# EXAMPLES OF SPHERICAL TWISTS

## EXAMPLE 1:

X - AN ELLIPTIC CURVE

$\mathcal{E} = \mathcal{O}_p$, THE SKYSCRAPER SHEAF OF A POINT $p \in X$.

$\forall$ POINT $q \in X$ WE HAVE

1) $p \neq q \Rightarrow \mathcal{O}_q \in \mathcal{E}^{\perp} \Rightarrow T_{\mathcal{E}}(\mathcal{O}_q) = \mathcal{O}_q$

2) $p = q \Rightarrow T_{\mathcal{E}}(\mathcal{O}_q) = T_{\mathcal{E}}(\mathcal{E}) = \mathcal{E}[-\cancel{\dim X}^{\,0} + 1] = \mathcal{O}_p$

BY COMPUTATION IN THE LAST LECTURE

QUESTION: IS $T_{\mathcal{E}}$ THE IDENTITY FUNCTOR?

ANSWER: NO, BUT THE ABOVE IMPLIES THAT $T_{\mathcal{E}} = (-) \otimes \mathcal{L}$ FOR SOME $\mathcal{L} \in \operatorname{Pic} X$. WHICH $\mathcal{L}$?

T. Bridgeland, "Equivalences of triangulated categories...", Prop. 4.2 & Lemma 4.3

FACT: THE FUNCTORIAL EXACT TRIANGLE

WORKS FOR ANY X

$$R\mathrm{Hom}(E,-) \otimes_{\mathbb{C}} E \to \mathrm{id}_{D(X)} \to T_E \qquad (\dagger)$$

IS CONSTRUCTED FROM THE EXACT TRIANGLE

$$E^{\vee} \boxtimes E \xrightarrow{\varepsilon} \mathcal{O}_{\Delta} \to \mathrm{Cone}(\varepsilon) \xrightarrow{[1]} \qquad (\dagger\dagger)$$

OF FOURIER-MUKAI KERNELS IN $D(X \times X)$.

$X \xrightarrow{\Delta} X \times X$, $x \mapsto (x,x)$

RECALL: ANY $M \in D(X \times X)$ INDUCES $D(X) \xrightarrow{\Phi_M} D(X)$ WHERE $\Phi_M := R\pi_{2*}(M \overset{L}{\otimes} \pi_1^*(-))$

HERE $\varepsilon$ IS THE COMPOSITION

$$E^{\vee} \boxtimes E \xrightarrow{\text{RESTRICT TO } \Delta} \Delta_*(E^{\vee} \otimes E) \xrightarrow{\Delta_* \mathrm{eval}} \Delta_* \mathcal{O}_X.$$

APPLY THIS TO $E = \mathcal{O}_p$:

$$\mathcal{O}_p \simeq \mathrm{Cone}(\underbrace{\mathcal{O}_X(-p) \hookrightarrow \mathcal{O}_X}_{\text{LOC. FREE RES-N}}) \Rightarrow \mathcal{O}_p^{\vee} \simeq \mathrm{Cone}(\mathcal{O}_X \hookrightarrow \mathcal{O}_X(p))[-1] \simeq \mathcal{O}_p[-1]$$

THUS $E^{\vee} \boxtimes E \simeq \mathcal{O}_{p,p}[-1] \simeq \Delta_* \mathcal{O}_p[-1]$ & $E^{\vee} \otimes E \simeq \{\mathcal{O}_X \to \mathcal{O}_X(p)\} \otimes \mathcal{O}_p \simeq \{\mathcal{O}_p \xrightarrow{0} \mathcal{O}_p\} \simeq \mathcal{O}_p \oplus \mathcal{O}_p[-1]$

$\Rightarrow$ RESTRICT TO $\Delta$ $\rightsquigarrow$ $\Delta_* \mathcal{O}_p[-1] \xrightarrow{0 \oplus \mathrm{id}} \Delta_* \mathcal{O}_p \oplus \Delta_* \mathcal{O}_p[-1]$

