L3

Q: What is $\mathrm{Aut}\,D(X)$ when $\omega_X \simeq \mathcal{O}_X$?

[Beauville-Bogomolov]: Let X be sm. proj. with $c_1(X)=0$. Then there exists a finite étale cover $\tilde{X} \to X$ which decomposes as:

$$\tilde{X} \simeq \prod_i A_i \times \prod_j Y_j \times \prod_k Z_k$$

where the

- A_i are simple ab. vars (i.e. not isogenous to a prod. of ab. var of lower dim.)
- Y_j are strict Calabi-Yau vars of dim ≥ 3 (i.e. simply conn & $h^{i,0} = h^0(\Omega^i_Y) = 0$ $\forall\, 0 < i < \dim Y$)
- Z_k are compact hyperkähler vars or irred. holo. sympl. vars. (i.e. simply conn. & unique non-deg holo 2-form σ, $H^0(\Omega^2_Z) \simeq \mathbb{C}\sigma$)

"governed by $\mathbb{S}^n$ & $\mathbb{P}^n$"
$H^*(Y,\mathcal{O}) \simeq H^*(\mathbb{S}^n, \mathbb{C})$
$H^*(Z,\mathcal{O}) \simeq H^*(\mathbb{P}^n, \mathbb{C})$
mirror symm. predicts Lag. fib's $Y \to \mathbb{S}^n$, $Z \to \mathbb{P}^n$

{and of course none on AVs of dim > 1 since everything deforms, i.e. $\dim \mathrm{Ext}^1 \geq 2$}

$\to$ no sph. vbs on CYs of dim > 2 since $\mathrm{Ext}^2(E,E) \xrightarrow{tr} H^2(\mathcal{O}_Y) \simeq \mathbb{C}$.
$\leadsto$ so a new, and different, notion is needed for HKs. (depending on $\mathbb{P}^n$)

$\Rightarrow$ A, Y, Z are the building blocks of all compact $\mathbb{C}$ vars with trivial canonical bundle. ($\omega_X \simeq \mathcal{O}_X$).

$\leadsto$ Q': Can we describe $\mathrm{Aut}\,D(A)$, $\mathrm{Aut}\,D(Y)$, $\mathrm{Aut}\,D(Z)$?
✓ Orlov (for $\mathrm{Aut}\,D(A)$); wide open ! (for $\mathrm{Aut}\,D(Y)$, $\mathrm{Aut}\,D(Z)$)

Examples: "The biggest source of examples of hyperkählers comes from moduli spaces of Giesker-stable sh. on ab. or K3 surf."

① Let $\mathcal{M}_S^H(v)$ = moduli space of H-stable sheaves on a K3 surf S with primitive Mukai vector.

[Mukai, Huybrechts, O'Grady, Yoshioka]

Then $\mathcal{M}_S^H(v)$ is hyperkähler and $\mathcal{M}_S^H(v) \overset{\text{def}}{\rightsquigarrow} S^{[\frac{v^2}{2}+1]}$

② Let
$$\begin{array}{ccc} K_A^H(v) & \hookrightarrow & \mathcal{M}_A^H(v) \\ \downarrow & & \downarrow (\det \times \det \Phi_P) \\ (e,\hat{e}) & \hookrightarrow & A \times \hat{A} \end{array}$$
be the moduli space of H-stable sh. with trivial determinant & 'codeterminant'

Then $K_A^H(v)$ is HK and $K_A^H(v) \overset{\text{def}}{\rightsquigarrow} K_{\frac{v^2}{2}-1}$ where K_n is the generalised Kummer variety, i.e.
$$\begin{array}{ccc} m^{-1}(e) =: K_n & \hookrightarrow & A^{[n+1]} \\ \downarrow & & \downarrow m = \mathrm{Alb} \\ e & \hookrightarrow & A \end{array}$$

③ Two sporadic examples of dim^n ten and six due to O'Grady. Let v be a primitive Mukai vector with $v^2 = 2$ then $\mathcal{M}_S^H(2v)$ and $K_A^H(2v)$ admit sympl. resol^{n}s: $\widetilde{\mathcal{M}} \longrightarrow \mathcal{M}$ & $\widetilde{K} \longrightarrow K$

obtained by blowing up the (red) sing. locus $\mathrm{Sym}^2\mathcal{M}(v)$ & $\mathrm{Sym}^2 K(v)$ resp.

⇝ Up to deformation, these are <u>all</u> the hyperkähler vars we know!

<u>Def^n</u>: An obj $E \in D(X)$ is a $\mathbb{P}^n$-obj if $\mathrm{Ext}^*(E,E) \simeq H^*(\mathbb{P}^n, \mathbb{C}) \simeq \mathbb{C}[h]/h^{n+1}$ (as gr. rings) $(E \otimes \omega_X = E)$

<u>Examples</u>: ① Suppose X = HK of dim^n $2n$ and $\mathbb{P}^n \subset X$ with $N_{\mathbb{P}^n/X} \simeq \Omega_{\mathbb{P}^n}$ (automatic by Mukai) then $\mathcal{O}_{\mathbb{P}^n}(k) \in D(X)$ is a $\mathbb{P}^n$-object.

