Algebra of the infrared and Fukaya-Seidel categories with coefficients in perverse schobers

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Introduction

Aim of the talk is to give an introduction to the recent joint work with Misha Kapranov and Lev Soukhanov "Perverse schobers and the algebra of the infrared", arXiv:2011.00845. It develops further an earlier paper of Kapranov, Kontsevich and myself "Algebra of the infrared and secondary polytopes", arXiv:1408.2673. Recent paper has two different motivations: one from $2d$ QFT (physics) and another one from symplectic topology (mathematics). From the physics perspective: we categorify the algebra of the infrared of Gaiotto-Moore-Witten which underlies the IR limits of $2d, \mathcal{N} = (2,2)$ theories. From the mathematics perspective: we propose a generalization of the notion of Fukaya-Seidel category to the case of perverse schobers. Perverse schobers categorify perverse sheaves. For applications to Fukaya-Seidel categories we will need perverse sheaves and their categorification on the complex line $\mathbb{C}$ only.

Conceptual approach to perverse schobers on general real surfaces will be developed in the joint project with Dyckerhoff, Kapranov and Schechtman (first paper has already appeared: "Perverse sheaves on Riemann surfaces as Milnor sheaves", arXiv:2012.11388). For the purposes of this talk it suffices to use the elementary approach proposed by Kapranov and Schechtman in their very first paper on perverse schobers arXiv:1411.2772.

Plan.

1.Motivations.

2. Reminder on the algebra of the infrared and perverse sheaves.

3. Categorification of 2 with applications to FS categories.

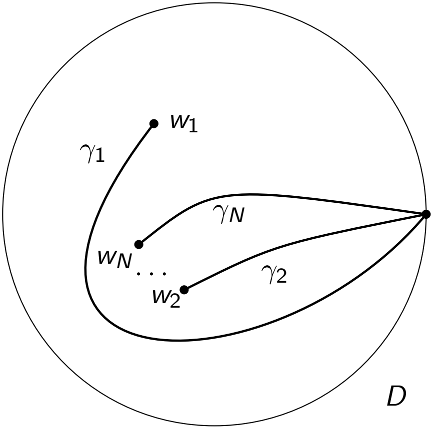
Motivations

Physics: for massive 2-dimensional theories with $(2,2)$ supersymmetry, the set A of vacua is discrete and embedded into the complex plane parametrizing charges of the supersymmetry algebra: $A = \{w_1, \cdots, w_N\} \subset \mathbb{C}$. Each vacuum has its category of D-branes, so we have have triangulated categories $\Phi_1, \cdots, \Phi_N$. The tunnelling between the vacua gives rise to *transport functors*

$$M_{ij} : \Phi_i \longrightarrow \Phi_j.$$

Mathematics: Landau-Ginzburg model (=Fukaya-Seidel category $\mathcal{FS}(X, W)$) gives rise to similar data. Then Φ_i are "local" FS categories generated by "vanishing thimbles", at least for Morse potential W. Transport functors M_{ij} are induced by symplectic parallel transports of vanishing Lagrangian spheres along e.g. straight segments $[w_i, w_j]$. Notice that alternatively we can choose a generic point ∞, and take symplectic transport of the Lagrangian spheres to this point first. A priori the answer depends on the choice of paths to infinity which form a "spider" graph. But the triangulated category $\mathcal{FS}(X, W)$ is independent of choices.

Example of the spider graph



Comments

Let X be a complex Kähler manifold, $dim_{\mathbb{C}} X = n$, and $W : X \to \mathbb{C}$ an analytic function (potential) with isolated Morse critical points $Crit(W) := \{x_1, ..., x_k\} \subset X$ and pairwise distinct critical values $A := \{w_1, ..., w_k\}$. Assume that the symplectic structure on X is exact: $\omega = d\alpha$, and there exists a nowhere vanishing holomorphic volume form $\Omega^{n,0}$ (Calabi-Yau structure).

For each $1 \leqslant j \leqslant k$ and a small circle of radius ε centered at w_j we have a local system $\mathcal{F}_j$ of the Fukaya categories with fibers which are Fukaya categories $= \mathcal{F}(W^{-1}(w_j + \varepsilon \cdot e^{i\theta}))$. Each fiber category is endowed with an autoequivalence coming from the action of the monodromy. In a similar way one defines a local system $\mathcal{FS}_j$ of Fukaya-Seidel categories, consisting of Lagrangian submanifolds in $W^{-1}(|w - w_j| < \varepsilon)$ with boundaries on $W^{-1}(w_j + \varepsilon \cdot e^{i\theta})$. Choice of paths γ_i gives rise to the point on the small circle about w_i, and Φ_i is the fiber of $\mathcal{FS}_j$ at this point. Transport functors M_{ij} in this model appear when we move the category Φ_i at w_i to "infinity" using symplectic connection and then move the resulting category to w_j.

Local Fukaya categories

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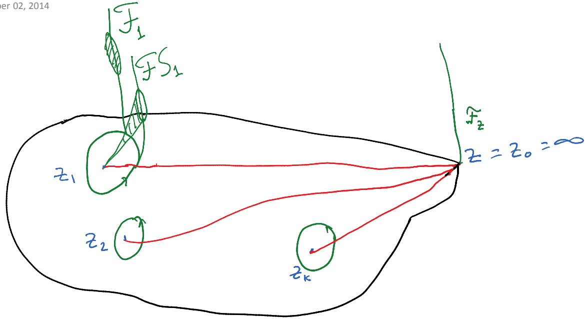


Analogy with perverse sheaves

The categories Φ_i are analogs ("categorifications") of the *spaces of vanishing cycles* of a perverse sheaf $\mathcal{F}$ on $\mathbb{C}$ at the singular points w_i. The idea of viewing $\mathcal{F}$ in terms of the transport data ("quiver") (Φ_i, M_{ij}) goes back to the 1996 paper of S. Gelfand, R. MacPherson and K. Vilonen on the classification of perverse sheaves. It is remarkable that this type of data appears in physics as "tunnelling data" between vacua. So the shortest summary of our paper would be: GMW=categorification of GMV. The categorification of GMV leads naturally to the notion of perverse schober on $\mathbb{C}$ with singularities at $A = \{w_i\}$.

Role of Fourier transform

The approach to Fukaya-Seidel categories (=Landau-Ginzburg models) proposed by GMW involves straight intervals. Their arguments come from physics. Mathematically this means that we look at perverse sheaves and subsequently perverse schobers *with the intent of making the Fourier transform*. More precisely, in the physical tunnelling picture, it is the rectilinear intervals $[w_i, w_j]$ that play a distinguished role, so the transport happens along such intervals. But the approach of Gelfand-MacPherson-Vilonen involves the transport along curved paths passing through some common faraway point ∞ (referring to the geometry of Russia we call it "Vladivostok").

