How to calculate A-Hilb $\mathbb{C}^n$ for $\frac{1}{r}(a, b, 1, \dots, 1)$

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Abstract

Our first result says, in the coprime case, that the n-fold Gorenstein orbifold $\mathbb{C}^n/A$ where $A = \frac{1}{r}(a, b, 1, \dots, 1)$ (with $n-2$ repeats) has a crepant resolution if and only if the point nearest the $(x_1 = 0)$ face is the junior point $\frac{1}{r}(1, d, c, \dots, c)$ with $r = 1 + d + (n-2)c$, and the Hirzebruch–Jung continued fraction of $\frac{r}{d}$ has every entry congruent to 2 modulo $n-2$. A version of the Nakamura–Craw–Reid algorithm then calculates the A-Hilbert scheme, with some fun for large n. This paper overlaps with the first author's Warwick PhD thesis [D].

An interesting feature of the paper is a new method of mapping between the A-Hilbert schemes of diagonal groups whose toric treatment involves lattice subcones of one another, although the groups themselves may be completely unrelated. This gives new insight and new results even in the usual $\mathrm{SL}(3, \mathbb{C})$ case.

The website `www.warwick.ac.uk/staff/T.Logvinenko/Traps` contains addenda and other back-up material related to this paper.

Contents

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1 Introduction

Being an overview and description of the main results, without too many technical details, and sketching the layout of the paper.

For $n = 2$ or 3 the quotient $\mathbb{C}^n/G$ by a finite subgroup $G \subset \mathrm{SL}(n,\mathbb{C})$ has a crepant resolution – in fact by [BKR], the moduli construction G-Hilb $\mathbb{C}^n$ is a preferred choice. The question has been studied for $n \geq 4$, mostly inconclusively (we discuss this briefly in Section 6). The best answer seems to be that a crepant resolution of $\mathbb{C}^n/G$ sometimes exists, but often does not.

We treat here the case of restricted diagonal subgroups $A \subset \mathrm{SL}(n,\mathbb{C})$ defined in Section 2 below; these include the cyclic groups $A = \frac{1}{r}(a, b, 1^{n-2})$ with $r = a+b+n-2$. Our first result Theorem 2.5 is an elementary criterion for these to admit a crepant resolution; groups of this form have been studied previously by Dais, Haus and Henk [DHH], but our criterion is simpler and more elegant. Our study is motivated by the very interesting calculation of A-Hilb $\mathbb{C}^n$ in these cases, and the currently open problem of giving a useable moduli interpretation to a resolution or partial resolution.

1.1 Restricted groups and their junior simplex

A *restricted group* or *restricted diagonal subgroup* $A \subset \mathrm{SL}(n, \mathbb{C})$ is given by the usual toric recipe starting from an overlattice $L \supset \mathbb{Z}^n$ of finite index, of the form

$$\overline{L} = \mathbb{Z}^n \subset L = \mathbb{Z}^n + (\text{fractional part}) \subset L_{\mathbb{R}} = \mathbb{R}^n_{\langle x_{1\dots n}\rangle}, \tag{1.1}$$

with fractional part restricted by

$$x_3 \equiv x_4 \equiv \dots \equiv x_n \mod \mathbb{Z}. \tag{1.2}$$

It follows of course that the fractional part has ≤ 2 generators. One may think of the cyclic case $\frac{1}{r}(a, b, 1^{n-2})$ with $r = a + b + n - 2$ as fairly typical; in this case the group is the cyclic group $A = \boldsymbol{\mu}_r \hookrightarrow \mathrm{SL}(n, \mathbb{C})$ given by $\mathrm{diag}(\varepsilon^a, \varepsilon^b, \varepsilon, \dots, \varepsilon)$ for $\varepsilon \in \boldsymbol{\mu}_r$.

We write (x_1, x_2, x_3^{n-2}) for a point of $L_{\mathbb{R}} = \mathbb{R}^n$ with last $n-2$ coordinates equal. These form a copy of $\mathbb{R}^3 \subset \mathbb{R}^n$, called the *restricted subspace* $\mathbb{R}^3_r$; we write $L_r = \mathbb{R}^3_r \cap L$ for the *restricted lattice*. Restricted groups are treated chiefly in terms of the three dimensional lattice L_r and its restricted junior simplex Δ, as we now explain.

Recall that the *junior simplex* of $L_{\mathbb{R}} = \mathbb{R}^n$ is the $(n-1)$-simplex obtained as the convex hull $\langle e_1, \dots, e_n \rangle$ of the basis vectors. Equivalently, it is the intersection of the affine hyperplane $\sum x_i = 1$ with the first orthant of $L_{\mathbb{R}}$ (or with the unit cube of $\overline{L} = \mathbb{Z}^n$).

Lemma 1.1 *Every point of the junior simplex of $L_{\mathbb{R}}$ satisfying the restriction* (1.2) *is either one of the standard basis vectors e_i of $\overline{L}$, or is in the triangle $\Delta = \langle e_1, e_2, A' \rangle$, where $A' = \frac{1}{n-2}(0, 0, 1^{n-2})$.*

Definition 1.2 The triangle Δ is the *restricted junior simplex* of A. We write $L_{\mathrm{rj}} = \langle \Delta \rangle \cap L$; it is an affine lattice $L_{\mathrm{rj}} \cong \mathbb{Z}^2$, that we call the *restricted junior lattice.*

Proof The first orthant of $L_{\mathbb{R}}$ is defined by $x_i \geq 0$. If any of the x_i reaches 1, the junior condition $\sum x_i = 1$ implies that x is one of the standard basis vectors e_i. On the other hand, if all $0 \leq x_i < 1$, the restriction $x_i \equiv x_j \bmod \mathbb{Z}$ implies that $x_i = x_j$ for $i, j \geq 3$, so that $x \in \mathbb{R}^3_r$. Intersecting the junior simplex of $\mathbb{R}^n$ with $\mathbb{R}^3_r$ gives Δ. $\square$

Notice that A' is the intersection of Δ with the codimension 2 subspace $\mathbb{R}^{n-2} = \mathbb{R}e_3 + \cdots + \mathbb{R}e_n$ (given by $x_1 = x_2 = 0$), but is not necessarily a lattice point of L (see Corollary 2.2); we draw it as an empty ring to reiterate the point. A line through A' in Δ represents a hyperplane of $\mathbb{R}^n$ through $\mathbb{R}^{n-2}$.

We draw subdivisions of the triangle Δ into basic triangles partly out of habit. More importantly, however, a crepant resolution $Y \to \mathbb{C}^n/A$, if it exists, is given by a toric fan subdividing the first orthant of $L_{\mathbb{R}}$ using only junior lattice points. It turns out that this fan in $L_{\mathbb{R}}$ is mapped faithfully by a fan Σ subdividing Δ. More precisely, define a *crepant basic cone* of L to be a cone $\sigma = \langle v_1, \ldots, v_n \rangle$ in $L_{\mathbb{R}}$ generated by a $\mathbb{Z}$-basis of L consisting of junior lattice points v_i in the first orthant of L. Under the standard toric dictionary, such a cone σ corresponds to an affine space $\mathbb{C}^n$ having a crepant birational toric morphism to $\mathbb{C}^n/A$.

Lemma 1.3 *Any crepant basic cone $\sigma = \langle v_1, \ldots, v_n \rangle$ is of one of two kinds: up to reordering, either*

(i) the first three vertices $v_1, v_2, v_3 \in \Delta \cap L_{\mathrm{rj}}$ form a $\mathbb{Z}$-basis of the affine lattice L_{rj}, and

$$\{v_4, \ldots, v_n\} = \{e_3, \ldots, \widehat{e_i}, \ldots, e_n\} \quad \textit{for some } i \textit{ with } 3 \le i \le n.$$

Then $A' \notin \sigma$. Or

(ii) $v_1, v_2 \in \Delta \cap L_{\mathrm{rj}}$ are such that v_1, v_2, w base L_r, where $w = (0, 0, 1^{n-2})$, and $v_3, \ldots, v_n = e_3, \ldots, e_n$. In this case $A' \in \sigma \cap \Delta$.

For the proof, see the start of Section 2.

Definition 1.4 (link) Our main method, as in [CR], is to use Hirzebruch–Jung continued fractions around the three vertices e_1, e_2, A' of Δ. For each of these, we define its *link* in L as the sequence of vectors to the successive lattice points on the boundary of the Newton polygon, defined as the convex hull of the points of $\Delta \cap L_{\mathrm{rj}}$ (other than the vertex in question, obviously). See the start of Section 3 for detailed notation and a numerical example.

As we describe in Remark 2.3, if $A' \in L$, everything reduces to a finite diagonal group $B \subset \mathrm{SL}(3, \mathbb{C})$, and is completely understood. A recurring leitmotiv of this paper, and its main difference compared to [CR] is that working with the link of A' when $A' \notin L$ involves passing between *two different lattices* L and $L' = L + \mathbb{Z}A'$.

To discuss crepant resolutions of $\mathbb{C}^n/A$, we strengthen the assertion of Lemma 1.3, (ii) by restricting u, v to the link of A' (see Figure 1.1).

Definition 1.5 We define *internal* and *external triangles* of Δ as follows:

internal: a triangle $v_1v_2v_3$, with $v_1, v_2, v_3 \in \Delta$ a $\mathbb{Z}$-basis of the affine lattice L_{rj}; it corresponds to $n-2$ crepant basic cones in Lemma 1.3, (i).

external: a triangle $A'v_1v_2$ with v_1v_2 a primitive line interval of the boundary of the convex hull of $\Delta \cap L$ (with no internal lattice points). it corresponds to a single crepant basic cones in Lemma 1.3, (ii).

The following observation is the starting point for our conditions for the existence of crepant resolutions in Section 2.

Lemma 1.6 *A crepant basic cone appearing in a crepant resolution must be internal or external in the sense of Definition 1.5.*

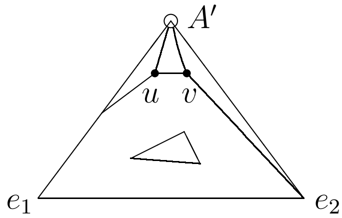

Figure 1.1: Every crepant basic cone appearing in a crepant resolution is *either* a basic triangle in the affine lattice $\Delta \cap L$ coned with $\langle e_3, \dots, \widehat{e_i}, \dots, e_n \rangle$, *or* a primitive boundary interval u, v of the convex hull of $\Delta \cap L$ coned with $\langle e_3, \dots, e_n \rangle$.

A lattice triangle in L_{rj} with no lattice points other than its vertices is automatically basic for L_{rj}, and gives rise to $n-2$ internal basic simplices. As everyone knows, a lattice polygon in the plane can be subdivided into basic triangles (usually in very many ways). Thus the existence of a crepant resolution of $\mathbb{C}^n/A$ reduces to the question of which external simplices are basic. As Figure 1.1 suggests, the problem only concerns the Newton polygon of lattice points around A' in Δ. Our first main result Theorem 2.5 gives several equivalent criteria for a crepant resolution to exist. The *coprime case* is the cyclic group $\frac{1}{r}(a,b,1^{n-2})$ with $r=a+b+n-2$ and a,b coprime to r. The result is then that a crepant resolution exists if and only if the point of L_{rj} nearest the $(x_1=0)$ face is a junior point $P_c=\frac{1}{r}(1,d,c^{n-2})$ (that is, $r=(n-2)c+d+1$), and every entry of the Hirzebruch–Jung continued fraction of $\frac{r}{d}$ is congruent to 2 modulo $n-2$.

1.2 New feature: the trap

The 2-generator subgroup $(\mathbb{Z}/r)^{\oplus 2}$ with fractional part

$$\tfrac{1}{r}(r-1,1,0^{n-2})\oplus\tfrac{1}{r}(r-n+2,0,1^{n-2}) \tag{1.3}$$

is the opposite extreme to the coprime case. It corresponds to the maximal r-torsion subgroup $\boldsymbol{\mu}_r\times\boldsymbol{\mu}_r\subset\mathrm{SL}(n,\mathbb{C})$ compatible with the restriction (1.2). It is the key new feature of A-Hilb $\mathbb{C}^n$ for restricted groups, and we call it a *trap* (for isosceles trapezium).

We write $r=(n-2)c+\overline{r}$ with $0\le\overline{r}<n-2$. The case $\overline{r}=0$ is rather trivial (see Remark 2.3): then $A'\in L$ and there are no external triangles, so that a crepant resolution exists, and everything reduces to taking cones over the known construction for $\mathrm{SL}(3,\mathbb{C})$.