Indeed, $\mathrm{Ext}_X^q(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \simeq \Lambda^q N_{\mathbb{P}^n/X} \simeq \Omega^q_{\mathbb{P}^n}$

and so the local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q_X(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})) \Longrightarrow \mathrm{Ext}^{p+q}_X(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})$$

provides a ring isom: $\mathrm{Ext}^*_X(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \simeq H^*(\mathbb{P}^n, \Omega^*_{\mathbb{P}^n}) \simeq H^*(\mathbb{P}^n, \mathbb{C})$.

② Any line b. $\mathcal{L}$ on $X = HK$ is a $\mathbb{P}^n$-object since $\mathrm{Ext}^*_X(\mathcal{L}, \mathcal{L}) \simeq H^*(X, \mathcal{O}_X) \simeq H^*(\mathbb{P}^n, \mathbb{C})$.

Given a $\mathbb{P}^n$-obj $E \in D(X)$, we can view the generator $h \in \mathrm{Ext}^2(E,E)$ as a morphism $h: E \to E[2]$ in the derived category.

Similarly, using the natural isom. $\mathrm{Ext}^2(E,E) \simeq \mathrm{Ext}^2(E^\vee, E^\vee)$, $h \longmapsto h^\vee$

we obtain a natural map $h^\vee: E^\vee \to E^\vee[2]$ and can thus consider the morphism:

$$H: E^\vee \boxtimes E[-2] \xrightarrow{h^\vee \boxtimes 1 - 1 \boxtimes h} E^\vee \boxtimes E \quad \text{on } X \times X.$$

We can complete this morphism to a distinguished triangle and splice it together with the spherical twist triangle from last time:

$$\begin{array}{ccccc}
 & & T_E[-1] & \longrightarrow & P_E[-1] \\
 & \overset{\text{unique lift}}{\nearrow} & \downarrow & & \downarrow \\
E^\vee \boxtimes E[-2] & \xrightarrow{H} & E^\vee \boxtimes E & \longrightarrow & \mathrm{cone}(H) \\
 & & \downarrow {\scriptstyle tr} & & \downarrow \\
 & & \mathcal{O}_\Delta & = & \mathcal{O}_\Delta
\end{array}$$

note: We have been writing T_E for the cone of the natural map $\mathcal{H}om(E,F) \otimes E \xrightarrow{ev} F$ when E is sph. but the cone makes sense for any $E \in D(X)$ (but may not give an equivalence)

To see that there is a unique lift of H, we apply $\mathrm{Hom}(-, \mathcal{O}_\Delta)$ to the middle Δ.

to get:

$$f \longmapsto f\circ H = h - h = 0$$

$$\begin{array}{ccccccc} Ext^{-1}(E^\vee \boxtimes E[-2], \mathcal{O}_\Delta) & \to & Hom(cone(H), \mathcal{O}_\Delta) & \to & Hom(E^\vee \boxtimes E, \mathcal{O}_\Delta) & \to & Hom(E^\vee \boxtimes E[-2], \mathcal{O}_\Delta) \\ \cong & & & & \cong & & \cong \\ Ext^1(E,E) = 0 & & & & Ext^0(E,E) & & Ext^2(E,E) \end{array}$$

- boundary map being zero implies middle map is surj, i.e. we can always lift H.
- Since $Ext^{-1}(E^\vee \boxtimes E[-2], \mathcal{O}_\Delta) = 0$ then the middle map is actually an isom and so the lift of H is unique.

⤳ In other words, the trace map factorises over cone $\to \mathcal{O}_\Delta$.

Now we can define $P_E := cone(E^\vee \boxtimes E[-1] \xrightarrow{\tilde{H}[1]} T_E)$ which, by the octahedral axiom, is equiv to
$$cone(cone(H) \longrightarrow \mathcal{O}_\Delta) \in D(X \times X),$$
i.e. the kernel of P_E is obtained from a <u>double</u> cone constr$^{\wedge}$.

- Notice that the induced actions $P_E^K : K(X) \to K(X)$ and $P_E^H : H^*(X,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ are both equal to the identity since $[cone(H)] = [E^\vee \boxtimes E] + [E^\vee \boxtimes E[-1]] = 0$ in K-theory.