Relevance of the Fourier transform for Landau-Ginzburg theory associated with a holomorphic function $W : Y \to \mathbb{C}$ can be also seen from the perspective of the corresponding oscillatory integrals. A typical oscillatory integral

$$I(\hbar) = \int_{\Gamma \subset Y} e^{\frac{i}{\hbar} W(y)} \Omega(y) dy,$$

can be seen as the result of first integrating over the fibers of W and then taking the single variable Fourier transform. Mathematically, the construction of the Fukaya-Seidel category with its dependence on the direction to infinity can be seen as a categorification (involving perverse schobers rather than sheaves) of the Geometric Fourier Transform of Laumon which produces a local system on the dual complex plane with the origin removed. Perverse sheaves correspond to D-modules with regular singularities. The Fourier transform still has regular singularities on $\mathbb{C}$ (hence defines a perverse sheaf), but it has a higher order pole at the origin (or at infinity, depending on conventions). One can describe the corresponding Stokes data in terms of the transport maps.

Picard-Lefsctetz identities

An important part in the physics story is played by the wall-crossing functors constructed by GMW. In fact they are built into the very structure of a perverse schober. Just as for a perverse sheaf we have abstract *Picard-Lefschetz identities* describing the change in the curvilinear transport with the variation of the path, for a perverse schober, we have *Picard-Lefschetz triangles* which are exact triangles in appropriate triangulated categories of functors. In fact, it is the arrows in the Picard-Lefschetz triangles which give the most important ingredient of the Algebra of the Infrared, namely, the already mentioned Maurer-Cartan element.

Algebra of the infrared and FS category

It was observed by GMW and made mathematically precise in our paper with Kapranov and Kontsevich that the FS category can be alternatively described as the triangulated envelope of the category of A_∞-modules over a certain A_∞-algebra.
The idea of construction is the following. First one constructs an L_∞-algebra $\mathfrak{g}$ generated roughly speaking by triangulations of the polygon $Conv(\{w_i\})$. Then one constructs an A_∞-algebra R in terms of convex polygons with vertices at $Conv(\{w_i\})$ and a point at infinity in a given direction. There is a morphism of L_∞-algebras $\mathfrak{g} \to C^\bullet(R, R)$ (Hochschild cochain complex of R). The desired A_∞ structure on R is defined as a perturbation of the initial one. The latter depends only on the set of triangualtions of $Conv(\{w_i\})$ (more precisely, on the structure of the secondary polytope of this convex hull), while in order to define the former one has to specify a Maurer-Cartan element in $\mathfrak{g}$.

The Maurer-Cartan element can be described either analytically in terms of the moduli spaces of solutions to certain non-linear equations (ζ-instanton equations of GMW) or algebraically in terms of a collection of Picard-Lefschetz relations (wall-crossing formulas). It is not clear how to generalize analytic approach to the case of schobers. In the paper with Kapranov and Soukhanov we used the second approach. Instead of the A_∞-algebra in the category of vector spaces we have an algebra in a certain dg-category of functors.

Next several slides are devoted to a short reminder of the above-mentioned L_∞ and A_∞ algebras from our paper with Kapranov and Kontsevich (it gives the IR description of the conventional FS category). Conventional FS category corresponds to the Lefschetz perverse schober associated with W (fiber of the Lefschetz schober at $z \in \mathbb{C} - \{w_i\}$ is the Fukaya category of $W^{-1}(z)$). These several slides are not necessary for understanding perverse sheaves and perverse schobers, but it helps to understand what kind of structure we would like to categorify in order to obtain the IR description of the FS category with coefficients in a perverse schober.

Reminder on L_∞-algebras

L_∞-algebras are also known as *homotopy Lie algebras*. Such an algebra $\mathfrak{g}$ is a $\mathbb{Z}$-graded vector space, i.e. $\mathfrak{g} = \oplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ endowed with “higher Lie brackets"

$$\lambda_n : \wedge^n \mathfrak{g} \to \mathfrak{g}[2-n],$$

(everything is understood in $\mathbb{Z}$-graded sense) satisfying an infinite system of quadratic relations. In case if only λ_2 is non-trivial, we get usual Lie algebras. In case if λ_1, λ_2 are non-trivial we get a DGLA with $\lambda_1 = d$ being a differential. In these two cases the Jacobi identity for λ_2 is satisfied. In general it is satisfied up to higher homotopies only, hence the name.
Equivalently, such an L_∞-structure on $\mathfrak{g}$ is given by a differential Q on the free commutative graded algebra $Sym(\mathfrak{g}^*[-1])$ (in fact it should be completed).

Reminder about A_∞-algebras and Hochschild cochain complex

Definition of the notion of A_∞-algebra is similar, but this time we use free tensor algebras without symmetrization. Then for an A_∞-algebra R we have "higher products"

$$m_n : R^{\otimes n} \to R[2-n]$$

satisfying quadratic relations. Usual associative algebras correspond to the case when only m_2 is non-trivial. DGAs correspond to the case when m_1, m_2 are non-trivial with $m_1 = d$ being the differential. For an A_∞-algebra R one can define its Hochschild cochain complex $C^\bullet(R,R) = \oplus_{n \geqslant 1} Hom(R^{\otimes n}, R)$ as a complex or even as a DGLA (after a shift of the grading).

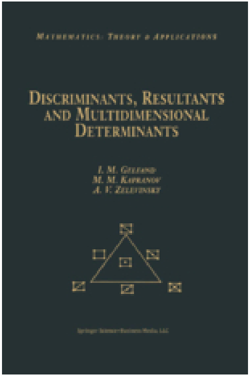
A_∞-categories

The notion of A_∞-algebra can be upgraded to the notion of A_∞-category. Such categories appear naturally as "categories of branes" in physics. L_∞-algebras naturally appear in deformation theory of A_∞-algebras (or categories). The reason for that is the fact that shifted Hochschild complex $C^\bullet(R,R)[1]$ is a DGLA, and in the deformation theory one can replace the controlling DGLA by any quasi-isomorphic L_∞-algebra.

Both L_∞ and A_∞ structures have natural geometric interpretation in terms of formal graded manifolds (non-commutative in A_∞ case) endowed with a "homological" vector field Q of degree $+1$ such that $[Q,Q]=0$. (see my joint paper with Kontsevich "Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I", arXiv:math/0606241)

Polytopes and marked polytopes

Standard reference for the foundational material is the classical book by I. Gelfand, M. Kapranov, A. Zelevinsky "Discriminants, Resultants and Multidimensional Determinants".