Otherwise, the link of A' consists of the collinear points $\frac{1}{r}(i,\overline{r}-i,c^{n-2})$ for $i\in[0,\dots,\overline{r}]$, and our criterion says that a crepant resolution exists if and only if $\overline{r}=1$ (or 0, as just described). Thus the interesting case is $\overline{r}=1$. Then the fractional part of L has alternative generators

$$\tfrac{1}{r}(1,0,c^{n-2})\oplus\tfrac{1}{r}(0,1,c^{n-2}). \tag{1.4}$$

Now the existence of a crepant resolution is not in doubt. In contrast, A-Hilb $\mathbb{C}^n$ in this case is neither crepant nor (if $n\ge5$) a resolution. It is defined by a subdivision of the trap that seems somewhat exotic at first sight (see Figure 1.2). Section 5 treats this in detail; Figure 1.2 illustrates the case $n=6$, $c=3$ and $r=13$ as a brief foretaste. The figure is flanked on either side by a pseudoregular triangle of side c and the lower shelves consist of $n-3$ alleys of parallelograms interleaved with pseudoregular triangles of side $c-1$. The parallelograms are subdivided into 4, each with an age 2 point over the centre (that is, a divisor of discrepancy 1). These alleys just continue downwards harmlessly for larger values of c. At the top, the alleys converge into a foyer, where the fun really starts (compare Figure 5.1).

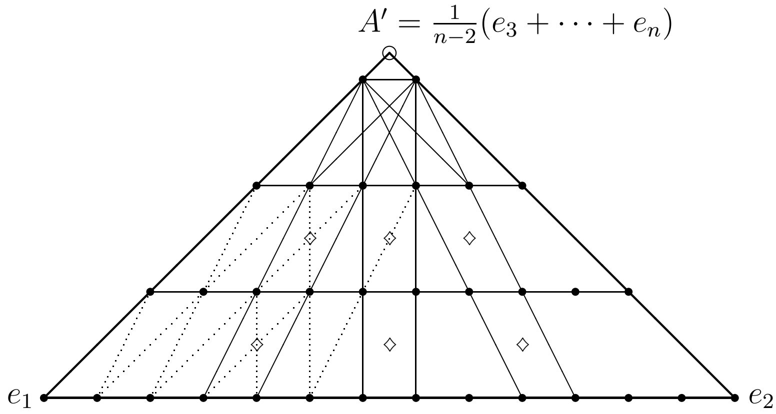

Figure 1.2: The trap constructing A-Hilb $\mathbb{C}^6$ for $A = \frac{1}{13}(12, 1, 0^4) \oplus \frac{1}{13}(9, 0, 1^4)$. Bullet points are junior lattice points; the midpoints $\diamond$ of the parallelograms have age 2. The intersection points over the foyer have age 2, 3 or 4.

1.3 The coarse subdivision

Our main result Theorem 3.3 states that for a restricted group A, if $\mathbb{C}^n$ admits a crepant resolution, an appropriate modification of the algorithm of Nakamura [Na] and Craw–Reid [CR] calculates A-Hilb $\mathbb{C}^n$. The construction consists first of a *coarse subdivision* of the restricted junior simplex Δ into blocks, either regular triangles or traps, based on a knock-out competition between lines out of the three vertices e_1, e_2 and A'; the lines and their relative strengths in the game are determined as in [CR] by appropriate use of Hirzebruch–Jung continued fractions. Figure 1.3 is an impressionistic sketch giving the overall idea. Then A-Hilb $\mathbb{C}^n$ is obtained by taking the regular tesselation of each regular triangle, and the corresponding A-Hilb decomposition of each trap (as in Figure 1.2 and Section 5).

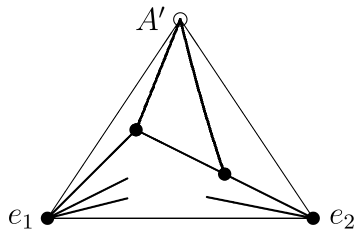

Figure 1.3: Coarse subdivision (sketch)

There are a couple of minor differences with the $\mathrm{SL}(3, \mathbb{C})$ case of [CR] that take some getting used to,[1] arising chiefly from the fact that A' in our figures is not a lattice point, but represents the codimension 2 axis $\mathbb{R}^{n-2}$; it appears weighted by $n-2$ in many calculations, counting for the $n-2$ variables $z_{3...n}$. A triangle bounded by two lines out of A' is a trap; when it is basic, it is an external triangle in the sense of Lemma 1.3, and is a trap with $c=0$. If we draw our figures to scale, these triangles appear to be smaller than the internal triangles in the ratio $1 : n-2$ (see triangle $A'P_{10}P_{11}$ in Figure 1.4 or the tiny cell at the top of Figure 1.2). Lines leading in to A' have $n-1$ as their final tag to indicate a straight line, rather than the familiar 2 from toric geometry.

Example 1.7 We draw the coarse subdivision for $n=5$, $A=\frac{1}{39}(4,32,1^3)$ as Figure 1.4. The figure is in the junior restricted plane except for A', and is drawn to scale. The point nearest the e_2A' side is $P_{10}=\frac{1}{39}(1,8,10^3)$; by Theorem 2.5, since all the entries of $\frac{39}{8}=[5,8]$ are $\equiv 2 \bmod n-3$, the quotient $\mathbb{C}^5/A$ has a crepant resolution.

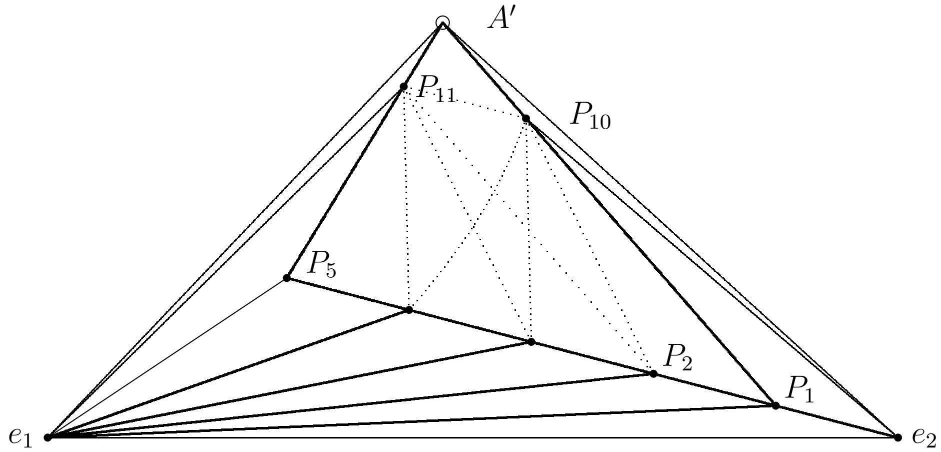

Figure 1.4: Coarse subdivision for $n=5$, $A=\frac{1}{39}(4,32,1^3)$

The junior lattice points are

$$P_1=(4,32,1^3),\ P_2=(8,25,2^3),\ P_3=(12,18,3^3),\ P_4=(16,11,4^3),$$
$$P_5=(20,4,5^3),\ P_{10}=(1,8,10^3),\ P_{11}=(5,1,11^3).$$

[1]This is explained to some extent by the discussion in Section 3, esp. Remark 3.1: internal triangles are viewed in L_{rj}, but external ones in L'_j. This footnote should be assimilated into the main text.

The continued fraction $\frac{39}{4} = [10, 4]$ documents the link of e_2 (the vectors to e_1, P_1, P_{10} and A'), with $10e_2P_1 = e_2e_1 + e_2P_{10}$ and $4e_2P_{10} = e_2P_1 + e_2A'$. In the same way, $\frac{39}{32} = [2, 2, 2, 2, 3, 4]$ describes the link of e_1. The continued fraction $\frac{39}{8} = [5, 8]$ describes the link of A'; namely the vectors $A'e_2$, $A'P_{10}$, $A'P_{11}$ and $A'e_1$ satisfy the tag relations

$$5A'P_{10} = A'e_2 + A'P_{11} \quad \text{and} \quad 8A'P_{11} = A'P_{10} + A'e_1 \tag{1.5}$$

Exercise 1.8 To get a better idea, we recommend that you try for yourself a numerical case such as $n = 5$, with Newton polygon around A' given by $[5, 2, 8]$, so that $A = \frac{1}{67}(4, 60, 1^3)$ (with the lattice points $\frac{1}{67}(1, 15, 17^3)$, $\frac{1}{67}(5, 8, 18^3)$ and $\frac{1}{67}(9, 1, 19^3)$ topping two adjacent traps), or $[8, 8]$ giving $A = \frac{1}{63}(1, 8, 18^3)$ and a trap with two shelves. These are fun and not too demanding – the hard part is typesetting the resulting figure. Corollary 2.6 says that essentially every case arises from this simple kind of trick.

1.4 The new stuff

We could in principle do the whole calculation of A-Hilb $\mathbb{C}^n$ by brute force, following Nakamura [Na] and Craw–Reid [CR]. This involves writing out

(i) the fan Σ of A-Hilb $\mathbb{C}^n$;

(ii) for every cone σ in Σ, the affine piece of A-Hilb $\mathbb{C}^n$ corresponding to it, together with the A-clusters it parametrises; and

(iii) every A-cluster a priori, to check that we have everything.

This is all perfectly feasible, and we have done it in enough numerical cases to know it works in general, but it is the unimaginative way to go, involving as it does a disproportionate volume of notation and calculations.

Instead, we proceed by a number of reduction steps, corresponding to making a subdivision of part of the restricted junior simplex and giving it a moduli interpretation. This introduces some new ideas that are interesting in their own right, and provides insight even in the known $\mathrm{SL}(3, \mathbb{C})$ cases. In particular, while [CR] makes substantial use of the coarse subdivision to calculate A-Hilb $\mathbb{C}^3$, there is no interpretation on the level of relations between the moduli functor for the group A and the group $(\mathbb{Z}/r)^2$ corresponding to a regular triangle in the coarse subdivision.

The picture for the convex hull around A' is a convex plane lattice polygon, and suggests at once relations with dimer models. But methods based on dimer models only apply up to now to subgroups of $\mathrm{SL}(3,\mathbb{C})$, whereas our problem area relates more closely to subgroups of $\mathrm{GL}(3,\mathbb{C})$ such as $\frac{1}{r}(a,b,1)$, where $r = n-2+a+b$ and $\frac{1}{r}(a,b,1^{n-2})$ has a crepant resolution.

1.5 Layout of the paper

Section 2 elaborates on the material of 1.1, and proves our first main result Theorem 2.5 containing the necessary and sufficient conditions for $\mathbb{C}^n/A$ to have a crepant resolution. 2.4 shows how to use the Hirzebruch–Jung form of this criterion to list all the cases to which our methods apply: they are the cases with $A' \in L$ that we view as trivial (see Remark 2.3), or in a family parametrised by Hirzebruch–Jung continued fractions $[a_1,\dots,a_k]$ with all $a_i = 2 \bmod n-2$. 2.5 discusses a certain reduction of $A\text{-Hilb}\,\mathbb{C}^n$ for a restricted group A in $\mathrm{SL}(n,\mathbb{C})$ to constructions involving groups $\overline{A}$ in $\mathrm{GL}(3,\mathbb{C})$ and $\overline{\overline{A}}$ in $\mathrm{GL}(2,\mathbb{C})$ (typically, in the coprime case, $\frac{1}{r}(a,b,1)$ and $\frac{1}{r}(a,b)$).

Section 3 describes our knock-out game, following 1.3 and [CR]. It contains our main result Theorem 3.3: for a restricted group A admitting a crepant resolution, the knock-out game computes $A\text{-Hilb}\,\mathbb{C}^n$.

Section 4 is concerned with computing $A\text{-Hilb}\,\mathbb{C}^n$ by reduction steps that cut up the functor $A\text{-Hilb}\,\mathbb{C}^n$ according to parts of the fan, as mentioned in 1.4.

Section 5 computes $A\text{-Hilb}\,\mathbb{C}^n$ for an individual trap.

Section 6 contains some final remarks on the significance of our results and their possible generalisations.

2 Crepant resolution for restricted groups

2.1 Crepant basic cones and external triangles

We complete the material of 1.1, starting with the proofs of Lemma 1.3 and Lemma 1.6.