Explicitly, the P-twist P_E assoc. to E acts on objs $F \in D(X)$ as follows:
$$P_E(F) = cone(cone(Ext^{*-2}(E,F) \otimes E \xrightarrow{H} Ext^*(E \otimes F) \otimes E) \longrightarrow F)$$

For example, if $F \in E^\perp := \{ G \mid Ext^*(E,G) = 0 \}$ then $\Phi_{cone(H)}(F) = 0$ and therefore $P_E(F) = \Phi_{\mathcal{O}_\Delta}(F) = F$.

Similarly, if we apply P_E to E then cone(H) is the cone on
$$E[-2] \otimes H^*(\mathbb{P}^n, \mathbb{C}) \xrightarrow{h} E \otimes H^*(\mathbb{P}^n, \mathbb{C})$$

i.e. if $H^*(\mathbb{P}^n, \mathbb{C}) \simeq \mathbb{C}[h]/h^{n+1}$ then $\mathrm{cone}(H) = E \oplus E[-2n-1]$

$$\Rightarrow \mathrm{cone}(\mathrm{cone}(H) \to \mathcal{O}_\Delta) \simeq \mathrm{cone}(E \oplus E[-2n-1] \to E) \simeq E[-2n].$$

That is, $P_E : E \mapsto E[-2n]$ & $E^\perp \to E^\perp$.

Since $E \cup E^\perp$ is a spanning class this immediately shows that P_E is fully faithful: $\mathrm{Hom}^i(F, F') \xrightarrow{\sim} \mathrm{Hom}^i(P_E(F), P_E(F'))$ $\forall i$.

We are implicitly assuming $\omega_X \simeq \mathcal{O}_X$ throughout and so this is actually an equiv: $P_E(F \otimes \omega_X) \simeq P_E(F) \otimes \omega_X$ $\forall F \in$ spanning cl.
[Bridgeland's criterion]

$\rightsquigarrow$ To summarise, every $\mathbb{P}^n$-obj gives rise to an auto $P_E \in \mathrm{Aut}\, D(X)$.
$\hookrightarrow$ "Huybrechts-Thomas twist"

Relationship between $\mathbb{P}^n$-objects and spherical objects

In dim > 2 the notions are genuinely different. However, in many examples, $\mathbb{P}^n$-objects should be thought of as "hyperplane sections" of spherical objects.

More precisely, suppose

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \mathcal{X} \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & C \end{array}$$

$\mathcal{X}$ ← smooth family
C ← smooth curve. (with parameter t, say)

The family $\mathcal{X}$, viewed as a deformation of X, induces the Kodaira-Spencer class $\kappa(\mathcal{X}) \in H^1(X, T_X)$, which is (by def^n) the ext^n class of the normal bundle seq: $T_X \to T_{\mathcal{X}}|_X \to \mathcal{O}_X$
(mult^n by t induces trivialisation of the normal b.)

The sequence can be dualised to get $\mathcal{O}_X \to \Omega_{\mathcal{X}}|_X \to \Omega_X$ and the KS class $\kappa(\mathcal{X}) \in H^1(X, T_X) \simeq \mathrm{Ext}^1(\mathcal{O}_X, T_X) \simeq \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ can be viewed as its boundary morphism

$$\kappa(\mathcal{X}) : \Omega_X \longrightarrow \mathcal{O}_X[1].$$

Now, let $\Delta \subset X \times X$ be the diagonal and 2Δ be its double:

$$\left.\begin{array}{ccccc} \mathcal{I}_\Delta^2 & \longrightarrow & \mathcal{I}_\Delta & \longrightarrow & \Omega_\Delta \\ \| & & \downarrow & & \downarrow \\ \mathcal{I}_\Delta^2 & \longrightarrow & \mathcal{O}_{X\times X} & \longrightarrow & \mathcal{O}_{2\Delta} \\ & & \downarrow & & \downarrow \\ & & \mathcal{O}_\Delta & = & \mathcal{O}_\Delta \end{array}\right\}$$

where $\Omega_\Delta := \Delta_* \Omega_X = \mathcal{I}_\Delta / \mathcal{I}_\Delta^2$

View as seq of FM kernels on $X \times X$ and apply to any object $E \in D(X)$ to get the following natural triangle:

$$E \otimes \Omega_X \longrightarrow J(E) \longrightarrow E \xrightarrow{A(E)} E \otimes \Omega_X[1]$$

the ext^1 class is (by def^{n}) the Atiyah class

$$A(E) \in \mathrm{Ext}^1(E, E \otimes \Omega_X)$$

[$J(E) := \pi_{2*}(\pi_1^*(E) \otimes \mathcal{O}_{2\Delta})$ is called the first jet space of E]

The product $A(E) \cdot \kappa(\mathcal{X}) \in \mathrm{Ext}^2(E, E)$ can be described as the composition:

$$\begin{array}{ccccc} C[-1] & \longrightarrow & \Omega_{\mathcal{X}}|_X & \longrightarrow & J(E) \\ \downarrow & & \downarrow & & \| \\ E[-1] & \xrightarrow{A(E)} & E \otimes \Omega_X & \longrightarrow & J(E) \\ {\scriptstyle A(E)\cdot\kappa(\mathcal{X})}\downarrow & \circlearrowleft & \downarrow {\scriptstyle 1\otimes\kappa(\mathcal{X})} & & \\ E[1] & = & E \otimes \mathcal{O}_X[1] & & \end{array}$$

(obstruction to deforming E sideways to first order to neighbouring fibres in the family.)