Let $A \subset \mathbb{R}^d$ be a finite set, $Q = Conv(A)$ be the convex hull (we will need only $d = 2$). Assume that affine span of A is the whole space, so Q is a d-dimensional convex polytope. If we want to remember the set A we will call the pair (Q, A) marked polytope. Then we can speak about marked subpolytopes $(Q', A') \subset (Q, A)$ as well as about polyhedral subdivisions $(Q, A) = \cup_\nu (Q_\nu, A_\nu)$ (here Q has to be the union of Q_ν, but this is not required for A and A_ν). A triangulation is a polyhedral subdivision into d-simplicies such that the intersection of two simplices is a common face (it can be empty).

Example

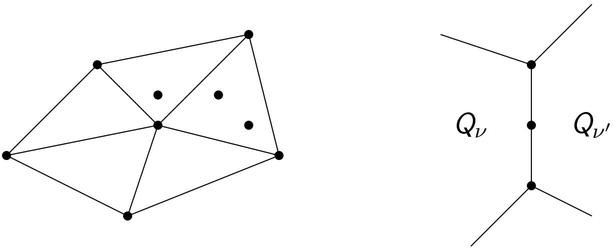


Figure: A triangulation and a polyhedral subdivision.

Secondary polytopes

Recall that a triangulation T of (Q, A) is called *regular* if if there is a continuous convex function $f : Q \to \mathbb{R}$ such that:

- f is affine-linear on each simplex of T.
- f is not affine-linear on any subset of Q which is not contained in a simplex of T. In other words, f breaks along each codimension 1 simplex which is a common face of two different d-dimensional simplices of T.

We denote by $\Sigma(A)$ the *secondary polytope* of A. Thus $\Sigma(A)$ is a convex polytope (it can be realized as a subset in $\mathbb{R}^A$, the space of all functions $A \to \mathbb{R}$) whose vertices φ_T are in bijection with regular triangulations T of (Q, A). It has the remarkable *factorization property*: faces of $\Sigma(A)$ correspond to regular subdivisions in such a way that the face F_P corresponding to a regular subdivision $P = (Q_\nu, A_\nu)$, has the form $F_P = \prod_\nu \Sigma(A_\nu)$.

The L_∞-algebra

To a polytope P we associate its chain complex over any ground field $\mathbf{k}$:

$$C_\bullet(P) = \oplus_{F \subset P} or(F)[dim\, F],$$

where $or(F) = H_c^{dim\, F}(int(F), \mathbf{k})$ is the *orientation line* of the face F. If we take $P = \Sigma(A)$, this gives us the graded vector space

$$V := \oplus_{A' \subset A, |A'| \geqslant d+1} or(\Sigma(A'))[dim\, \Sigma(A')].$$

Notice that if $\mathbf{k} = \mathbb{R}$ then $or(\Sigma(A')) = \bigwedge^{max}(\mathbb{R}^{A'}/Aff(\mathbb{R}^d))$.

Proposition

The graded vector space $\mathfrak{g} := \mathfrak{g}_A = V^*[-1]$ *carries a natural* L_∞*-structure. Moreover, it is nilpotent, i.e.* $\lambda_n = 0$ *for sufficiently large* n.

Comments on the Proposition

Suppose that A is affinely generic (i.e. each subset of $d+1$ elements generates a d-simplex). Then:
1. $\mathfrak{g}^{\leqslant 0} = 0$.
2. $\mathfrak{g}^1$ is spanned by d-simplices, i.e. by marked subpolytopes (Q', A') such that $|A'| = d+1$.
3. $\mathfrak{g}^2$ corresponds to *circuits*, which are subsets $A' \subset A$ such that $|A'| = d+2$ and $\Sigma(A')$ is an interval.
The differential and higher brackets λ_n roughly correspond to composing a bigger convex polytope from smaller ones. They come from the "big" differential Q on $Sym(V)$ which can be derived from a (compatible with each other) chain differentials on all $C_\bullet(\Sigma(A'))$. The differential Q makes $Sym(V)$ into a DGA, which implies the Proposition.

Example: four points in the convex position

Let $d = 2$, and let A consists of 4 points, forming the vertices of a convex 4-gon. Then $\mathfrak{g}^1$ has dimension 4, with the basis vectors corresponding to the 4 triangles a, b, c, d. The space $\mathfrak{g}^2$ is 1-dimensional, with the basis vector corresponding to the 4-gon itself, while $\mathfrak{g}^{\geqslant 3} = 0$. Thus is has five basis vectors: e_a, e_b, e_c, e_d of degree 1 and e_A of degree 2, with the only non-zero brackets being

$$[e_a, e_b] = [e_c, e_d] = e_A.$$

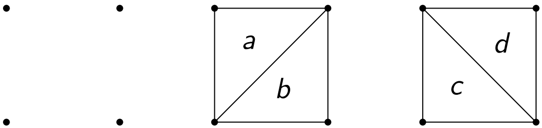


Figure: Four points in convex position

If we add to the story a half-plane containing the set A, the orthogonal direction to the boundary line specified by a non-zero complex number ζ defines a point "at infinity". Then we can play the same game as above, by allowing one vertex of the convex polygon to be at infinity. In this way we arrive to an A_∞-algebra $R_{A,\zeta}$. Main observation is that the L_∞-algebra $\mathfrak{g}_A$ acts on $R_{A,\zeta}$. Next slide contains the figure illustrating this action. After that we will continue with perverse sheaves and their categorification.

Figure of infinite polytopes

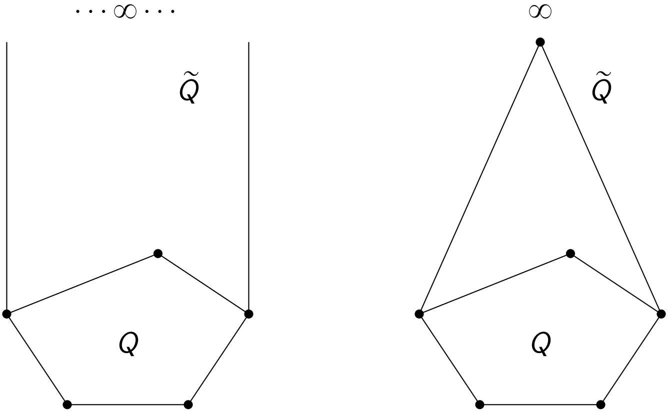


Figure: ∞ as a point at the projective infinity vs. as a finite point far away.

Review of perverse sheaves on complex line

Elementary approach to schobers is a categorification of the quiver description of perverse sheaves on a disc. Then we will obtain a categorification of the set of relations which follow from basic Picard-Lefschetz triangles for perverse sheaves. The collection of these relations for perverse sheaves can be thought of as the “baby algebra of the infrared".