Proof Let $\sigma = \langle v_1,\dots,v_n\rangle$ be any crepant basic cone. By Lemma 1.1 each generator v_i is one of $e_3,\dots,e_n$ or lies in L_r, and L_r has rank 3 so at most three of the v_i are in L_r. Therefore, σ is based either by $n-3$ elements $e_3, \dots, \widehat{e_i}, \dots, e_n$ together with $u, v, w \in L_{\mathrm{rj}}$, or by all $n-2$ basis vectors $e_3, \dots, e_n$ together with $u, v \in L_{\mathrm{rj}}$.

Our restriction (1.2) on the fractional part of L is equivalent to

$$L = L_r \oplus \langle e_3, \dots, \widehat{e_i}, \dots, e_n \rangle \quad \text{for any } i = 3, \dots, n; \tag{2.1}$$

for, given $x = (x_i) \in L$, we can change all the x_j for $i \neq j$ by integers to make them equal to x_i. Therefore $u, v, w \in L_r$ form a $\mathbb{Z}$-basis of L_r if and only if together with $e_3, \dots, \widehat{e_i}, \dots, e_n$ they base L. This gives Lemma 1.3.

For the proof of Lemma 1.6, suppose that Σ is a fan of crepant basic cones, and let $\sigma = \langle A', u, v, e_3, \dots, e_n \rangle$ be a cone of Σ with A', u, v as in Lemma 1.3, (ii). The assertion of Lemma 1.6 is that $u, v \in L$ must be successive boundary points of the Newton polygon of $L_r j \cap \Delta$.

If not, we may suppose that v (say) is internal to the Newton polygon; then the vector $A'v$ must cross the Newton boundary, say between u_j and u_{j+1} with $j \geq i$. The triangle $u_j u_{j+1} v$ has vertices in L_{rj}; it cannot be covered by cones in Σ by definition of fan, since $A'v$ is a side of $A'uv$. This implies that at least one of the cones of Σ involved in covering $u_j u_{j+1} v$ is not basic, which contradicts the assumption on Σ.

We now treat the key question left hanging in 1.1: which external triangles give crepant basic cones? For this, write $w = \sum_{i=3}^n e_i = (0, 0, 1^{n-2}) \in L_r$.

Lemma 2.1 *Two vectors $u, v \in L_r$ together with $e_3, \dots, e_n$ form a $\mathbb{Z}$-basis of L if and only if u, v, w is a $\mathbb{Z}$-basis of L_r.*

Proof Suppose first that $u, v, e_3, \dots, e_n$ is a $\mathbb{Z}$-basis of L. Then some integral linear combination $w_0 = \sum_{i=3}^n m_i e_i$ with $m_i \in \mathbb{Z}$ completes u, v to a $\mathbb{Z}$-basis of L_r. The converse also holds. For $w_0 \in L_r$ implies that all the m_i are equal, and for u, v, w_0 to be a $\mathbb{Z}$-basis of L_r, this common multiple can only be ± 1. (For this, use (2.1) and write e_i as a combination of u, v, w_0, $e_3, \dots, \widehat{e_i}, \dots, e_n$.) $\square$

A crepant resolution $Y \to \mathbb{C}^n/A$ subdivides the positive orthant of $\mathbb{R}^n$ as a fan of crepant basic cones (see 1.1). By Lemma 1.3 and Figure 1.1, any crepant basic cone cuts the restricted junior simplex Δ in an internal or external basic triangle.

Corollary 2.2 *Suppose that a crepant resolution $Y \to \mathbb{C}^n/A$ exists and write Σ for the corresponding fan in Δ. Then one of the following two cases hold:*

(I) Σ has an external triangle uvA' with $u, v \in \Delta$. Then $w = (0, 0, 1^{n-2})$ is primitive in L.

(II) Σ has no external triangle. Then $A' = \frac{1}{n-2}(0, 0, 1^{n-2}) \in L$.

Proof (I) is already implicit in Lemma 2.1: if u, v, w is part of a basis of L then w is primitive. For (II), points near A' must be covered by some triangle, necessarily internal. The only way this can happen is that $A' \in L$. $\square$

Remark 2.3 We view the case $A' = \frac{1}{n-2}(0, 0, 1^{n-2}) \in L$ as trivial. There are no external triangles, so a crepant resolution is automatic. Moreover, Δ in Figure 1.1 is a lattice triangle, so everything reduces to a diagonal subgroup of $\mathrm{SL}(3, \mathbb{C})$ followed by a bit of coning over it. More precisely, in this case the fractional part of L consists of vectors $\frac{1}{r}(a, b, 1^{n-2})$ with r, a, b all divisible by $n-2$; the reduced group in $\mathrm{SL}(3, \mathbb{C})$ has fractional part $\frac{1}{r'}(a', b', 1)$, where we write $r = (n-2)r'$, $a = (n-2)a'$ and $b = (n-2)b'$ for each point of L. We can calculate A-Hilb $\mathbb{C}^3$ by [CR], and then prove that A-Hilb $\mathbb{C}^n$ is a crepant resolution of $\mathbb{C}^n/A$.

2.2 The main criterion

We build up our main result Theorem 2.5 in steps, according to the quantity of notation involved. It is clear from what we have already said that for a restricted group A, the following two conditions are equivalent:

(1) There exists a crepant resolution.

(2) Every primitive boundary interval uv of the Newton polygon of $L \cap \Delta$ around A' in Figure 1.1 has $\{u, v, (0, 0, 1^{n-2})\}$ a $\mathbb{Z}$-basis of L_r. (In other words, by Lemma 2.1, every external triangle uvA' corresponds to a crepant basic cone.)

Remark 2.4 In Figure 1.1, each internal triangle of Δ yields $n-2$ crepant basic cones of $L_{\mathbb{R}}$; while the basic triangulation clearly exists, it is far from unique. By contrast, each external triangle corresponds to at most one crepant basic cone of $L_{\mathbb{R}}$; its existence is the main issue, but it is unique if it exists.

Our next trick needs one further preliminary, a *Tale of Two Lattices*. The occurrence in our arguments of the point $A' = \frac{1}{n-2}(0,0,1^{n-2}) \in \Delta$ hints at the fact that the restricted space $\mathbb{R}^3_r$ introduced in 1.1 hosts two different lattices, the restricted lattice $L_r = L \cap \mathbb{R}^3_r$ and the slightly bigger *projected lattice* $L' = \alpha(L)$, the image of L under the averaging map

$$\alpha\colon L_{\mathbb{R}} = \mathbb{R}^n \to \mathbb{R}^3 \quad \text{given by} \quad (x_1,\dots,x_n) \mapsto \Big(x_1, x_2, \tfrac{1}{n-2}\sum\nolimits_{i=3}^n x_i\Big). \tag{2.2}$$

This is of course the linear map taking $e_i \mapsto A'$ for $i \geq 3$. The restricted junior simplex Δ of Figure 1.1 is both the intersection of the junior simplex of $L_{\mathbb{R}}$ with $\mathbb{R}^3_r$ and its projection under α, which maps the last $n-2$ vertices e_i to A', together with the $(n-3)$-simplex they span.

If $(0,0,1^{n-2})$ is primitive then $L'/L_r \cong \mathbb{Z}/(n-2)$, with $A' \in \Delta$ as a generator. Write $L'_j = L' \cap \langle\Delta\rangle$ for the intersection of L' with the plane of the restricted junior simplex; since $A' \in \Delta$ this is also an overlattice of L_{rj} of index $n-2$. As well as the Newton polygon of L_{rj} of Figure 1.1, we can also draw the Newton polygon of L'_j in Δ around A'. Then (1–2) above are equivalent to the following condition:

(3) *Either* $A' \in L$; *or* $(0,0,1^{n-2})$ is primitive and the link of A' calculated in L' and in L coincide.

Proof Assume (2), and suppose that $A' \notin L$. Let Σ be a basic triangulation of Δ; since $A' \notin L$, Σ must contain at least one external triangle uvA'. Then by Lemma 2.1 $\{u, v, (0,0,1^{n-2})\}$ is a $\mathbb{Z}$-basis of L_r, and in particular $(0,0,1^{n-2})$ is primitive.

Since L'_j is a finer lattice, its Newton boundary around A' is contained in the external triangles of Σ. However, an external triangle uvA' that corresponds to a basic cone only intersects L'_j in its vertices; in fact, since α maps the junior simplex of $L_{\mathbb{R}}$ to Δ, the triangle uvA' must be basic in $\Delta \cap L'$ because it is the image of a basic cone of L. This proves (2) $\implies$ (3).

For the converse, if $u, v \in L_{\mathrm{rj}}$ and uv is a primitive interval in the Newton boundary of $\Delta \cap L'_j$, then uvA' is a basic triangle of L'_j. Hence $\{u, v, A'\}$ is a $\mathbb{Z}$-basis of L' and, since $(0,0,1^{n-2})$ is primitive, $\{u, v, (0,0,1^{n-2}\}$ is a $\mathbb{Z}$-basis of L_r. This gives (3) $\implies$ (2). $\square$

2.3 The criterion in Hirzebruch–Jung form

We expressed (3) in terms of the link of A' in L'_j. This is given by a standard Hirzebruch–Jung continued fraction calculation, which provides our next criterion (4) below. Although the notation is something of a pain, the point is very simple: the existence of a crepant resolution implies that the first two vectors v_0, v_1 of the link of A' in L'_j go from A' to points of $L \subset L'$. Once the vectors v_0, v_1 are in place, the remaining Newton boundary lattice points are given by the usual recurrence relation $v_{i-1} + v_{i+1} = a_i v_i$. Because $L'/L \cong \mathbb{Z}/(n-2)$ is generated by A', if $v_{i-1}, v_i \in L$ then $v_{i+1} \in L \iff a_i \equiv 2 \mod n-2$. Here we take v_0 to be the first lattice point of L' along the $A'e_2$ axis, and v_1 the next point on the link of A' in L'; if $v_0 = (0, \eta_0, -\zeta_0)$ and $v_1 = (\xi_1, \eta_1, -\zeta_1)$ with $\xi_i, \eta_i, \zeta_i \in \mathbb{Q}$ then the relevant Hirzebruch–Jung continued fraction is the expansion of $\eta_0/\eta_1 = [a_1, a_2, \dots]$.

Suppose that $(0, 0, 1^{n-2}) \in L$ is primitive. Because the sublattice $(x_1 = 0) \cap L$ has rank 2, if it has any fractional part, it is cyclic and generated by some point $P_0 = \frac{1}{s}(0, 1, e^{n-2})$. Then $\overrightarrow{A'P_0}$ is the first vector v_0 in the link of A' mentioned above (the case $e = 0$, $s = 1$ gives $P_0 = e_2$). By the $\mathrm{SL}(n, \mathbb{C})$ assumption on A, $1 + (n-2)e$ is divisible by s. For a crepant resolution to exist, P_0 must be junior; thus

$$P_0 = \tfrac{1}{s}(0, 1, e^{n-2}) \quad \text{with } s = 1 + (n-2)e. \tag{2.3}$$

Next, for $v_1 = \overrightarrow{A'P_1}$, we argue as follows: in order for P_0P_1A' to be a basic external triangle we must have

$$P_1 = \tfrac{1}{rs}(1, d, c^{n-2}) \quad \text{with } rs = 1 + (n-2)c. \tag{2.4}$$

Thus the numerical form of condition (3) is as follows:

(4) *Either* $A' \in L$; *or* $(0, 0, 1^{n-2})$ is primitive in L, P_0 and P_1 are junior, as expressed in (2.3–2.4), and the Hirzebruch–Jung expansion of $\frac{r}{d}$ has every entry congruent to 2 modulo $n-2$.

Conversely, if these conditions hold, we can calculate the Newton boundary of A' in $L' \cap \Delta$ and check that (3) holds. This proves our first main result.

Theorem 2.5 *Let A be a restricted group. Then the above conditions (1–4) are equivalent.*

2.4 Inflation, or *The Origin of Traps*

Inflation of a subgroup of a finite subgroup of a torus $A_1 \subset \mathbb{G}_m^k$ refers to taking the inverse image s^{-1} of A_1 under the sth power map

$$(\mathbb{G}_m)^k \to (\mathbb{G}_m)^k \quad \text{given by } g \mapsto g^s. \tag{2.5}$$

Then of course $s^{-1}A_1$ has order s^k times the order of A_1. In our case, because we are dealing with restricted diagonal subgroups of $\mathrm{SL}(n,\mathbb{C})$, the torus in question is $(\mathbb{G}_m)^2$.