$\Delta_*(\mathrm{EVAL}) \rightsquigarrow \Delta_*(\mathcal{O}_p \oplus \mathcal{O}_p[-1] \xrightarrow{0 + \alpha} \mathcal{O}_X)$ WHERE $\alpha$ COMES FROM

$$\mathcal{O}_X \to \mathcal{O}_X(p) \to \mathcal{O}_p \xrightarrow{\alpha} \mathcal{O}_X[1].$$

THUS $E^{\vee} \boxtimes E \xrightarrow{\varepsilon} \Delta_* \mathcal{O}_X$ BECOMES $\Delta_*(\mathcal{O}_p[-1] \xrightarrow{\alpha} \mathcal{O}_X)$.

$\therefore$ FM KERNEL OF $T_E = \mathrm{Cone}(\varepsilon) = \Delta_* \mathcal{O}_X(p)$

WE CONCLUDE THAT $T_E = (-) \otimes \mathcal{O}_X(p)$ ← LINE BUNDLE

## EXAMPLE 2:

X - AN ELLIPTIC CURVE

E - $\mathcal{O}_X$, THE STRUCTURE SHEAF

THE MAP $E^{\vee} \boxtimes E \xrightarrow{\varepsilon} \Delta_* \mathcal{O}_X$ BECOMES

$$\mathcal{O}_{X \times X} \simeq \mathcal{O}_X^{\vee} \boxtimes \mathcal{O}_X \xrightarrow{\text{RESTR. TO } \Delta} \Delta_*(\mathcal{O}_X^{\vee} \otimes \mathcal{O}_X) \xrightarrow{\Delta_*(\mathrm{iso})} \Delta_* \mathcal{O}_X$$

AND THUS $\mathrm{Cone}(\varepsilon) \simeq \mathrm{Cone}(\mathcal{O}_{X \times X} \xrightarrow{\text{restr}} \mathcal{O}_{\Delta}) \simeq \mathcal{I}_{\Delta}[1]$, IDEAL SHEAF OF $X \overset{\Delta}{\hookrightarrow} X \times X$

$\therefore T_E(-) \simeq \pi_{2*}(\mathcal{I}_{\Delta} \otimes \pi_1^*(-))$. ← WHAT DOES THIS DO?

$\forall p \in X$ $T_E(\mathcal{O}_p) \simeq i^*_{(p,x)} \mathcal{I}_{\Delta} \simeq \mathcal{O}_X(-p)$  $X \xrightarrow{i_{p,x}} X \times X$, $q \mapsto (p,q)$

WE HAVE

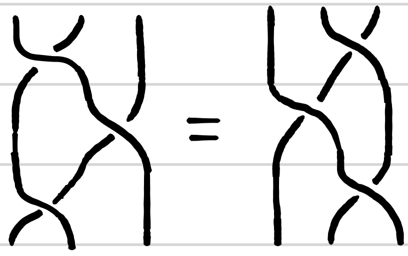
$$X \simeq \mathrm{Pic}^{-1}(X) \overset{\text{CHOICE OF } p_0 \in X}{\simeq} \mathrm{Pic}^0(X) =: \hat{X}$$

$$p \mapsto \mathcal{O}_X(-p) \mapsto \mathcal{O}_X(p_0 - p)$$

THIS IDENTIFIES $X \times X$ WITH $X \times \hat{X}$, AND $\mathcal{I}_{\Delta}$ WITH THE POINCARE LINE BNDL.

THUS $T_E$ IS (A SHIFT OF) THE ORIGINAL FM TRANSFORM $D(X) \xrightarrow{PLB} D(\hat{X})$.

Recall: The braid gp $B_{m+1}$ on $(m+1)$-strands is generated by elts $\beta_1, \dots, \beta_m$ such that



$\leftrightarrow$ $\beta_i \cdot \beta_{i+1} \cdot \beta_i = \beta_{i+1} \cdot \beta_i \cdot \beta_{i+1} \quad \forall\, 1 \leq i \leq m$

$\beta_i \cdot \beta_j = \beta_j \cdot \beta_i$ if $|i-j| \geq 2$.