Perverse sheaves on the disk

Let $X = D = \{|z| \leqslant 1\}$ be the unit disk in $\mathbb{C}$ and $A = \{0\}$. We recall the classical description of the category $\mathrm{Perv}(D, 0)$ over any ground field $\mathbf{k}$ of characteristic zero.
$\mathrm{Perv}(D, 0)$ is equivalent to the category $\mathcal{P}$ of diagrams

$$a : \Phi \to \Psi, b : \Psi \to \Phi$$

of $\mathbf{k}$-vector spaces are linear maps such that $T_\Psi := \mathrm{Id}_\Psi - ab$ is an isomorphism (equivalently, $T_\Phi = \mathrm{Id}_\Phi - ba$ is an isomorphism). This can be generalized to the case of several singular points. Then we have several a_i, b_i, T_i. This gives the quiver description of the category $Perv(D, A)$ of perverse sheaves on the disc with singularities at the finite subset $A \subset D$.

Perverse sheaves with several singular points

Let $X = D$ be a unit disc and $A = \{w_1, \cdots, w_N\}$ a collection of distinct points in X. For any $w \in A$ we have the natural *circle of directions*

$$S^1_w = \big(T_w X - \{0\}\big)/\mathbb{R}^*_{>0}.$$

Let $\mathcal{F} \in \mathrm{Perv}(X, A)$. For any $i = 1, \cdots, N$ we can restrict $\mathcal{F}$ to a small disk around w_i and associate to it local systems $\Phi_i = \Phi_{w_i}(\mathcal{F})$, $\Psi_i = \Psi_{w_i}(\mathcal{F})$ on $S^1_{w_i}$.

Figure: The transport map.

Transport maps

The stalks

$$\Phi_{i,\alpha} = \underline{\mathbb{H}}^1_\alpha(\mathcal{F})_{w_i}, \quad \Phi_{j,\alpha} = \underline{\mathbb{H}}^1_\alpha(\mathcal{F})_{w_j}$$

are just the stalks of the local systems Φ_i, Φ_j at the tangent directions $\mathrm{dir}_i(\alpha)$ and $\mathrm{dir}_j(\alpha)$ respectively. The stalk

$$\Psi_\alpha = \Gamma(\alpha - \{w_i, w_j\}, \underline{\mathbb{H}}^1_\alpha(\mathcal{F}))$$

is identified with the stalk $\mathcal{F}_p$ at any intermediate point $p \in \alpha$. Note that this identification depends on the chosen orientations of X and α which give a *co-orientation* of α, i.e., a particular numeration of the 2-element set of "sides" of $\Sigma - \alpha$ near p (change of either orientation incurs a minus sign in the identification). These spaces are connected by the maps

$$\Phi_{i,\alpha} \underset{b_{i,\alpha}}{\overset{a_{i,\alpha}}{\rightleftarrows}} \Psi_\alpha \underset{b_{j,\alpha}}{\overset{a_{j,\alpha}}{\leftrightarrows}} \Phi_{j,\alpha},$$

obtained from the description of $\mathcal{F}$ on small disks near w_i and w_j. We define the *transport map* along α as $m_{ij}(\alpha) = b_{j,\alpha} \circ a_{i,\alpha} : \Phi_{i,\alpha} \longrightarrow \Phi_{j,\alpha}$.

Picard-Lefschetz formulas

We now describe what happens when a path crosses a marked point. That is, we consider a situation as in Figure where a path γ' from w_i to w_k approaches the composite path formed by β from w_i to w_j and α from w_j to w_k. After crossing w_j, the path γ' is changed to γ.

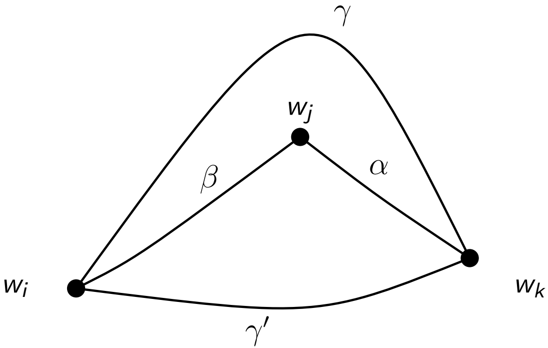


Figure: The Picard-Lefschetz situation.

Proposition (Abstract Picard-Lefschetz identity)

We have the equality of linear operators $\Phi_i \to \Phi_k$:

$$m_{ik}(\gamma') = m_{ik}(\gamma) - m_{jk}(\alpha) m_{ij}(\beta).$$

Fourier transform of perverse sheaves

Let $A = \{w_1, \cdots, w_N\} \subset \mathbb{C}$. Let us assume that A is in linearly general position including infinity. For any $w_i \in A$, we identify $S^1_{w_i}$, the circle of directions at w_i, with the unit circle $S^1 = \{|\zeta| = 1\} \subset \mathbb{C}$. Let us denote by

$$\zeta_{ij} = \frac{w_i - w_j}{|w_i - w_j|} \in S^1 \tag{0.4}$$

the slope of the intervals $[w_i, w_j]$.

Let $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A)$. We denote by $\Phi_i(\mathcal{F})$ the stalk of the local system $\Phi_i(\mathcal{F})$ at the horizontal direction $1 \in S^1_{w_i} \simeq S^1$. For any $i \neq j$ we define the *rectilinear transport map* $m_{ij} = m_{ij}(\mathcal{F}) : \Phi_i(\mathcal{F}) \to \Phi_j(\mathcal{F})$ as the composition

$$\Phi_i(\mathcal{F}) = \Phi_i(\mathcal{F})_1 \xrightarrow{T_{1,\zeta_{ji}}} \Phi_i(\mathcal{F})_{\zeta_{ji}} \xrightarrow{m_{ij}([i,j])} \Phi_j(\mathcal{F})_{\zeta_{ij}} \xrightarrow{T_{\zeta_{ji},1}} \Phi_j(\mathcal{F})_1 = \Phi_j(\mathcal{F}).$$

Here $m_{ij}([w_i, w_j])$ is the transport map along the rectilinear interval $[w_i, w_j]$, $T_{1,\zeta_{ji}}$, resp. $T_{\zeta_{ij},1}$ is the monodromy from 1 to ζ_{ji}, resp. from ζ_{ij} to 1, taken in the counterclockwise direction, if $\mathrm{Im}(w_i) < \mathrm{Im}(w_j)$, and in the clockwise direction, if $\mathrm{Im}(w_i) > \mathrm{Im}(w_j)$.

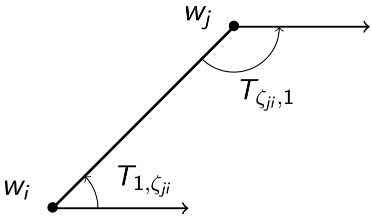


Figure: Rectilinear transport.