If a restricted group A is not cyclic, it is of the form

$$s^{-1}(sA) \quad \text{for some } s. \tag{2.6}$$

That is, $A \cong \mathbb{Z}/rs \oplus \mathbb{Z}/s$, its image under (2.5) is a cyclic restricted group $sA \cong \mathbb{Z}/r$, and A itself is the restricted $\mathrm{SL}(n,\mathbb{C})$ inverse image of sA under the sth power map $(\mathbb{G}_m)^2 \to (\mathbb{G}_m)^2$.

One sees that if A has a crepant resolution then $s \equiv 1 \bmod n-2$ and sA also has a crepant resolution. Conversely, if A has a crepant resolution, its sth inflation also has a crepant resolution for any $s \equiv 1 \bmod n-2$.

At the level of the blocks of A-Hilb$\,\mathbb{C}^n$, inflation performs a further tesselation of each regular triangle, and replaces a trap of height c by one of height $sc+\frac{s-1}{n-2}$; to put it more simply, the group $(\mathbb{Z}/r)^2$ corresponding to the trap is replaced by $(\mathbb{Z}/rs)^2$. In particular, an external basic cone is replaced by a trap of height $\frac{s-1}{n-2}$. This is where traps come from.

For example, the case $A = \frac{1}{39}(4,32,1^3)$ of Figure 1.4 inflates by a factor of $s=4$ to give $s^{-1}A = \frac{1}{156}(1,8,49^3) \oplus \frac{1}{4}(0,1,1^3)$.

We obtain a nice classification of the restricted groups A we are interested in.

Corollary 2.6 *Every restricted group A for which $\mathbb{C}^n/A$ admits a crepant resolution is given by the following recipe.*

(I) Fix $n \geq 3$; fix integers $a_1,\dots,a_k$, each $\equiv 2$ mod $n-2$.

(II) Compute $\frac{r}{d} = [a_1,\dots,a_k]$ and $c = \frac{r-1-d}{n-2}$; one checks that c is always an integer.

(III) In the cyclic case the group is $A = \frac{1}{r}(1,d,c^{n-2})$.

(IV) Every example is an inflation $s^{-1}A$ of this A by some $s \equiv 1$ mod $n-2$.

The reflected continued fraction $\frac{r}{d*} = [a_k, \ldots, a_1]$ has the same r and $dd^* \equiv 1 \bmod r$. It follows that the two sides e_1A' and e_2A' of Δ have the same index s in L, and everything can be phrased equally well in terms of the points $\frac{1}{s}(1, 0, e^{n-2})$, $\frac{1}{rs}(d^*, 1, c^{*n-2})$ corresponding to v_0, v_1 on the $x_2 = 0$ side.

2.5 Reduction to $\overline{A} \subset \mathrm{GL}(3, \mathbb{C})$ and $\overline{\overline{A}} \subset \mathrm{GL}(2, \mathbb{C})$

Since we draw triangulations of Δ, and the link of A' in L' essentially controls most of the action, it is natural to look for a reduction of the calculations to subgroups of $\mathrm{GL}(3, \mathbb{C})$ and $\mathrm{GL}(2, \mathbb{C})$. For this, let $A \subset \mathrm{SL}(n)$ be a restricted group. We write $x, y, z_{3\ldots n}$ for coordinates on $\mathbb{C}^n$. Write $\overline{A} \subset \mathrm{GL}(3)$ for the action of A on $\mathbb{C}^3_{\langle x,y,z\rangle}$ where $z = z_i$ and $\overline{\overline{A}} \subset \mathrm{GL}(2)$ for the quotient group of A acting on $\mathbb{C}^2_{\langle x,y\rangle}$. In other words, take a typical element $\frac{1}{r}(a, b, 1, \ldots, 1)$ of $A \subset \mathrm{SL}(n, \mathbb{C})$ to

$$\tfrac{1}{r}(a, b, 1) \in \overline{A} \subset \mathrm{GL}(3) \quad \text{and} \quad \tfrac{1}{r}(a, b) \in \overline{\overline{A}} \subset \mathrm{GL}(2). \tag{2.7}$$

The map $A \to \overline{A}$ is an isomorphism, whereas $A \to \overline{\overline{A}}$ acts by deflation: if $A = \mathbb{Z}/rs \oplus \mathbb{A}/s$ as described in 2.4 then $\overline{\overline{A}} \cong \mathbb{Z}/r$.

As usual, write $Z \subset \mathbb{C}^n$ for an A-cluster, $I_Z \subset \mathbb{C}[x, y, z_{3\ldots n}]$ for its ideal, and $H^0(\mathcal{O}_Z) = \mathbb{C}[x, y, z_{3\ldots n}]/I_Z$ for its coordinate ring. The restriction (1.2) means that A acts on the z_i for $i = 3, \ldots, n$ by a common eigenvalue (or character). The distinction of Lemma 1.3 between the external triangles of Δ and the rest of Δ expresses A-Hilb $\mathbb{C}^n$ as a union of two regimes: for every A-cluster Z, either

internal: one of the z_i is basic in $H^0(\mathcal{O}_Z)$; or

external: some monomial in x and y in the same eigenspace as the z_i (say $x^m y^n$) is basic in $H^0(\mathcal{O}_Z)$,

or both. Both conditions are open. The first defines an open set of A-Hilb $\mathbb{C}^n$ that is a fibre bundle over $\mathbb{P}^{n-3}_{\langle z_{3\ldots n}\rangle}$ with fibre $\overline{A}$-Hilb $\mathbb{C}^3$. The second defines a fibre bundle over $\overline{\overline{A}}$-Hilb $\mathbb{C}^2$ (the monomial $x^m y^n$ may vary with the affine piece) with fibre $\mathbb{C}^{n-2}_{z_{3\ldots n}}$.

It is well known that $\overline{\overline{A}}$-Hilb $\mathbb{C}^2$ is the minimal resolution of singularities of $\mathbb{C}^2/\overline{\overline{A}}$. In other words, in the open set governed by the external regime,

A-Hilb $\mathbb{C}^n$ is nonsingular, locally of the form $(\overline{\overline{A}}\text{-}\operatorname{Hilb}\mathbb{C}^2) \times \mathbb{C}^{n-2}$, and one sees that the natural morphism to $\mathbb{C}^n/A$ that is birational and crepant.

As yet, for a diagonal subgroup $B \subset \mathrm{GL}(3, \mathbb{C})$, we only have useable results about B-Hilb $\mathbb{C}^3$ in a limited number of cases (see Section 6). For a general restricted group, it may well be reducible or even have components of dimension $> n$. In the rest of the paper, we compute $\overline{A}$-Hilb $\mathbb{C}^3$ in the case of a restricted group A for which the conditions (1–4) of Theorem 2.5, and show in particular that it is a normal toric variety birational to $\mathbb{C}^3/\overline{A}$.

Given this, the argument here proves that for a restricted group A having a crepant resolution, A-Hilb $\mathbb{C}^n$ is a normal toric variety.

3 Knock-out game and the fan of A-Hilb $\mathbb{C}^n$

This section treats the fan of A-Hilb $\mathbb{C}^n$, adapting the argument of [CR] as sketched in 1.3. We omit some of the details that are mechanical adaptations of [CR]. We are mainly concerned in this section with the geometry of the fan, and the detailed proof that it computes A-Hilb $\mathbb{C}^n$ is left to later.

We assume that we are in the cyclic case $\frac{1}{r}(a, b, 1)$, and from now we abbreviate the $n-2$ repetitions of the last coordinate (in all our calculations, the third coordinate is weighted by $n-2$). Write $\frac{1}{r}(a, b, 1)$, $\frac{1}{r}(1, d, c)$ and $\frac{1}{r}(d^*, 1, c^*)$ for the points nearest the three faces of the triangle. We start by fixing notation for the vectors out of e_1, e_2 and A' that we use to construct the coarse subdivision. Our notation imitates closely the propellor diagram of [CR, 2.1, Figure 3]. In particular, we follow the cyclic ordering and the $\pm$ signs from [CR], so that the cycle

$$f_0, f_1, \dots, f_\alpha,\ -h_0, -h_1, \dots - h_\gamma,\ g_0, g_1, \dots, g_\beta,\ -f_0, \dots \tag{3.1}$$

partitions the plane.

out of e_1: the vectors $f_0, f_1, \dots, f_\alpha$ start with

$$f_0 = \overrightarrow{e_1 e_2} \quad \text{and} \quad f_1 = \overrightarrow{e_1 \tfrac{1}{r}(a, b, 1)}. \tag{3.2}$$

(see below for a numerical example). The entries a_i in the continued fraction $\frac{r}{b} = [a_1, \dots, a_{\alpha-1}]$ give the usual recurrence relation $a_i f_i = f_{i-1} + f_{i+1}$, and we assign each intermediate vector f_i the *strength* a_i. The sequence ends with

$$f_{\alpha-1} = \overrightarrow{e_1 \tfrac{1}{r}(d^*, 1, c^*)} \quad \text{and} \quad f_\alpha = (n-2) \times \overrightarrow{e_1 A'}. \tag{3.3}$$

out of A': the vectors $h_0, h_1, \dots, h_\gamma$, starting with

$$h_0 = \overrightarrow{A'e_1} = -\frac{1}{n-2} f_\alpha \quad \text{and} \quad h_1 = \overrightarrow{A' \tfrac{1}{r}(d^*, 1, c^*)} = h_0 + f_{\alpha-1} \in L'. \tag{3.4}$$

We only consider multiples mh_i for $m \equiv 1 \bmod n-2$, so that, although $h_i \in L'$, we always have $A' + mh_i \in L_{\mathrm{rj}}$. The sequence of h_i ends with $h_\gamma = \overrightarrow{A'e_2}$.

out of e_2: the vectors $g_0, \dots, g_\beta$ starting from

$$g_0 = (n-2) \times \overrightarrow{e_2 A'} = -(n-2) h_\gamma \quad \text{and} \quad g_1 = \overrightarrow{e_2 \tfrac{1}{r}(1, d, c)}, \tag{3.5}$$

and ending with $g_\gamma = \overrightarrow{e_2 e_1} = -f_0$.

Remark 3.1 The main difference from the setup of [CR] is the factors $\frac{1}{n-2}$ and $n-2$ in (3.4) and (3.5) on passing into and out of the segment from A'. The sectors around e_1 and e_2 are internal, and are concerned with bases and change of basis in the restricted junior lattice of L_{rj} of Definition 1.2, whereas the sector around A' is external, concerned with bases in the bigger projected lattice L' of 2.2. In the internal sector one takes f_α, whereas in the external section the appropriate thing to take is $h_0 = -\frac{1}{n-2} f_\alpha$. There is more of this to come.

This difference means that we do not get regular triangles involving A' in exactly the same way as is [CR], but neither are we supposed to: the blocks involving A' are traps.

For example, the case $n = 5$ and $\frac{1}{39}(4, 32, 1)$ of Figure 1.4 has

$$\begin{array}{llll}
f_0 = e_1 e_2 & = -39, 39, 0 & \\
f_1 = e_1 P_1 & = -35, 32, 1 & (2) \\
f_2 = e_1 P_2 & = -31, 25, 2 & (2) \\
f_3 = e_1 P_3 & = -27, 18, 3 & (2) \\
f_4 = e_1 P_4 & = -23, 11, 4 & (2) \\
f_5 = e_1 P_5 & = -19, 4, 5 & (3) \\
f_6 = e_1 P_{11} & = -34, 1, 11 & (4) \\
f_7 = 3 \times e_1 A' & = -117, 0, 39 &
\end{array}
\qquad
\begin{array}{lll}
h_0 = A' e_1 & = 39, 0, -\frac{39}{3} & \\
h_1 = A' P_{11} & = 5, 1, -\frac{6}{3} & (5) \\
h_2 = A' P_{10} & = 1, 8, -\frac{9}{3} & (8) \\
h_3 = A' e_2 & = 0, 39, -\frac{39}{3} & \\
g_0 = 3 \times e_2 A' & = 0, -117, 39 & \\
g_1 = e_2 P_{10} & = 1, -31, 10 & (4) \\
g_2 = e_2 P_1 & = 4, -7, 1 & (10) \\
g_3 = e_2 e_1 & = 39, -39, 0 &
\end{array} \tag{3.6}$$

Here we omit the denominator $\frac{1}{39}$ throughout. The (a_i) denote tags. Between the blades of the propellor the transitions are

$$h_0 = -\tfrac{1}{n-2} f_\alpha, \quad h_1 = h_0 + f_{\alpha-1}, \quad g_0 = -(n-2)h_\gamma, \quad g_1 = g_0 + h_{\gamma-1} \tag{3.7}$$

and $f_0 = -g_\beta$, $f_1 = f_0 + g_{\beta-1}$.

