U}^*)\right\rangle$.

If $n = 2m+1$ this decomposition is rectangular with $\mathcal{A}_0 = \ldots = \mathcal{A}_{2m} = \langle \mathcal{O}, \mathcal{U}^*, \dots, S^{m-1}\mathcal{U}^*\rangle$.

If $n = 2m$ then $\mathcal{A}_0 = \ldots = \mathcal{A}_{m-1} = \langle \mathcal{O}, \mathcal{U}^*, \dots, S^{m-1}(\mathcal{U}^*)\rangle$ and $\mathcal{A}_m = \ldots = \mathcal{A}_{2m-1} = \langle \mathcal{O}, \mathcal{U}^*, \dots, S^{m-2}(\mathcal{U}^*)\rangle$.

For other $\mathrm{Gr}(k,W)$ similar Lefschetz decompositions were constructed by Fonarev.

The homological projective dual to $\mathrm{Gr}(2,W)$ must be a variety Y and $g\colon Y \to \mathbb{P}(\Lambda^2W^*)$ such that $\mathrm{Crit}(g)$ is

$$X^\vee = \mathrm{Pf}(W^*) = \left\{\lambda \in \mathbb{P}(\Lambda^2W^*) \mid \mathrm{rk}(\lambda) \le 2(\left\lfloor \frac{n}{2}\right\rfloor - 1)\right\}.$$

For $n = \dim(W) \le 5$ the Pffafian variety $\mathrm{Pf}(W^*)$ itself is the HPD of X. For $n > 5$, however, $\mathrm{Pf}(W^*)$ becomes singular. If we define

$$\mathrm{Pf}_t(W^*) = \left\{\lambda \in \mathbb{P}(\Lambda^2W^*) \mid \mathrm{rk}(\lambda) \le 2t\right\}$$

then the singular locus of $\mathrm{Pf}(W^*)$ is $\mathrm{Pf}_{\lfloor \frac{n}{2}\rfloor - 2}(W^*)$.

Denote by $G = \mathrm{Gr}(2(\lfloor \frac{n}{2}\rfloor - 1), W^*)$ and by K the tautological bundle on G. We can construct a resolution

$$\begin{array}{ccc} \mathbb{P}_G(\Lambda^2K) & = & \left\{(\lambda, P) \in \mathbb{P}(\Lambda^2W^*) \times G \mid \mathrm{Im}(\lambda) \subset P\right\} \\ \big\downarrow & & \\ \mathrm{Pf}(W^*). & & \end{array}$$

We check the rank of the Grothendieck group to see whether $\mathbb{P}_G(\Lambda^2K)$ can be the HPD of X. The Lefschetz decomposition $\mathcal{A}_\bullet$ of X consists of $\binom{n}{2}$ objects in a $\lfloor \frac{n}{2}\rfloor$ cells high and $\binom{n}{2} = \dim(\Lambda^2W^*)$ cells wide rectangle. Therefore the expected rank of $K_0(Y)$ is $(\lfloor \frac{n}{2}\rfloor - 1)\binom{n}{2}$. Thus

- For $n = 6$ the expected rank of $K_0(Y)$ is 30.
- For $n = 7$ the expected rank of $K_0(Y)$ is 42.

On the other hand

$$\mathrm{rk}\left(K_0\left(\mathbb{P}_G(\Lambda^2 K)\right)\right) = \mathrm{rk}(K_0(G)) \times \mathrm{rk}\left(K_0(\text{fiber})\right) = \mathrm{rk}(K_0(G)) \times \mathrm{rk}(\Lambda^2 K).$$

For $n = 6, 7$ the fiber is $\mathbb{P}^5$. Thus, we see that

- For $n = 6$ the rank of $K_0\left(\mathbb{P}_G(\Lambda^2 K)\right)$ is 90.
- For $n = 7$ the rank of $K_0\left(\mathbb{P}_G(\Lambda^2 K)\right)$ is 210.