We also define

$$m_{ii} = b_i a_i : \Phi_i(\mathcal{F}) \longrightarrow \Phi_i(\mathcal{F}),$$

where a_i and b_i are the maps in the standard (Φ, Ψ)-diagram $\Phi_i(\mathcal{F}) \underset{b_i}{\overset{a_i}{\rightleftarrows}} \Psi_i(\mathcal{F})$, representing $\mathcal{F}$ near w_i in the horizontal direction.

Thus the data $(\Phi_i(\mathcal{F}), m_{ij})$ form an object of the category $\mathcal{M}_N$, whose objects are diagrams consisting of

a) Vector spaces Φ_i, $i = 1, \cdots, N$.

b) Linear operators $m_{ij} : \Phi_i \to \Phi_j$ given for all i, j and such that $\mathrm{Id}_{\Phi_i} - m_{ii}$ is invertible, and we have the *functor of rectilinear transport data*

$\Phi^{\|} = \Phi^{\|}_A : \overline{\mathrm{Perv}}(\mathbb{C}, A) \longrightarrow \mathcal{M}_N, \quad \mathcal{F} \mapsto (\Phi_i(\mathcal{F}), m_{ij}).$

Here the bar denotes the quotient by the subcategory of local systems.

Proposition

The functor $\Phi^{\|}$ is an equivalence of categories.

Baby infrared relations

Let us describe the Fourier transform of the (Φ, Ψ)-diagram $\check{\Phi} \underset{\check{b}}{\overset{\check{a}}{\rightleftarrows}} \check{\Psi}$, of the Fourier transform $\check{\mathcal{F}}$ of a perverse sheaf $\mathcal{F} \in \mathrm{Perv}(\mathbb{C}, A)$. One can show that $\check{\mathcal{F}} \in \mathrm{Perv}(\mathbb{C}, 0)$ and

$$\check{\Psi} \;=\; \Psi(\check{\mathcal{F}}) \;:=\; \bigoplus_{i=1}^{N} \Phi_i(\mathcal{F}), \qquad \check{\Phi} \;=\; \Phi_0(\check{\mathcal{F}}) \;:=\; \Psi(\mathcal{F}),$$

$$\check{b} = \sum_{i=1}^{N} a_i, \quad \check{a} = (\check{a}_1, \cdots, \check{a}_N), \quad \check{a}_i \;=\; b_i\, T_{i-1,\Phi} \ldots T_{1,\Phi}.$$

In terms of transport maps (this is incarnation of the infrared algebra) the same maps look as follows.

Proposition

We have

$$\check{b} = \sum_i a_i, \check{a}_i = b_i + \sum_{k>1} (-1)^{k-1} \sum_{j_1 < \ldots < j_k = i} m_{j_{k-1}, i}\, m_{j_{k-2}, j_{k-1}} \cdots m_{j_1, j_2}\, b_{j_1},$$

where $m_{ij} : \Phi_i \to \Phi_j$ is the rectilinear transport map for $\mathcal{F}$, and $(j_1 < \cdots < j_k = j)$ run over sequences such that $w_{j_1}, \cdots, w_{j_k} = w_i$ is a left convex path.

The term “left convex path" is explained on the next slide.

We say that a sequence $b_1, \cdots, b_l$ of complex numbers is a *ζ-convex path*, if the intervals $[b_1, b_2]$, $[b_2, b_3], \cdots, [b_{l-1}, b_l]$ are successive edges of the polygon

$\mathrm{Conv}^\zeta\{b_1, \cdots, b_l\} := \mathrm{Conv}\left(\bigcup_{b_i}(b_i + \zeta \cdot \mathbb{R}_+)\right)$. Left convex path corresponds to $\zeta = 1$.

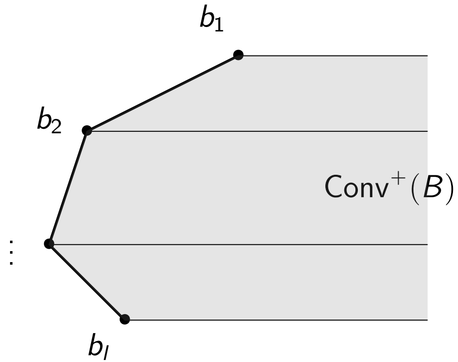


Figure: A left convex path.

Categorification: schobers on $(D, 0)$ and spherical functors.

Let Φ, Ψ be (dg-enhanced) triangulated categories and $a : \Phi \to \Psi$ be an exact (dg-) functor. Assume that a has left and right adjoints ${}^*a, a^* : \Psi \to \Phi$ with the corresponding unit and counit maps

$$e : \mathrm{Id}_\Psi \longrightarrow a \circ ({}^*a), \quad \eta : ({}^*a) \circ a \longrightarrow \mathrm{Id}_\Phi,$$
$$e' : \mathrm{Id}_\Phi \longrightarrow (a^*) \circ a, \quad \eta' : a \circ a^* \longrightarrow \mathrm{Id}_\Psi.$$

Using the dg-ehnancement, we define the functors

$$T_\Psi = \mathrm{Cone}(e)[-1], \quad T_\Phi = \mathrm{Cone}(\eta),$$

so that we have exact triangles of functors

$$T_\Psi \xrightarrow{\lambda} \mathrm{Id}_\Psi \xrightarrow{e} a \circ ({}^*a) \xrightarrow{f} T_\Psi[1], \quad T_\Phi[-1] \xrightarrow{g} ({}^*a) \circ a \xrightarrow{\eta} \mathrm{Id}_\Phi \to T_\Phi.$$

Proposition (Spherical functor package)

In the following list, any two properties imply the two others:

(i) T_Ψ *is an equivalence (quasi-equivalence of dg-categories).*

(ii) T_Φ *is an equivalence.*

(iii) *The composite map*

$$T_\Phi \circ a^*[-1] \xrightarrow{g \circ a^*} {}^*a \circ a \circ a^* \xrightarrow{{}^*a \circ \eta'} {}^*a$$

is an isomorphism (quasi-isomorphism of dg-functors).

(iv) *The composite map*

$${}^*a \xrightarrow{e' \circ ({}^*a)} (a^*) \circ a \circ ({}^*a) \xrightarrow{a^* \circ f} a^* \circ T_\Psi[1]$$

is an isomorphism. □

The functor a is called *spherical*, if the conditions of Proposition are satisfied. This is a categorification of the fact that the monodromy $T_\Phi = id_\Phi - ba$ was invertible in case of perverse sheaves.

Definition

A *perverse schober* $\mathfrak{S}$ on $(D, 0)$ is a datum of a spherical functor $a : \Phi \to \Psi$ between enhanced triangulated categories.