We call the triangular elements of the coarse subdivision *blocks*; they are either regular triangles with e_1 or e_2 as a vertex, or traps with A' as a vertex. We describe them in terms of certain triangles of vectors chosen from $\{f_{0\ldots\alpha}, g_{0\ldots\beta}, h_{0\ldots\gamma}\}$. The definition involves as case division, and it is useful to see this in a numerical case first. Consider $n = 5$ and $\frac{1}{39}(4, 32, 1)$, with the vectors enumerated in (3.6). The blocks of its coarse subdivision in Figure 1.4 correspond to the triples of vectors

$$\begin{gathered}(f_i, f_{i+1}, g_2) \quad \text{for } i = 0\ldots, 4, \quad (f_5, f_6, 3h_1), \quad (g_1, g_2, 3h_2) \\ (h_0, h_1, f_6), \quad (h_1, h_2, g_2), \quad (h_2, h_3, g_1).\end{gathered} \tag{3.8}$$

The distinction is again between internal and external triangles. The last three triples of (3.8) have two h_i, so are triangles with A' as a vertex, and we take them in the projected lattice L'_j. Those on the top line do not have A' as a vertex, so we measure them in units of L_{rj}, multiplying h_1 and h_2 by $n-2$.

Definition 3.2 A *block regular triple* is a set of 3 vectors v_1, v_2, v_3 chosen from $\{f_{0\ldots\alpha}, g_{0\ldots\beta}, h_{0\ldots\gamma}\}$ such that

$$\pm v_1 \pm v_2 \pm v_3 = 0 \tag{3.9}$$

(that is, they form a triangle in Δ) and one of the following conditions hold (compare [CR, (2.1)])

(a) no h_i appears and any two of the vectors base L_{rj}, or

(b) one h_i appears as the multiple $(n-2)h_i$ together with two f_j or two g_k; these three form a basic triangle of L_{rj}, or

(c) f_j, g_k, h_i has one of each three kinds, any two of which base L_{rj}, or

(d) two consecutive h_i appear together with one of f_j or g_k and any two of them base the bigger projected lattice L'_j.

Theorem 3.3 (Main result) *Let A be a restricted group and assume that the criterion of Theorem 2.5 holds. There is a unique coarse decomposition of the restricted junior simplex Δ into blocks, each of which is bounded by a block regular triple. Every such triple appears as one block in this coarse dcomposition.*

Then A-$\mathrm{Hilb}\,\mathbb{C}^n$ performs a regular tesselation of each regular triangle into crepant basic triangles, and cuts up each trap by A-Hilb decomposition described in Section 5. The resulting fan computes A-$\mathrm{Hilb}\,\mathbb{C}^n$.

4 Reduction steps

4.1 The ice cream functor

Let $G \subset \mathrm{GL}(3, \mathbb{C})$ be a finite diagonal subgroup and write $Q(G)$ for its McKay quiver. We view $\mathbb{C}^3$ as the scheme $\mathrm{Spec}[x_1, x_2, x_3]$. Let $\overline{M} = \mathbb{Z}^3$ be the lattice of monomial exponents, where we identify $(m_1, m_2, m_3) \in \mathbb{Z}^3$ with the monomial $x_1^{m_1} x_2^{m_2} x_3^{m_3}$. Let $M \subset \overline{M}$ be the sublattice of invariant monomials. The lattice quotient $\overline{M}/M$ is naturally isomorphic to the character group $G^\vee$ of G. We write $[x_1^a x_2^b x_3^c]$ for the character of G corresponding to a Laurent monomial $x_1^a x_2^b x_3^c$. The quiver $Q(G)$ is identified with the natural quiver structure on $\mathbb{Z}^3/M$ with arrows corresponding to multiplication by x_1, x_2 and x_3.

Applying $\mathrm{Hom}(-, \mathbb{Z})$ to $M \subset \overline{M}$, we have $\overline{L} \subset L$, where $\overline{L} = \overline{M}^\vee$ and the overlattice L is *the lattice of weights*. Since G is finite, each map $M \to \mathbb{Z}$ defined by an element of L extends to a map $\mathbb{Z}^3 \to \mathbb{Q}$, so we identify elements of L with triples $(q_1, q_2, q_3) \in \mathbb{Q}^3$ and identify $L \otimes \mathbb{R}$ with $\mathbb{R}^3$.

Let σ_+ be the positive octant in $L \otimes \mathbb{R}$. Let $\sigma = \langle e_1, e_2, e_3 \rangle$ be some nondegenerate triangular cone within σ_+. Let K be the sublattice of L spanned by e_1, e_2 and e_3. As σ is nondegenerate, K is isomorphic to $\mathbb{Z}^3$. Let N be the $\mathbb{Z}$-dual of K, it is an overlattice of M also isomorphic to $\mathbb{Z}^3$. Denote by t_1, t_2 and t_3 the basis of N dual to e_1, e_2 and e_3. As above, N is naturally a sublattice of $M \otimes \mathbb{Q} = \mathbb{Q}^3$, so each element of N can be identified with a monomial $x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$ with fractional powers $\beta_i \in \mathbb{Q}$.

Write $d = [L : K]$. Then t_i^d all lie in M and d is minimal with such property. The field $\mathbb{C}(N) = \mathbb{C}(t_1, t_2, t_3)$ is therefore a Kummer extension of $\mathbb{C}(M) = K^G(\mathbb{C}^3)$, whose Galois group H is an Abelian group of order $|d|$. Let $\mathbb{C}^3_N$ denote $\mathrm{Spec}\,\mathbb{C}[t_1, t_2, t_3]$, then H is naturally a diagonal subgroup of

$\mathrm{GL}(\mathbb{C}^3_N)$. Let $Q(H)$ denote the McKay quiver of H. Similar to above, we identify $Q(H)$ with the natural quiver structure on the lattice quotient N/M.

Let (β_{ij}) be the rational numbers such that $t_i = \prod x_j^{\beta_{ij}}$. Denote by β the resulting change of basis map $N \otimes \mathbb{Q} \to \mathbb{Z}^3 \otimes \mathbb{Q}$. Since σ is a nondegenerate triangular cone, this map is invertible and we denote by $\alpha = (\alpha_{ij})$ its inverse. By our assumptions σ lies inside the positive octant σ_+, hence $\alpha_{ij} \geq 0$ for all $i, j \in \{1, 2, 3\}$.

Now define the *ice cream map* $\psi\colon \mathbb{Z}^3 \to N$ as the map of sets that sends $m \in \mathbb{Z}^3$ to the rounddown $\lfloor \alpha(m) \rfloor$ of $\alpha(m)$. In other words,

$$x_1^{m_1} x_2^{m_2} x_3^{m_3} \mapsto t_1^{\lfloor \alpha(m)_1 \rfloor} t_2^{\lfloor \alpha(m)_2 \rfloor} t_3^{\lfloor \alpha(m)_3 \rfloor}. \tag{4.1}$$

This map is not additive, that is, not a lattice homomorphism. However, since M is a sublattice of both $\mathbb{Z}^3$ and N, for any $n \in M$ the change of basis map α merely maps its integer x_i-coordinates n_i to its integer t_i-coordinates $\alpha(n)_i$. The fact that $\alpha(n)_i$ are integers implies that

$$\psi(m + n) = \psi(m) + \alpha(n) \quad \text{for all } m \in \mathbb{Z}^3. \tag{4.2}$$

Therefore ψ descends to a map $\mathbb{Z}^3/M \to N/M$, that is, to a map $G^\vee \to H^\vee$. In other words, ψ sends vertices of the McKay quiver of G to the vertices of the McKay quiver of H.

Let $\mathbb{C}Q(G)$ and $\mathbb{C}Q(H)$ be the path algebras of the McKay quivers of G and H with the commutator relations $x_i \circ x_j = x_j \circ x_i$ and $t_i \circ t_j = t_j \circ t_i$. We now extend ψ to a $\mathbb{C}$-linear map $\mathbb{C}Q(G) \to \mathbb{C}Q(H)$. The commutator relations ensure that $\mathbb{C}Q(G)$ is a $\mathbb{C}$-vector space generated by pairs (χ, x^m) where $\chi \in G^\vee$ and $x^m \in \mathbb{Z}^3_{\geq 0}$, and similarly for $\mathbb{C}Q(H)$. Identifying $G^\vee$ with $\mathbb{Z}^3/M$ and $H^\vee$ with N/M, we define for any $[l] \in \mathbb{Z}^3/M$ and $m \in \mathbb{Z}^3_{\geq 0}$

$$\psi([l], x^m) = \big([\psi(l)], t^{\psi(l+m) - \psi(l)}\big). \tag{4.3}$$

To show that this is well defined, we need to first show that $\psi(l+m) - \psi(l)$ lies in $N_{\geq 0}$, that is, $t^{\psi(l+m)-\psi(l)}$ is a regular monomial in t_1, t_2 and t_3. For this, it suffices to show that $\alpha(m) \in N_{\geq 0} \otimes \mathbb{Q}$, as then the roundown of $\alpha(l+m)$ has to be greater or equal than that of $\alpha(l)$ in every coordinate. But we have $m \in \mathbb{Z}^3_{\geq 0}$ and, as noted above, $\alpha_{ij} \geq 0$ since $\sigma \subset \sigma_+$. Hence $\alpha(m) \in N_{\geq 0}$ as required. Then we need to show is that the expresion $\psi(l+m) - \psi(l)$ in (4.3) is independent of the choice of $l \in \mathbb{Z}^3$ to represent $[l] \in \mathbb{Z}^3/M$. But for any other $l' \in \mathbb{Z}^3$ with $n = l' - l \in M$ we have

$$\begin{aligned}\psi(l+m) - \psi(l) &= \psi(l' + m - n) - \psi(l' - n) = \\ &= \psi(l' + m) - \alpha(n) - \psi(l') + \alpha(n) = \psi(l' + m) - \psi(l')\end{aligned}$$

as required.

While $\psi\colon \mathbb{C}Q(G) \to \mathbb{C}Q(H)$ is a $\mathbb{C}$-linear map, it is not, in fact, an algebra homomorphism. This is because on the vertex sets ψ is, generally, a coarsening: it can send several different vertices of $Q(G)$ to the same vertex in $Q(H)$. It can therefore send paths in $Q(G)$ whose ends do not match up and whose product, therefore, is zero, to paths in $Q(H)$ which do match up and hence give nonzero product. However, it is easy check that for any paths $p, q \in Q(G)$ with $pq \neq 0$ we do have $\psi(pq) = \psi(p)\psi(q)$. This allows us to define the *ice cream functor*

$$\Psi\colon \ \mathbf{Mod}\text{-}\mathbb{C}Q(H) \to \mathbf{Mod}\text{-}\mathbb{C}Q(G) \tag{4.4}$$

by defining the image of a representation (V_χ, α_q) of $Q(H)$ to be $(V_{\psi(\chi)}, \alpha_{\psi(q)})$. This generalises the construction by Ishii and Ueda in [IU09, Section 6], that was carried out in terms of dimer models and thus only applies to $G \subset \mathrm{SL}(3, \mathbb{C})$.

Note that it follows from the above definition that Ψ sends representations of $Q(H)$ with dimension vector $(1, \dots, 1)$ to the representations of $Q(G)$ with dimension vector $(1, \dots, 1)$. In other words Ψ sends H-constellations to G-constellations.