If $j : X \hookrightarrow \mathcal{X}$ as above then HT show that there is a functorial isom $C \simeq j^* j_* E$, i.e. the boundary map of the std triangle

$$\underbrace{E \otimes \mathcal{O}_X(-X)[1]}_{E[1]} \longrightarrow j^* j_* E \longrightarrow E \xrightarrow{A \cdot K} \underbrace{E \otimes \mathcal{O}_X(-X)[2]}_{E[2]}$$

can be identified with $A(E) \cdot K(\mathcal{X}) \in \mathrm{Ext}^2(E,E)$.

If $E \in D(X)$ is a $\mathbb{P}^n$-object st. $A(E) \cdot K(\mathcal{X}) \neq 0$ (i.e. does not deform sideways in a 1-dim$^{\ell}$ family) then $j_* E \in D(\mathcal{X})$ is sph.

Indeed, apply $\mathrm{Hom}(-, E)$ to the above triangle to get

$$\mathrm{Ext}^k_X(E,E) \longrightarrow \mathrm{Ext}^k_X(j^* j_* E, E) \longrightarrow \mathrm{Ext}^{k-1}_X(E,E) \xrightarrow{\delta} \mathrm{Ext}^{k+1}_X(E,E)$$

$$\mathrm{Ext}^k_X(j^* j_* E, E) \cong \mathrm{Ext}^k_{\mathcal{X}}(j_* E, j_* E)$$

The boundary morphism δ is given by cup-product with $A(E) \cdot K(\mathcal{X})$ which, by assumption, can be taken to be the degree 2 generator of $\mathrm{Ext}^*(E,E)$. Therefore, δ is an isom for $1 \leq k \leq 2n-1$ and hence

$$\mathrm{Ext}^*_{\mathcal{X}}(j_* E, j_* E) \simeq \mathbb{C} \oplus \mathbb{C}[-2n-1].$$

<u>Example</u>: If $\mathbb{P}^n \subset X$ with $N_{\mathbb{P}^n/X} \simeq \Omega_{\mathbb{P}^n}$ then we can identify the formal nbhd of $\mathbb{P}^n \subset X$ with the formal nbhd of $\mathbb{P}^n \subset |\Omega_{\mathbb{P}^n}|$ in the linearised normal bundle, i.e. assoc. affine bundle. In this linear situation, we can use the Euler seq

$$\begin{array}{ccccc} \Omega_{\mathbb{P}^n} & \xrightarrow{j} & |\mathcal{O}(-1)^{\oplus n+1}| & =: & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & |\mathcal{O}| \simeq \mathbb{A}^1 & =: & C \end{array}$$

In such a situation, where a $\mathbb{P}^n$-obj becomes sph on an ambient space, the associated twists intertwine with one another. That is, we have a commutative diagram of the form:

$$\begin{array}{ccc} D(X) & \xrightarrow{\ j_*\ } & D(\mathcal{X}) \\ P_E \big\downarrow & \circlearrowright & \big\downarrow T_{j_*E} \\ D(X) & \xrightarrow[\ j_*\ ]{} & D(\mathcal{X}) \end{array}$$

P-twist $\rightsquigarrow$ P_E; spherical twist $\rightsquigarrow$ T_{j_*E}

$$\boxed{j_* \circ P_E \simeq T_{j_*E} \circ j_*}$$

↳ The spherical twist becomes the $\mathbb{P}^n$-twist on the special fibre $\mathcal{X}_0 = X$.

[Proof of this result is an application of Chen's Lemma].

- If $E \in D(X)$ is a $\mathbb{P}^1$-obj, i.e. $\dim X = 2$, then $P_E \simeq T_E^{\,2}$.
  - ↳ this will be easier to prove once we've introduced $\mathbb{P}$-fnctrs.

Can rephrase Bridgeland's conj for K3s as saying:

$$\mathrm{Aut}^\circ D(K3) = \langle \mathbb{P}\text{-twists} \rangle$$

i.e. gen^d by $\mathbb{P}$-twists around $\mathbb{P}^1$-objs.

⇝ Would like to generalise this to a hyper-Bridgeland conj:

$$\mathrm{Aut}^\circ D(HK) = \langle \mathbb{P}\text{-twists} \rangle$$

We will see next time that it is not enough if we only consider twists around $\mathbb{P}$-objects but might be plausible if we allow twists around "$\mathbb{P}$-functors".