This definition can be naturally generalized to the case of several singular points. Then one has Ψ and several Φ_i's with the spherical functors associated with each pair a_i, b_i. Let me recall this definition after Kapranov-Schechtman in the case of arbitrary surfaces. As I have already mentioned this definition will be sufficient for the purposes of this talk.

Definition of perverse schober

Let $(X, A = \{w_1, \cdots, w_N\})$ be a stratified surface and $S^1_{w_i}$ be the circle of directions at w_i. Let D_i be a small disk around w_i and $D_i^\circ = D_i \backslash \{w_i\}$ be the punctured disk. Thus we have a homotopy equvalence $D_i^\circ \to S^1_{w_i}$ which is "homotopy canonical", i.e., defined uniquely up to a contractible space of choices.

Definition

A perverse schober on (X, A) is a datum consisting of:

- A local system $\mathfrak{S}_0 = \mathfrak{S}|_{X \backslash A}$ of pre-triangulated categories on $X \backslash A$.
- For each i, a spherical local system $\mathbf{a}_i : \Phi_i(\mathfrak{S}) \to \Psi_i(\mathfrak{S})$ on $S^1_{w_i}$ (or, equivalently, on D_i°).
- An identification $\mathfrak{S}_0|_{D_i^\circ} \simeq \Psi_i(\mathfrak{S})$ for each i.

Recall that Landau-Ginzburg model with the Morse potential W gives rise to a perverse schober, where Ψ_i is the Fukaya category of $W^{-1}(p), p \in S^1_{w_i}$ and Φ_i is the Fukaya-Seidel category generated by thimbles projected to $[w_i, p]$. Morse condition can be relaxed.

Let α be a piecewise smooth oriented path in X, joining w_i with w_j and avoiding other w_k. For a schober $\mathfrak{S} \in \mathrm{Schob}(X, A)$ we have the category $\Phi_{i,\alpha} = \Phi_i(\mathfrak{S})_{\mathrm{dir}_i(\alpha)}$ (the stalk of the local system $\Phi_i(\mathfrak{S})$ in the direction of α) and the similar category $\Phi_{j,\alpha}$. Denoting Ψ_α the stalk of the local system $\mathfrak{S}_0$ at any intermediate point of α, we have two spherical functors

$$\Phi_{i,\alpha} \xrightarrow{a_{i,\alpha}} \Psi_\alpha \xleftarrow{a_{j,\alpha}} \Phi_{j,\alpha}$$

and we define the *transport functor*

$$M_{ij}(\alpha) = {}^*a_{j,\alpha} \circ a_{i,\alpha} : \Phi_{i,\alpha} \longrightarrow \Phi_{j,\alpha}, \tag{0.10}$$

analogous to the transport map for perverse sheaves.

Let α^{-1} be the path obtained by reversing the direction of α, so α^{-1} goes from w_j to w_i. We have the transport functor $M_{ji}(\alpha^{-1}) = {}^*a_{i,\alpha} \circ a_{j,\alpha} : \Phi_{j,\alpha} \longrightarrow \Phi_{i,\alpha}$.

Proposition

We have identifications of functors

$$M_{ij}(\alpha) \simeq T_{\Phi_j} \circ M_{ji}(\alpha^{-1})^* [1] \simeq {}^*M_{ji}(\alpha^{-1}) \circ T_{\Phi_i}[-1].$$

Unitriangular monads

Definition

Let $\mathcal{V} = (\mathcal{V}_1, \cdots, \mathcal{V}_N)$ be a finite ordered sequence of pre-triangulated dg-categories.

A *unitriangular (dg-)monad* on $\mathcal{V}$ is a collection $M = (M_{ij} : \mathcal{V}_i \to \mathcal{V}_j)_{i<j}, M_{ii} = \mathrm{Id}_{\mathcal{V}_i}$ of dg-functors together with closed, degree 0 natural transformations, called *composition maps*

$$c_{ijk} : M_{jk} \circ M_{ij} \longrightarrow M_{ik}, \quad i < j < k,$$

such that for any $i < j < k < l$ the diagram below commutes (associativity condition):

$$\begin{tikzcd}
M_{kl} \circ M_{jk} \circ M_{ij} \arrow[r, "M_{kl} \circ c_{ijk}"] \arrow[d, "c_{jkl} \circ M_{ij}"'] & M_{kl} \circ M_{ik} \arrow[d, "c_{ikl}"] \\
M_{jk} \circ M_{ij} \arrow[r, "c_{ijl}"'] & M_{il}.
\end{tikzcd}$$

An M-algebra is a sequence of objects $V = (V_1, \cdots, V_N)$, $V_i \in \mathcal{V}_i$, together with closed, degree 0 morphisms *(action maps)*

$$\alpha_{ij} : M_{ij}(V_i) \longrightarrow V_j, \quad i < j,$$

such that for any $i < j < k$ the diagram below commutes:

$$\begin{tikzcd}
M_{jk}(M_{ij}(V_i)) \arrow[r, "M_{jk}(\alpha_{ij})"] \arrow[d, "c_{ijk,V_i}"'] & M_{jk}(V_j) \arrow[d, "\alpha_{jk}"] \\
M_{ik}(V_i) \arrow[r, "\alpha_{ik}"'] & V_k.
\end{tikzcd}$$

Let

$$\begin{tikzcd}
\Phi_1 \arrow[dr, "a_1"'] & \cdots & \Phi_N \arrow[dl, "a_N"] \\
& \Psi &
\end{tikzcd}$$

be a diagram of N spherical dg-functors with common target. It gives a uni-triangular dg-monad $M = M(a_1, \cdots, a_N) = \left(M_{ij} = a_j^* \circ a_i : \Phi_i \longrightarrow \Phi_j\right)_{i<j}$ on $\Phi := (\Phi_1, \cdots, \Phi_N)$. Compositions:

$$c_{ijk} : M_{jk} \circ M_{ij} = a_k^* \circ a_j \circ a_j^* \circ a_i \longrightarrow a_k^* \circ a_i = M_{ik}$$

coming from the counit $\eta_j : a_j \circ a_j^* \to \mathrm{Id}_{\Phi_j}$ by composing with a_k^* on the left and a_i on the right. In general a monad gives rise to the category of algebras over it. In our case it is a pre-triangulated category $\mathrm{Alg}_M = \langle \Phi_1, \cdots, \Phi_N \rangle$ with its standard semi-orthogonal decomposition.