For each triangular subcone σ of σ_+ the resulting functor Ψ_σ can be visualised as follows. The basis t_1, t_2 and t_3 of $M \otimes \mathbb{Q}$ dual to the generators of σ defines a slicing up of the whole of $\mathbb{R}^3 = M \otimes R$ into parallelepipeds with sides t_1, t_2, t_3. We then identify each element $n \in N$, that is, a Laurent monomial $t_1^{n_1} t_2^{n_2} t_3^{n_3}$, with the paralleliped $[n_1, n_1{+}1] \times [n_2, n_2{+}1] \times [n_3, n_3{+}1]$. In these terms, the ice cream map ψ described above sends each element $m \in \mathbb{Z}^3$ to the parallelepiped in N which contains m. Similarly, any path $([l], m)$ in the path algebra $\mathbb{C}Q(G)$ can be visualised as any of the paths which start at $l \in Z^3$ and consists of m_1 arrows in x_1-direction, m_2 arrows in x_2-direction and m_3 arrows in x_3-directions. Then the image of $([l], m)$ in $\mathbb{C}Q(H)$ under ψ is simply the coarsening of this path in $\mathbb{Z}^3$ to the corresponding path through the parallelepipeds of N. In particular, any single arrow $([l], x_i)$ of the McKay quiver of G gets sent to something other than the trivial path at the parallelepiped containing l if and only if the arrow $l \to l + x_i$ crosses one of the walls of this parallelepiped.

4.2 The maximal shift functor

While the ice cream functor Ψ defined in Section 4.1 is a natural and intuitive construction, it does not do its intended job. When σ is one of the blocks in the coarse subdivision of Δ described in Section 1.3, the ice cream functor Ψ_σ defined by σ does not send H-clusters to G-clusters. It does not even do this for the basic triangle blocks, that is, when H is trivial.

The key to solving this problem lies in the following. The ice cream functor Ψ was cooked up in Section 4.1 from the ice cream map $\psi\colon \mathbb{Z}^3 \to N$ in a way which only relied on the following two properties of ψ:

1. For any $m \in M$ we have
$$\psi(l+m) = \psi(l) + \alpha(m) \qquad \text{for all } l \in \mathbb{Z}^3. \tag{4.5}$$

2. For any $m \in \mathbb{Z}^3_{\geq 0}$ we have
$$\psi(l+m) - \psi(l) \in N_{\geq 0} \qquad \text{for all } l \in \mathbb{Z}^3. \tag{4.6}$$

Regarding the property (1): since m lies in the invariant lattice M we have $[l] = [l+m]$ in $G^\vee$. Hence, if for each character $\chi \in G^\vee$ we shift the value of ψ on all $l \in \mathbb{Z}^3$ with $[l] = \chi$ by some fixed $q_\chi \in N$, the property (1) is still satisfied. However, for an arbitrary set of shifts $\{q_\chi\}$ there is no guarantee that the shifted version of ψ still satisfies (2).

Definition 4.1 *For any $\chi \in G^\vee$ define $\overline{q}_\chi \in N$ to be the rounddown*
$$\lfloor \mu_\chi \rfloor = t_1^{\lfloor \mu_{\chi,1} \rfloor} t_2^{\lfloor \mu_{\chi,2} \rfloor} t_3^{\lfloor \mu_{\chi,3} \rfloor} \tag{4.7}$$
of the element μ_χ of $N \otimes \mathbb{Q}$ defined by
$$\mu_{\chi,i} = \min_{m \in \mathbb{Z}^3_{\geq 0}, [m] = \chi} e_i(m). \tag{4.8}$$

Lemma 4.2 *The map $\overline{q}\psi\colon \mathbb{Z}^3 \to N$ defined by*
$$\overline{q}\psi(m) = \psi(m) - \overline{q}_{[m]} \quad \textit{for } m \in \mathbb{Z}^3 \tag{4.9}$$
satisfies the properties (1) and (2) of the map ψ.

Proof It is clear that $\overline{q}\psi$ satisfies the property (1). We now demonstrate that it satisfies the property (2). Observe that any $n \in N$ lies in $N_{\geq 0}$ if and only if $e_i(n) \geq 0$ for all $i \in \{1,2,3\}$. It therefore suffices to show that for any $l \in \mathbb{Z}^3$ and $m \in \mathbb{Z}^3_{\geq 0}$ we have

$$e_i\big(\overline{q}\psi(l+m) - \overline{q}\psi(l)\big) \geq 0 \qquad \text{for all } i \in \{1,2,3\}. \tag{4.10}$$

By definitions of $\overline{q}\psi$ and ψ we have

$$\overline{q}\psi(l+m) = \psi(l+m) - \overline{q}_{[l+m]} = \lfloor \alpha(l+m) \rfloor - \lfloor \mu_{[l+m]} \rfloor \tag{4.11}$$

and therefore

$$e_i\big(\overline{q}\psi(l+m)\big) = \lfloor e_i(l+m) \rfloor - \lfloor \mu_{[l+m],i} \rfloor. \tag{4.12}$$

Since $\mu_{[l+m],i}$ is the minimal value of e_i on the monomials in $\mathbb{Z}^3_{\geq 0}$ of character $[l+m]$, the fractional part of $\mu_{[l+m],i}$ is the same as that of $e_i(l+m)$. This is because e_i takes integer values on all elements of N and, in particular, on the invariant monomials of M. We conclude that, in fact,

$$e_i\big(\overline{q}\psi(l+m)\big) = e_i(l+m) - \mu_{[l+m],i}. \tag{4.13}$$

Similarly

$$e_i(\overline{q}\psi(l)) = e_i(l) - \mu_{[l],i} \tag{4.14}$$

and hence

$$e_i\big(\overline{q}\psi(l+m) - \overline{q}\psi(l)\big) = \mu_{[l],i} + e_i(m) - \mu_{[l+m],i}. \tag{4.15}$$

Finally, let l' be a monomial in $\mathbb{Z}^3_{\geq 0}$ of character $[l]$ on which the value of e_i is minimised. Then $e_i(l') = \mu_{[l],i}$ and hence

$$\mu_{[l],i} + e_i(m) - \mu_{[l+m],i} = e_i(l') + e_i(m) - \mu_{[l+m],i} = e_i(l'+m) - \mu_{[l+m],i}. \tag{4.16}$$

Both l' and m lie in $\mathbb{Z}^3_{\geq 0}$ and $[l'] = l$. Hence $l'+m$ is a monomial in $\mathbb{Z}^3_{\geq 0}$ of character $[l+m]$. Therefore the value of e_i on $l'+m$ is greater than $\mu_{[l+m],i}$. We conclude that

$$e_i\big(\overline{q}\psi(l+m) - \overline{q}\psi(l)\big) \geq 0 \tag{4.17}$$

as required.

In fact, if we apply normalization by the shift of the trivial character, the shifts $\{\overline{q}_\chi\}_{\chi \in G^\vee}$ are maximal shifts after the application of which ψ still satisfies properties (1) and (2):

Lemma 4.3 *Let $\{q_\chi\}_{\chi \in G^\vee}$ be a set of shifts such that $\psi'(m) = \psi(m) - q_{[m]}$ satisfies the properties* (1) *and* (2). *Then for all $\chi \in G^\vee$*

$$q_\chi - q_{[0]} \leq \overline{q}_\chi - \overline{q}_{[0]}. \tag{4.18}$$

Proof First, note that $\overline{q}_{[0]} = 0$ since $0 \in \mathbb{Z}^3_{\geq 0}$ minimises the value of each e_i on all G-invariant monomials in $\mathbb{Z}^3_{\geq 0}$. Next, note that any m in $\mathbb{Z}^3_{\geq 0}$ we have

$$\psi'(m) - \psi'(0) = (\lfloor \alpha(m) \rfloor - q_{[m]}) - (0 - q_{[0]}) = \lfloor \alpha(m) \rfloor - (q_{[m]} - q_{[0]}). \quad (4.19)$$

Since ψ' satisfies property (2), the LHS must lie in $N_{\geq 0}$. Hence

$$e_i(q_{[m]} - q_{[0]}) \leq \lfloor e_i(m) \rfloor \qquad \text{for each } i \in \{1, 2, 3\}. \quad (4.20)$$

Choose any $\chi \in G^\vee$ and $i \in \{1, 2, 3\}$. By definition, $e_i(\mu_\chi)$ is the minimal value of e_i on the monomials of character χ in $\mathbb{Z}^3_{\geq 0}$. In particular, there exists some $m \in \mathbb{Z}^3_{\geq 0}$ with $[m] = \chi$ such that $e_i(\mu_\chi) = e_i(m)$. Hence, by above

$$e_i(q_{[m]} - q_{[0]}) \leq \lfloor e_i(\mu_\chi) \rfloor . \quad (4.21)$$

Since $\overline{q}_\chi$ was defined to be $\lfloor \mu_\chi \rfloor$, the claim follows.

Once the properties (1) and (2) are satisfied, the rest of the construction in Section 4.1 goes through. We therefore define:

Definition 4.4 *Define the maximal shift map*

$$\overline{q}\psi \colon \mathbb{C}Q(G) \to \mathbb{C}Q(H) \quad (4.22)$$

to be the $\mathbb{C}$-linear extension of $\overline{q}\psi \colon \mathbb{Z}^3 \to N$ given by

$$([l], m) \mapsto \big([\overline{q}\psi(l)], \overline{q}\psi(l+m) - \overline{q}\psi(l)\big). \quad (4.23)$$

Define the maximal shift functor

$$\overline{q}\Psi \colon \ \mathbf{Mod}\text{-}\mathbb{C}Q(H) \longrightarrow \mathbf{Mod}\text{-}\mathbb{C}Q(G) \quad (4.24)$$

by

$$(V_\chi, \alpha_q) \mapsto (V_{\psi(\chi)}, \alpha_{\psi(q)}). \quad (4.25)$$

4.3 G-Hilb via H-Hilbs

Let σ be a basic triangle in Δ which belongs to G-Hilb. That is, the usual construction yields a family of G-clusters parametrised by A_σ, the affine toric variety defined by σ and isomorphic to $\mathbb{C}^3$. Since σ is basic the group H defined by it is trivial. The H-clusters, therefore, are simply the point sheaves on A_σ. It follows from the argument in [L08, Prop. 3.17] that the maximal shift functor $\overline{q}\Psi_\sigma$ sends each of these point sheaves to the G-cluster parametrised by the corresponding point in A_σ. In other words, $\overline{q}\Psi_\sigma$ sends H-clusters to G-clusters.

A similar argument shows that the same is true for any non-basic triangle σ which is cut out by any three rays belonging to the Newton polygons of e_1, e_2 and A'. For such σ functor $\overline{q}\Psi_\sigma$ sends H-clusters to G-clusters and thus induces a map $H\text{-Hilb} \to G\text{-Hilb}$ which is an open embedding.

Suppose now that, as outlined in Section 1.3, we can subdivide the whole of Δ into triangular blocks, each of which is either a regular triangle or a trap and each of which is cut out by the rays from the Newton polygons of e_1, e_2 and A'. For each such triangular block let us further subdivide the corresponding triangular cone using the H-Hilb-subdivision: the tesselation described in [CR] for regular triangles and the subdivision described in Section 5 for traps. We obtain a subdivision Σ of Δ. Let Y denote the corresponding toric variety. By construction, for each coarse block σ the corresponding open affine piece U_σ is isomorphic to H-Hilb and therefore, via $\overline{q}\Psi_\sigma$, embeds as an open subset into G-Hilb. These embeddings are compatible with overlaps, i.e. given any two coarse blocks σ_1 and σ_2 the embeddings $\overline{q}\Psi_{\sigma_1}$ and $\overline{q}\Psi_{\sigma_2}$ restrict to the same embedding on $U_{\sigma_1} \cap U_{\sigma_2}$. Therefore the embeddings $\overline{q}\Psi_\sigma$ glue together to give an embedding of Y into G-Hilb. Since this embedding clearly contains the free G-orbits, we conclude that Y is the *coherent component* of G-Hilb, i.e. its unique irreducible component which contains the free G-orbits.

5 The trap and its $A\text{-Hilb}\,\mathbb{C}^n$

The trap $\frac{1}{r}(r-1,1,0) \oplus \frac{1}{r}(r-n+2,0,1)$ is the maximal r-torsion subgroup among restricted diagonal subgroups of $\mathrm{SL}(n,\mathbb{C})$. It is isomorphic to $(\mathbb{Z}/r)^{\oplus 2}$. We are really only interested in the case r and $n-2$ are coprime, and then the criterion says that a crepant resolution exists if and only if $r \equiv 1 \bmod n-2$. The points along the side e_1A' of the restricted simplex are $\frac{1}{r}(r-i(n-2),0,i)$ for $i = 0,1,\dots,[\frac{r}{n-2}]$, and by Theorem 2.5, we know that if r and $n-2$ are coprime, a crepant resolution exists if and only if the last point is $\frac{1}{r}(1,0,c)$ with $r = c(n-2)+1$.

This section is a complete brute force computation of $A\text{-Hilb}\,\mathbb{C}^n$ in this case.