We can construct a noncommutative resolution if there exists a relative Lefschetz decomposition of D over Z:

Lemma 3.1. *Suppose there exists a Z-linear Lefschetz decomposition*

$$D(D) = \langle \mathcal{D}_{j-1}(1-j), \ldots, \mathcal{D}_1(-1), \mathcal{D}_0 \rangle$$

Then:

1. *i_* is fully faithful on $\mathcal{D}_k$ for $k \geq 1$,*
2. *$i_*(\mathcal{D}_{j-1}(1-j)), \ldots, i_*(\mathcal{D}_1(-1))$ are semi-orthogonal,*
3. *$D(\widetilde{Y}) = \langle i_*(\mathcal{D}_{j-1}(1-j)), \ldots, i_*(\mathcal{D}_1(-1)), \mathcal{C} \rangle$ where $\mathcal{C} = \{F \in D(\tilde{Y}) \mid i^* F \in \mathcal{D}_0\}$.*
4. *Suppose, additionally, that $\mathcal{D}_0$ is the Karoubi completion of $\langle i^* E \otimes p^* D(Z) \rangle$ where E is a vector bundle on $\tilde{Y}$. Suppose also that E is tilting over Y, i.e. $\mathbf{R}\pi_* \mathcal{E}nd(E)$ is a single sheaf of algebras on Y. Then*
$$\mathcal{C} \cong D(Y, \pi_* \mathcal{E}nd(E)).$$

Let $n = 7$. Then $Z = \mathrm{Gr}(2, 7)$ and D is a fiber bundle over Z with fiber $\mathrm{Gr}(2, 5)$. There exists a rectangular Lefschetz decomposition of $D = F(2, 4; W^*)$ with

$$\mathcal{D}_0 = \ldots = \mathcal{D}_4 = \langle D(Z), D(Z) \otimes S^* \rangle$$

where S is the quotient of 4-dimensional tautological vector bundle by the 2-dimensional tautological vector bundle on $F(2, 4; W^*)$. Therefore

$$\mathrm{rk}\left(K_0(\mathcal{C})\right) = \mathrm{rk}\, K_0(\widetilde{Y}) - \left(\mathrm{rk}\, K_0(\mathrm{Gr}(2, 5)) - 2\right) \times \mathrm{rk}\left(K_0(Z)\right) = 210 - 168 = 42$$

and $(Y, \pi^*(\mathrm{End}(\mathcal{O}_{\tilde{Y}} \oplus \mathcal{K}^*)))$ is the noncommutative homological projective dual to $\mathrm{Gr}(2, 7)$. Here $\mathcal{K}$ is a certain bundle on $\tilde{Y}$ which restricts to S on D. Analogously for $n = 6$.

Conjecture 3.2. *For $n > 7$ the homological projective dual of $\mathrm{Gr}(2, W)$ is an appropriate noncommutative resolution of the Pffafian variety $\mathrm{Pf}(W^*)$.*

NB: For $n > 7$ the exceptional fibre is much more complicated.

By the work of Hori Homological Projective Duality is related to non-linear sigma models. From string theory it follows that an appropriate noncommutative resolution of $\mathrm{Pf}_k(W)$ should be homological projective dual of a noncommutative resolution of $\mathrm{Pf}_{\lfloor \frac{n}{2} \rfloor - k}(W^*)$. These resolutions are known for $\mathrm{Pf}_2(8) \leftrightarrow \mathrm{Pf}_2(8)$ and $\mathrm{Pf}_2(9) \leftrightarrow \mathrm{Pf}_2(9)$. However, the proof of the duality is not known.

We have already seen that $\Sigma_1 = \mathbb{P}(W) \subset \mathbb{P}(S^2 W)$ has the homological projective dual $Y = (\mathbb{P}(S^2 W^*), \underline{Cl_0})$. We conjecture that the homological projective dual of $(\Sigma_k W, \underline{Cl_0})$ is $(\Sigma_{n+1-k} W^*, \underline{Cl_0})$.

Other know examples of homological projective duals are

- $\mathrm{OGr}(5, 10) \leftrightarrow \mathrm{OGr}(5, 10)$,
- $\mathrm{LGr}(3, 6) \leftrightarrow Q_4 \subset \mathbb{P}^{13}$ – a twisted noncommutative resolution of the quartic hypersurface,
- $G_2\mathrm{Gr} \leftrightarrow$ a twisted noncommutative resolution of $Y \xrightarrow{2:1} \mathbb{P}^{13}$ ramified in a sextic,
- $\mathrm{Gr}(3, 6) \leftrightarrow$ a twisted noncommutative resolution of $Y \xrightarrow{2:1} \mathbb{P}^{19}$.