In particular for $U \in \Psi$ the collection $b(U) = (a_1^*(U), \cdots, a_N^*(U))$ is naturally an M-algebra. The action map

$$\alpha_{ij} : M_{ij}(a_i^*(U)) \;=\; a_j^* \, a_i \, a_i^* \,(U) \longrightarrow a_j^*(U)$$

is obtained from the counit $a_i \circ a_i^* \to \mathrm{Id}$. This gives a dg-functor

$$b : \Psi \longrightarrow \mathrm{Alg}_M \tag{0.13}$$

which we call the *Barr-Beck functor*.

Fourier transform and Fukaya-Seidel category

We assume that A is in linearly general position and $\mathfrak{S} \in \mathrm{Schob}(\mathbb{C}, A)$. We distinguish the horizontal direction ($\zeta = 1$) in each circle $S^1_{w_i}$ and denote $\Phi_i(\mathfrak{S})$ the stalk of the local system $\Phi_i(\mathfrak{S})$ at $1 \in S^1_{w_i}$. For any distinct $i, j \in \{1, \cdots, N\}$ we have the *rectilinear transport functor*

$$M_{ij} \ := \ M_{ij}([w_i, w_j]) : \Phi_i(\mathfrak{S}) \longrightarrow \Phi_j(\mathfrak{S}),$$

Let now $\mathfrak{S}$ is represented by a diagram of spherical functors $\{a_i : \Phi_i \to \Psi\}$ with respect to a spider-like graph K with the head of the spider at infinity in the direction of $\zeta, |\zeta| = 1$. Define the Fourier transform $\check{\mathfrak{S}}$ to be represented by the single Barr-Beck spherical functor

$$b : \Psi \longrightarrow \mathbb{F}_K(\mathbb{C}, \zeta; \mathfrak{S}) \ := \ \mathrm{Alg}_{M_K}.$$

Study of the Fukaya-Seidel category $\mathbb{F}_K(\mathbb{C}, \zeta; \mathfrak{S})$ can be therefore seen as study of the Fourier transform for schobers. The notation Alg_{M_K} means the category of algebras over the uni-triangular monad associated with rectilinear M_{ij} as above.

L_∞-algebra associated with a schober

Let $A = \{w_1, \cdots, w_N\} \subset \mathbb{C}$ be in linearly general position and $Q = \mathrm{Conv}(A)$. By a *subpolygon* in Q we mean a set of the form $Q' = \mathrm{Conv}(A')$, where $A' \subset A$ has cardinality at least 3.
Let $Q' \subset Q$ be a subpolygon with p vertices which we denote $w_{i_1}, \cdots, w_{i_p}$ in the clockwise cyclic order, starting from some chosen vertiex w_{i_1}. The choice of w_{i_1} being not essential, we will consider $i_1, \cdots, i_p$ as a cyclically ordered set identified with $\mathbb{Z}/p$, in particular, write $i_{p+1} = i_1$ etc. Let now $\mathfrak{S} \in \mathrm{Schob}(\mathbb{C}, A)$ be a schober with singularities in A.

We define the *(dg-)space of intertwiners* of $\mathfrak{S}$ associated to Q' as

$$(0.14) \qquad \mathbb{I}(Q') \;=\; \mathrm{Hom}^\bullet(M_{i_1,i_p}, M_{i_{p-1},i_p} \cdots M_{i_1,i_2}).$$

One can prove that $\mathbb{I}(Q')$ does not depend, up to a canonical identification, on the choice of the initial vertex w_{i_1}.

Proposition

Let $i_r, i_{r+1}, \cdots, i_s$ be a cyclic interval in $\{i_1, \cdots, i_p\}$. Then we have a canonical identification

$$\mathbb{I}(Q') \;=\; \mathrm{Hom}^\bullet\Big(M_{i_{r+1},i_r} T^{-1}_{i_{r+1}} M_{i_{r+2},i_{r+1}} T^{-1}_{i_{r+2}} \cdots T^{-1}_{i_{s-1}} M_{i_s,i_{s-1}}, \\ M_{i_{r-1},i_r} M_{i_{r-2},i_{r-1}} \cdots M_{i_s,i_{s-1}}\Big)[-r+s+1].$$

In particular, for $r = s = 1$ (cyclic interval of length 0) we have

$$\mathbb{I}(Q') \;=\; \mathrm{Hom}^\bullet(T_{\Phi_1}, M_{i_p,i_1} M_{i_{p-1},i_p} \cdots M_{i_1,i_2})[1].$$

Assume that A is in sufficiently general position. For any subset $A' \subset A$ with $|A| \geqslant 3$ we define $Q' = \mathrm{Conv}(A')$ and $\mathbb{I}(A') = \mathbb{I}(Q')$.

The secondary polytope $\Sigma(A')$ has dimension $|A'| - 3$ and its faces $F_{\mathcal{P}}$ are labelled by regular polygonal decompositions $\mathcal{P} = (Q''_\nu, A''_\nu)$ of (Q, A). Further, an inclusion of faces $F_{\mathcal{P}'} \subset \overline{F}_{\mathcal{P}}$ means that we have a refinement relation $\mathcal{P}' \prec \mathcal{P}$ and so the map

$$\Pi_{\mathcal{P}',\mathcal{P}} : \mathbb{I}(\mathcal{P}') \longrightarrow \mathbb{I}(\mathcal{P}). \tag{0.16}$$

One shows that the maps $\Pi_{\mathcal{P}',\mathcal{P}}$ are transitive for triple refinement $\mathcal{P}'' \prec \mathcal{P}' \prec \mathcal{P}$ and so define a cellular complex of sheaves $\mathcal{N}_{A'}$ on $\Sigma(A')$ as on my slides from last time. That is, the stalk of $\mathcal{N}_{A'}$ on $F_{\mathcal{P}}$ is $\mathbb{I}(\mathcal{P})$ and the generalization map from $F_{\mathcal{P}'}$ to $F_{\mathcal{P}}$ is $\Pi_{\mathcal{P}',\mathcal{P}}$. The complex $\mathcal{N}_{A'}$ is factorizing, i.e., for any regular polygonal decomposition $\mathcal{P} = (Q''_\nu, A''_\nu)$ as above, the restriction of $\mathcal{N}_{A'}$ to $F_{\mathcal{P}} = \prod \Sigma(A''_\nu)$ is identified with $\boxtimes_\nu \mathcal{N}_{A''_\nu}$.
Let us now define $\mathfrak{g}_{A'} =$
$\mathbb{I}(A') \otimes \mathrm{or}(\Sigma(A'))[-\dim \Sigma(A') - 1], \quad \mathfrak{g}_{Q'} := \mathfrak{g}_{Q' \cap A}, \; Q' \subset (Q, A).$

Theorem

The differentials in the cellular cochain complexes of the $\mathcal{N}_{A'}$ unite to make the dg-vector space $\overset{\bullet}{\mathfrak{g}} = \bigoplus_{(Q',A')\subset(Q,A)} \mathfrak{g}_{A'}$ into a $\mathcal{L}ie_\infty$-algebra. The subspace $\mathfrak{g} = \bigoplus_{Q'\subset(Q,A)} \mathfrak{g}_{Q'}$ is a $\mathcal{L}ie_\infty$-subalgebra in $\overset{\bullet}{\mathfrak{g}}$. □

The second part of the Infrared Algebra formalism is an A_∞ algebra associated to a choice of a direction towards infinity. In our schober setting this will be not an algebra in the usual sense (i.e., living in the category of vector spaces) but an algebra in an appropriate category of functors, i.e., a monad similar to the one considered previously.