5.1 Notation and terminology

The case we treat here is dim $= n$ and $c =$ number of shelves. Then $r = (n-2)c+1 = (n-2)(c-1)+n-1$, etc., and $A = (\mathbb{Z}/r)^{\oplus 2}$ has generators

$$\textit{left shoulder} \quad L = \tfrac{1}{r}(1,0,c) \text{ and } \quad \textit{right shoulder } M = \tfrac{1}{r}(0,1,c),$$

or alternatively $\frac{1}{r}(r-1,1,0) \oplus \frac{1}{r}(r-n+2,0,1)$. The trap is divided by c *shelves*, with the *foyer* sitting over the *top shelf*, the line from $\frac{1}{r}(n-1,0,c-1)$ to $\frac{1}{r}(0,n-1,c-1)$. The lattice points along the top shelf are

$$Q_i = \frac{1}{r}(n-i, i-1, c-1) \quad \text{for } i = 1, \dots, n; \tag{5.1}$$

that is, the third coordinate $c-1$ is fixed, and in the first two coordinates, the lattice points are the same as for the well known subgroup $\frac{1}{n-1}(n-2,1)$ of $\mathrm{SL}(2,\mathbb{C})$. The Q_i are junior points. A posteriori, it turns out that the

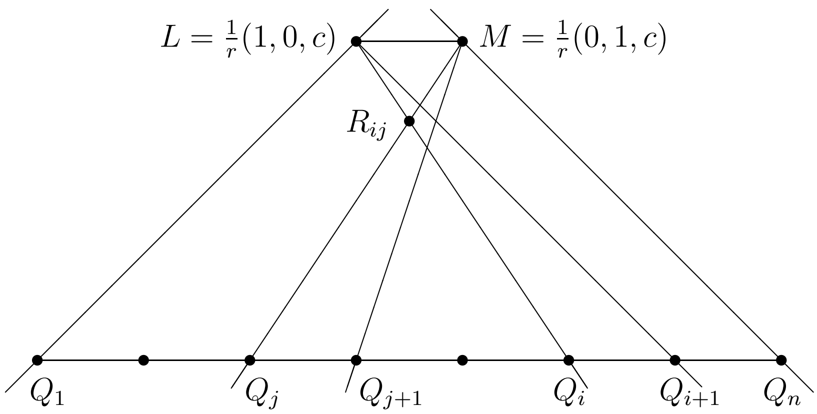

Figure 5.1: The foyer for $A = \frac{1}{r}(r-1,1,0) \oplus \frac{1}{r}(r-n+2,0,1)$

way A-Hilb $\mathbb{C}^n$ divides the trap is basically independent of c, so we could in principle concentrate on the foyer. (This can also be proved independently, as in Section 4, or it follows from the direct calculations below.)

In A-Hilb $\mathbb{C}^n$, the trap is subdivided by the *spokes* from the two shoulders to the lattice points Q_i of the top shelf. The point is just that the two monomials

$$x^{c(i-1)} \quad \text{and} \quad y^{c(n-i-1)+1} z^{i-1} \tag{5.2}$$

generate an eigenspace of the A-action on $\mathbb{C}[x,y,z]$.

Lemma 5.1 *The A-invariant ratio between the two monomials of* (5.2) *is perpendicular to the spoke LQ_i.*

The assertion is a straightforward calculation: the lattice points $L = \frac{1}{r}(1,0,c)$ and $Q_i = \frac{1}{n-1}(n-i, i-1, c-1)$ evaluate to zero against the stated ratio. For every Z, we have to choose at most one of $x^{c(i-1)}$ or $y^{c(n-i-1)+1}z^{i-1}$ as a basis vector in $\mathcal{O}_Z$ and write the other as its multiple. Therefore the spoke LQ_i is a dividing line in the toric fan for $A\text{-Hilb}\,\mathbb{C}^n$: everything to the left takes $y^{c(n-i-1)+1}z^{i-1}$ as a basic monomial, and everything to the right $x^{c(i-1)}$. Between the spokes LQ_i and LQ_{i+1}, every cluster must have

$$\begin{aligned} y^{c(n-i-1)+1}z^{i-1} &= \alpha x^{c(i-1)} \quad \text{and} \\ x^{ci} &= \beta y^{c(n-i-2)+1}z^i \end{aligned} \tag{5.3}$$

In the same way every cluster between spokes MQ_j and MQ_{j+1} has

$$\begin{aligned} y^{c(n-j)} &= \gamma x^{c(j-2)+1}z^{n-j} \\ x^{c(j-1)+1}z^{n-j-1} &= \delta y^{c(n-j-1)} \end{aligned} \tag{5.4}$$

If we were thinking of the ratios $\alpha, \beta, \gamma, \delta$ as rational monomials given by (5.3–5.4) on an irreducible toric variety, we could just multiply the equations together to conclude that $\alpha\beta = \gamma\delta$ and $x^cy^c = \alpha\beta z = \gamma\delta z$. This argument is not valid, because a prior we cannot assume that $A\text{-Hilb}\,\mathbb{C}^n$ is irreducible or reduced; compare 6. We work here with the whole A-Hilbert functor, not just its birational component. The next lemma justifies the cancellation. We go through it formally in detail, because similar arguments appear throughout the calculation of $A\text{-Hilb}\,\mathbb{C}^n$.

Lemma 5.2 $\alpha\beta = \gamma\delta$ *and* $x^cy^c = \alpha\beta z = \gamma\delta z$.

Proof By definition of A and the current case assumptions, the monomial x^cy^c is in the 1-dimensional eigenspace of $H^0(\mathcal{O}_Z)$ based by z, so that we must have

$$x^cy^c = \lambda z \quad \text{for some } \lambda. \tag{5.5}$$

The issue is to determine λ. Multiply by $x^{c(i-1)}$, then substitute for x^{ci} on the left from the second equation of (5.3), then by the first:

$$\lambda x^{c(i-1)}z = x^{ci}y^c = \alpha y^{c(n-i-1)+1}z^i = \alpha\beta x^{c(i-1)}z. \tag{5.6}$$

Now $x^{c(i-1)}z$ is basic in $H^0(\mathcal{O}_Z)$, so that finally we are allowed to cancel and conclude that $\lambda = \alpha\beta$. The same argument proves that $\lambda = \gamma\delta$. $\square$

The two spokes LQ_i and MQ_j

- are divergent if $j > i + 1$;
- are parallel if $j = i + 1$;
- intersect at Q_i if $i = j$ with, obviously, Q_i of age 1;
- intersect at a point R_{ij} internal to the foyer if $i > j$, where

  $$R_{ij} = Q_i + (i-j)L = Q_j + (i-j)M, \tag{5.7}$$

  so in particular $\operatorname{age}(R_{ij}) = i + 1 - j$.

By construction R_{ij} is the cross product of

$$\begin{pmatrix} ci & -c(n-i-2)-1 & -i \\ c(j-2)-1 & -c(n-j) & n-j \end{pmatrix}, \tag{5.8}$$

but (5.7) is simpler and more useful. The spokes only intersect in the foyer: in the bottom of the trap they are parallel or divergent.

The 4 spokes in (5.3–5.4) bound a lozenge (parallelogram) if and only if $i \geq j+1$. Since by (5.1) the top shelf has n points $Q_1, \ldots, Q_n$ (including the

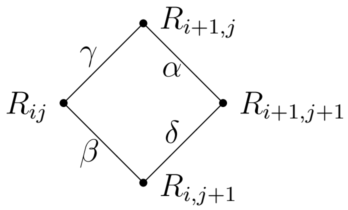

Figure 5.2: The lozenge with bottom point $R_{i,j+1}$, of age $i-j$

two endpoints), there are $\binom{n-3}{2}$ lozenges, that is none if $n = 4$, exactly one if $n = 5$, three if $n = 6$, etc. A lozenge is a plane parallelogram bounded by 4 spokes

$$LQ_i \text{ and } LQ_{i+1}, \quad MQ_j \text{ and } MQ_{j+1} \quad \text{with } i \geq j+1. \tag{5.9}$$

Its 4 vertices, read from bottom and left, are

$$R_{i,j+1}, R_{ij}, R_{i+1}j+1, R_{i+1,j} \tag{5.10}$$

and have age $i-j$, $i-j+1$, $i-j+1$, $i-j+2$. By (5.3–5.4), the four spokes correspond to the monomials

$$\begin{aligned} LQ_{i+1} &: \alpha = x^{ci}/y^{c(n-i-2)+1}z^i, \\ LQ_i &: \beta = y^{c(n-i-1)+1}z^{i-1}/x^{c(i-1)}, \\ MQ_j &: \gamma = y^c(n-j)/x^{c(j-2)+1}z^{n-j}, \\ MQ_{j+1} &: \delta = x^{c(j-1)+1}z^{(n-j-1)}/y^{c(n-j-1)}. \end{aligned} \tag{5.11}$$

Note that $\alpha\beta = \gamma\delta = x^c y^c/z$. Wherever possible, we write $\lambda = \alpha\beta = \gamma\delta$ to display the symmetry. In the lozenge picture,

5.2 The affine piece of A-Hilb $\mathbb{C}^n$ belonging to a lozenge

Take the 4 monomials $\alpha, \beta, \gamma, \delta$ of (5.11). Then the Jacobian determinant (or discrepancy) is computed by the product

$$xyz^{n-2} = \lambda^{i-j}\beta\delta. \tag{5.12}$$

Here $i-j$ is the age of the bottom $R_{i,j+1}$ of the lozenge, $\lambda = \alpha\beta = \gamma\delta$ the quantity just explained, and $\beta\delta$ the product of the two monomials on the bottom.

So far this is just notation and a little calculation to establish (5.12). The main assertion is the following:

Lemma 5.3 *The quadric* $(\alpha\beta = \gamma\delta) \subset \mathbb{A}^4_{\langle\alpha,\beta,\gamma,\delta\rangle}$ *parametrises an affine piece of the Hilbert scheme* A-Hilb $\mathbb{C}^n$*. The ideal* I_Z *of each cluster* Z *in this affine piece has minimal generators the monomial equations written out in (5.14–5.15), with coefficients the monomials*

$$\alpha, \gamma, \lambda, \beta\delta\lambda^{i-j-1}, \beta\delta\lambda^{i-j}, \quad \text{and } \lambda^k\beta, \lambda^k\delta \text{ for } k = 0, \dots, i-j. \tag{5.13}$$

(All of these divide $\beta\delta\lambda^{i-j}$*.) The equations are the five special ones*

$$\begin{gathered} x^c y^c = \lambda z, \quad z^{n-1} = \beta\delta\lambda^{i-j-1}x^{c-1}y^{c-1}, \quad xyz^{n-2} = \beta\delta\lambda^{i-j}, \\ x^{ci} = \alpha y^{c(n-i-2)+1}z^i, \quad y^{c(n-j)} = \gamma x^{c(j-2)+1}z^{n-j}, \end{gathered} \tag{5.14}$$

together with the two series of $i-j+1$ *equations*

$$\begin{aligned} x^{c(j-1+k)+1}z^{(n-j-1-k)} &= \lambda^k\delta y^{c(n-j-1+k)}, \\ y^{c(n-i-1+k)+1}z^{i-1-k} &= \lambda^k\beta x^{c(i-1+k)} \end{aligned} \tag{5.15}$$

for $k = 0, \dots, i-j$.

Example 5.4 For a big fat example, choose dimension $n = 11$ and number of shelves $c = 4$, so that $r = c(n-2)+1 = 37$, and consider the lozenge with $i = 6$ and $j = 3$. Its four sides LQ_i, LQ_{i+1}, MQ_j, MQ_{j+1} correspond to

$$x^{24} = \alpha y^{13} z^6, \quad y^{17} z^5 = \beta x^{20}, \quad y^{32} = \gamma x^5 z^8, \quad x^9 z^7 = \delta y^{28}. \tag{5.16}$$

We continue to write $\lambda = \alpha\beta = \gamma\delta$ for symmetry. A rigorous cancellation argument similar to the above proves that

$$x^4 y^4 = \lambda z, \quad z^{10} = \beta\delta\lambda^2 x^3 y^3, \quad xyz^9 = \beta\delta\lambda^3. \tag{5.17}$$

The two series of 4 monomial equation in x, z and y, z come from the δ and β equation by successive multiplication by λ:

$$\begin{aligned} x^9 z^7 &= \delta y^{28} & y^{17} z^5 &= \beta x^{20} \\ x^{13} z^6 &= \lambda\delta y^{24} & y^{21} z^4 &= \lambda\beta x^{16} \\ x^{17} z^5 &= \lambda^2\delta y^{20} & y^{25} z^3 &= \lambda^2\beta x^{12} \\ x^{21} z^4 &= \lambda^3\delta y^{16} & y^{29} z^2 &= \lambda^3\beta x^4 \\ x^{24} &= \alpha y^{13} z^6 & y^{32} &= \gamma x^5 z^8 \end{aligned} \tag{5.18}$$

Here the α and γ equations act as punctuation at the end of the series – you can check as an exercise that the next equation $x^{25} z^3 = \lambda^4 \delta y^{12}$ in the series is already in the ideal generated by the α equation and (5.17).