Let $\zeta \in \mathbb{C}$, $|\zeta| = 1$. Let us assume that no interval $[w_i, w_j]$ has direction ζ, i.e., none of the slopes ζ_{ij} from (0.4) is equal to ζ. Recall also the concept of ζ-convexity. It gives rise to the partial order $\leqslant_\zeta$ on the set A.

Let us number $A = \{w_1, \cdots, w_N\}$ according to this order, i.e., so that $w_i <_\zeta w_j$ for $i < j$.

We will be interested in ζ-convex polygons $P = \mathrm{Conv}^\zeta(B)$, $B \subset A$, $|B| \geqslant 2$. Such a polygon is unbounded and we can number its vertices $w_{i_1}, \cdots, w_{i_p}$ in clockwise direction so that the edges are

$$[\zeta\infty, w_{i_1}]\,[w_{i_1}, w_{i_2}], \cdots, [w_{i_p}, \zeta\infty], \quad i_1 < \cdots < i_p.$$

We denote $s(P) = w_{i_1},\ t(P) = w_{i_p}$ and call these vertices the *source* and *target* vertices of P. For $i < j$ let $\mathbb{P}(i,j)$ be the set of ζ-convex polygons P with $\mathrm{Vert}(P) \subset A$, $s(P) = w_i$ and $t(P) = w_j$. For $P \in \mathbb{P}(i,j)$ we consider the functor

$$\mathbb{M}(P) \;=\; M_{i_{p-1},i_p} \cdots M_{i_1,i_2} : \Phi_i \longrightarrow \Phi_j.$$

For any $i < j$ put

$$R_{ij} = R_{ij}^{\zeta} = \bigoplus_{P \in \mathbb{P}(i,j)} R_P, \quad R_P := \mathbb{M}(P)[1 - |P \cap A|].$$

Let $i < j < k$ and $P \in \mathbb{P}(j,k)$, $P' \in \mathbb{P}(i,j)$. If $P \cup P'$ is convex, then $P \cup P' \in \mathbb{P}(i,k)$, and the composition of functors gives rise to the map

$$c_{P,P'} : R_P \otimes R_{P'} \longrightarrow R_{P \cup P}.$$

Let us define the maps $c_{ijk} : R_{jk} \otimes R_{ij} \to R_{ik}$ by

$$c_{ijk}\big|_{R_P \otimes R_{P'}} = \begin{cases} c_{P,P'}, & \text{if } P \cup P' \text{ is convex;} \\ 0, & \text{otherwise.} \end{cases}$$

Since the c_{ijk} are given by composition of functors, they are associative. In other words, $R = R^{\zeta} = (R_{ij}, c_{ijk})$ is a unitriangular monad. This monad is a replacement, for the general schober situation, of the A_∞-algebra mentioned earlier. We now define the *deformation complex*, or *ordered Hochschild complex* of the monad R as

$$\overrightarrow{C}^{\bullet} = \overrightarrow{C}^{\geqslant 1}(R,R) \, = \, \Big\{ \bigoplus_{i<j<k} \mathrm{Hom}^{\bullet}(R_{jk}R_{ij}, R_{ik}) \to$$

$$\bigoplus_{i<j<k<l} \mathrm{Hom}^{\bullet}(R_{kl}R_{jk}R_{ij}, R_{il}) \to \cdots \Big\},$$

with the (horizontal) grading starting in degree 1.

Elements of the component

$$\overrightarrow{C}^p = \bigoplus_{i_0 < \cdots < i_{p+1}} \mathrm{Hom}^\bullet\left(R_{i_p, i_{p+1}} \cdots R_{i_0, i_1},\ R_{i_0, i_{p+1}}\right)$$

can be seen as $(p+1)$-ary "operations" having as inputs, the $p+1$ functors $R_{i_\nu, i_{n+1}}$ and as output, the functor $R_{i_0, i_{p+1}}$. Similarly to the case of the Hochschild of an associative (dg-)algebra, $\overrightarrow{C}^\bullet$ is a dg-Lie algebra with respect to the *Gerstenhaber bracket*

$$[f, g] = \sum_{\nu=1}^{p+1} f \circ_\nu g - (-1)^{\deg(f)\deg(g)} \sum_{\nu=1}^{q+1} g \circ_\nu f, \quad f \in \overrightarrow{C}^p,\ g \in \overrightarrow{C}^q.$$

Here $f \circ_\nu g$ is the "operadic composition", obtained by substituting the output of the operation g into the ν-th input of the operation f (whenever this makes sense, and defined to be zero otherwise).

As in the case with associative (dg-)algebras, a Maurer-Cartan element $\beta \in \overrightarrow{C}^1$ gives a unitriangular A_∞-deformation $R(\beta)$ of R. More precisely, β, being of total degree 1, consists of components

$$\beta^{(2)} = (\beta^{(2)}_{ijk}) \in \bigoplus_{i<j<k} \mathrm{Hom}^0(R_{jk}R_{ij}, R_{ik}), \quad \beta^{(3)} = (\beta^{(3)}_{ijkl}) \in \bigoplus_{i<j<k<l} \mathrm{Hom}^{-1}$$

The deformed monad $R(\beta)$ has:

(0) The same functors $R(\beta)_{ij} = R_{ij}$.

(1) The composition maps $c_{ijk}(\beta) : R_{jk}R_{ij} \to R_{ik}$ defined by $c_{ijk}(\beta) = c_{ijk} + \beta^{(2)}_{ijk}$.

(2) The $c_{ijk}(\beta)$ may not be strictly associative but the $\beta^{(3)}_{ijkl}$ define the homotopy for the associativity and so on to give a full A_∞-structure.

Then one can construct explicit morphism of the L_∞ algebra $\mathfrak{g}$ constructed above to the ordered Hochschild complex $\overrightarrow{C}^{\geqslant 1}$. Conjecturally, there is a Maurer-Cartan element which gives rise to the category equivalent to the previously defined FS category. Inn the case of Lefschetz schober all that is equivalent to GMW and our paper with Kapranov and Kontsevich.