A monomial basis of the resulting A-set is a stack of 10 layers according to powers of z. The monomials with no z form a monomial basis of the quotient $k[x, y]/(x^4 y^4, x^{24}, y^{32})$ giving $24 \times 32 - 20 \times 18 = 208$ monomials.

```
  (x^4*y^4, x^24,y^32)          208
with one z, repeat              208
with z^2  (x^4*y^4, x^24,y^29)  196
with z^3  (x^4*y^4, x^24,y^25)  180
with z^4  (x^4*y^4, x^21,y^21)  152
with z^5  (x^4*y^4, x^17,y^17)  120
with z^6  (x^4*y^4, x^13,y^17)  104
with z^7  (x^4*y^4, x^9,y^17)    88
with z^8  repeat                 88
with z^9  (x*y, x^9,y^17)        25
with z^10  (1)                 ======
                 total         1369
n-1 layers
sim. calc. proves Z is A-cluster
```

I used the following Magma code for Example 5.4

```
M := Matrix(Integers(), 3, [24,-13,-6, -20,17,5,
 -5,32,-8, 9,-28,7]);
N0 := Matrix(Integers(),
&cat[[[0,1,i,i],[0,0,i,i+1],[0,1,i,i+1]] : i in [0..3]]
cat [[1,0,0,0],[0,0,1,0],[0,0,1,1]]);
N := N0*M;
RR<x,y,z> := PolynomialRing(Rationals(),3);
I := [&*[RR.i^(N[j,i]) : i in [1..3] | N[j,i] gt 0]
 : j in [1.. NumberOfRows(N)]];
Sort(MinimalBasis(Ideal(I)));
Dimension(quo<RR|I>); 37^2;
for j in [1.. NumberOfRows(N)] do N0[j]; N[j]; end for;
```

5.3 The lower shelves

The trap of c shelves of Figure 1.2 has $n-3$ alleys of parallelograms leading into the foyer. Figure 5.3 illustrates the general case. By (5.3–5.4) and easy

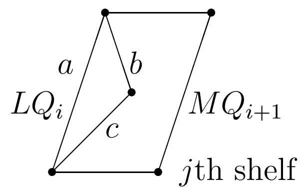

Figure 5.3: Parallelogram bounded by LQ_i, MQ_{i+1} and the jth and $(j+1)$st shelves

considerations, the sides of the parallelogram correspond to the ratios

$$\begin{gathered} y^{c(n-i-1)+1} z^{i-1} = \lambda x^{c(i-1)} \quad \text{and} \quad x^{c(i-1)+1} z^{n-i-1} = \mu y^{c(n-i-1)}, \\ x^{j+1} y^{j+1} = \sigma z^{(n-2)(c-j-1)+1} \quad \text{and} \quad z^{(n-2)(c-j)+1} = \tau x^j y^j, \end{gathered} \tag{5.19}$$

where i runs from 1 to $n-1$ and j from 0 to $c-1$. One derives a plausible relation $xyz^{n-2} = \lambda\mu = \sigma\tau$ (in fact this is provable, much as before). These equations do not themselves define A-clusters, nor do the equations from the dual cones if we cut the parallelogram along either diagonal (unfortunately, since either would give two crepant basic cones). To get cones parametrising A-clusters, we need to quarter the parallelogram by cutting it along *both* diagonals.

The second set of equations in (5.19) hints at what to do. Namely, the product of the equations would give (after cancelling) $xyz^{n-2} = \sigma\tau$. Since xyz^{n-2} corresponds to the canonical divisor of Y, if σ, τ together with some monomial φ were coordinates on a nonsingular model, the divisor φ would be a pole of the canonical class, which contradicts that Gorenstein quotient singularities are canonical. This suggests strongly that σ and τ must have a common divisor.

If we quarter our parallelogram, any of the 4 resulting basic discrepant triangles is dual to a monomial cone $\langle a, b, c\rangle$, where a is the orthogonal of one of the sides (say the left side LQ_i to be definite), and b, c correspond to two halfdiagonals (say from top right and bottom right, as in Figure 5.19). Then $\sigma = ab$, $\tau = ac$, and $xyz^{n-2} = a^2bc$ restores the discrepancy of the affine piece to what it should be. One checks that a, b, c and $n-3$ of the z_i then parametrise an affine piece of $A\text{-Hilb}\,\mathbb{C}^n$.

We do just one numerical case. The general case is just repetition with a lot of tedious messing around with exponents. Set $n = 6$, $c = 3$, $j = 0$ and $i = 2$, so that we are doing the parallelogram on the bottom left of Figure 1.2, with vertices $(10, 3, 0), (9, 4, 0), (7, 2, 1), (6, 3, 1)$. The 4 equations (5.19) for the sides of the parallelogram are

$$y^{10}z = \lambda x^3, \quad x^4z = \mu c^9, \quad xy = \sigma z^9, \quad z^{13} = \tau. \tag{5.20}$$

We calculate the dual monomials to the sides a, b, c of the left-hand triangle either directly by coordinate geometry, or by using $\lambda = a$, $\sigma = ab$, $\tau = ac$. We get the basic equations

$$x^4 = by^9z^{10}, \quad y^{10}z = ax^3, \quad x^3z^{12} = cy^{10}, \tag{5.21}$$

that imply also

$$xy = abz^9, \quad z^{13} = ac, \quad y^{11} = a^2bx^2z^8. \tag{5.22}$$

(and $xyz^4 = a^2bc$, which is redundant as a generator). One checks that these equations define a cluster depending on $\mathbb{C}^6$ with coordinates a, b, c (together with three of the ratios z_i/z). The verification involves simply writing out a monomial basis for $\mathbb{C}[x, y, z]$ modulo these relations: there are

14 monomials in x, y only, 13 monomials with z^i times a monomial in x, y only for $i = 1, \dots, 11$, and 12 monomials with z^{12}. One checks that they occupy the 13^2 characters of the group $A = 2\mathbb{Z}/13$.

6 Final comments

When does a crepant resolution exist? Let $A \subset \mathrm{SL}(4, \mathbb{C})$ be a diagonal subgroup. A crepant resolution is the same thing as a subdivision of the junior simplex into a fan of basic cones; the question of its existence has been studied, probably originally by Firla and Ziegler [FZ], [F]; compare Davis [D]. Although in this paper we have had a lot of fun with cases where a crepant resolution exists, it is an implicit consequence of our result that these form a vanishingly small proportion of all cases.

We are only allowed junior lattice points as the 1-skeleton of our fan, so we clearly need an adequate supply of them. One sees that a necessary condition is that every lattice points of age 2 in the unit cube is the sum of two juniors. Most reasonably small cases satisfying this condition have a crepant resolution. However, the condition is not sufficient; according to [FZ] the smallest counterexample is $\frac{1}{39}(1, 5, 25, 8)$ (or we can write it $\frac{1}{39}(1, 5, 25, 125)$ to display its cyclic symmetry).

The barycentric subdivision at a junior point is a crepant partial resolution. However, it may happen that a crepant resolution exists, but none that dominates a barycentric subdivision; the smallest example seems to be $\frac{1}{67}(1, 5, 8, 53)$ [D, 2.6].

One might look for a criterion for the existence of a crepant resolution based on determining a priori the basket of terminal singularities that appears on a minimal model. However, this also seems to be a misplaced hope; different minimal models may have quite different terminal singularities.

An orbifold has a stacky derived category with some of the same categorical properties as a nonsingular variety. A partial crepant resolution can be used to isolated different semiorthogonal components. One might expect the parts tied to terminal singularities to have some characterisation that makes them qualitatively different from the parts that spread over higher dimensional pieces of a partial resolution. If so, the distinction has so far eluded us.

Algorithm for A-Hilb $\mathbb{C}^4$ More general discussion of my algorithm for A-Hilb $\mathbb{C}^4$.

A-Hilb is an algorithmic construction: if Z is an A-cluster, the quotient ring $\mathcal{O}_Z = \mathbb{C}[x_{1\dots4}]/I_Z$ breaks up into 1-dimensional eigenspaces $b \in A^\vee$, each represented by a monomial x^{m_b}; every other monomial in $\mathbb{C}[x_{1\dots4}]$ is then a multiple of x^{m_b} modulo the ideal I_Z.

The notes `AH4.pdf` on the paper website describes a Magma implementation in a slightly specialised case. (The opposition also have a simpler toric algorithm based on Groebner bases for the birational component.) There are many cases for which a crepant resolution exists, but A-Hilb $\mathbb{C}^4$ is not a crepant resolution – in simple cases, it may be a blowup of a crepant resolution. For $n = 4$, the restricted groups $A = \frac{1}{r}(a, b, 1, 1)$ treated here have this property whenever their coarse subdivision includes a trap of height ≥ 1.

What do we know about A-Hilb for A in $\mathrm{GL}(3, \mathbb{C})$? Most of the material of this paper goes badly wrong if we do not have the assumptions (1–4) of Theorem 2.5.

When we cut up simplicial cones for a general diagonal groups in $A \subset \mathrm{GL}(3, \mathbb{C})$, some of the bits are things that we know how to do by the terminal case $\frac{1}{r}(1, a, r-a)$ treated in Kedzierski's thesis, and some we know how to do by the more recent $\frac{1}{r}(a, b, 1)$ with $a + b < r - 1$. We can also take the relative canonical model, which has $\frac{1}{r}(1, a, r-a)$ points and also divisors on which K is ample. It is possible that by doing this we get a treatment of A-Hilb $\mathbb{C}^n$ for a reasonable class of groups A in $\mathrm{GL}(3)$. The question is, how large?

The monstrous case $\frac{1}{30}(1, 6, 10, 13)$ More general discussion of the monstrous case $\frac{1}{30}(1, 6, 10, 13)$.

The notes `bad.pdf` on the paper website discuss the pathological example $\frac{1}{30}(1, 6, 10, 13)$, which has a crepant resolution, but for which A-Hilb $\mathbb{C}^4$ has a 5-dimensional component concentrated at the origin. The A-Hilb is locally disconnected, with local equations

$$b(def - 1) = b(f - ac) = 0 \quad \text{in} \quad \mathbb{A}^6_{\langle a,b,c,d,e,f\rangle}. \tag{6.1}$$

Whether it is actually disconnected is an interesting question. (This is raised as a question in recent work on multigraded Hilbert schemes.)

Historical note Nakamura originally suggested that calculating G-Hilb sometimes gives a crepant resolution. We take this idea further. Even when it does not, it often indicates useful directions for looking for a preferred class of resolutions. In any case, the point is to have a resolution with an interpretation as a moduli space (of A-equivariant sheaves, or of quiver representations).

Caution This paper uses at many points the assumption that a crepant resolution exists (the equivalent conditions of Theorem 2.5). Most of what we say here goes badly wrong otherwise.

Question on the link Is it true that link A' also computes bits of A-Hilb even when not basic (i.e., A-Hilb has singularities like $\frac{1}{\nu}(1,1)$ when the link of A' has those intervals in its Newton boundary?) For example, the trap like constructions with $k \equiv \nu \bmod n-2$? The question is about the special locus in A-Hilb where none of the z_i are basic in $\mathcal{O}_Z$, but are multiples of some $x^\alpha y^\beta$. (The special centre $z_i = 0$ that was blown up in earlier drafts.) This is maybe something like a $\frac{1}{\nu}(1,1)$ affine space bundle over $\overline{\overline{A}}$-Hilb?

Other subgroups Are there other general classes of subgroups of $\mathrm{SL}(n,\mathbb{C})$ for which a similar analysis is possible? For example, image a class of groups for which most of the junior triangle Δ is triangulated in a conventional way, leaving a neighbourhood of two or more vertices needing separate treatment with exterior triangles and traps.

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