An $A_m$-configuration of sph. objs in $D(X)$ consists of sph. objs $E_1, \dots, E_m \in D(X)$ s.t.

$$\bigoplus_k \dim \mathrm{Hom}^k(E_i, E_j) = \begin{cases} 1 & \text{if } |i-j|=1 \\ 0 & |i-j|>1 \end{cases}$$

[Seidel-Thomas]: Given such a configuration, the induced twists $T_i := T_{E_i}$ satisfy the braid rels, i.e. as above with $\beta_i = T_i$.

That is, an $A_m$-config. of sph. objs in $D(X)$ induces a gp hom: $B_{m+1} \hookrightarrow \mathrm{Aut}\, D(X)$ i.e. a rep$^{\underline{n}}$ of $B_{m+1}$ on $D(X)$.

$\Rightarrow$ Key: $E \in D(X)$ sph. obj & $\Phi \in \mathrm{Aut}\, D(X)$ then $T_{\Phi(E)} \simeq \Phi\, T_E\, \Phi^{-1}$.

$\boxed{\text{even dim}}$ $\boxed{X = K3}$

Recall: $\pi : \mathrm{Aut}\, D(X) \longrightarrow \mathrm{Aut}^+ H^*(X, \mathbb{Z})$ ; $\Phi_\mathcal{E} \longmapsto \Phi_\mathcal{E}^H$

where $\Phi_\mathcal{E}^H(\alpha) = \pi_{2*}(\pi_1^*(\alpha) \cdot v(\mathcal{E}))$ & $v(\mathcal{E}) := ch(\mathcal{E}) \cdot \sqrt{td\, X} \in H^0(X) \oplus NS(X) \oplus H^4(X)$

$NS(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$

Eg. If $E \in D(X)$ is sph. then $\pi(T_E) = T_E^H$ is the reflection in the hyperplane orthogonal to $v(E)$, i.e. $T_E^H : v \longmapsto v + \langle v(E) . v \rangle v(E)$

Mukai pairing

In other words, $(T_E^H)^2 = id_{H^*(X)} \Rightarrow T_E^2 \in \mathrm{Aut}^0 D(X)$;

$E \mapsto E[2-2\dim X]$ $E^\perp \mapsto E^\perp$ so cannot be id.

$\delta = (-2)$-class

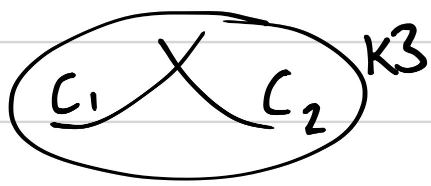
Expectation: "$\mathrm{Aut}^0 D(X)$ gen$^{\underline{d}}$ by sph. tw" $\Rightarrow$ see Tom's conj: $\mathrm{Aut}^0 = \pi_1(P^+ \setminus \bigcup \delta^\perp)$

Rmk: $\mathrm{Aut}\, D(X)$ has finite index inside $\mathrm{Aut}^+ H^*(X)$ but $\mathrm{Aut}^0 D(X)$ is not finitely gen$^{\underline{d}}$.

If the Mukai vectors $v(E_i)$ of an $A_m$-config. are lin. indep. then the braid gp action covers the Weyl gp action given by reflections in the hyperplanes orthog to $v(E_i)$:

$$\begin{tikzcd}
T_E^2 \arrow[r, squiggly] & PB_{m+1} \arrow[r, hook, "\sim"] \arrow[d] & \mathrm{Aut}^0 D(X) \arrow[d] & \textcolor{red}{?} \arrow[l, bend right, red] \\
T_E \arrow[r, squiggly] & B_{m+1} \arrow[r, hook] \arrow[d] & \mathrm{Aut}\, D(X) \arrow[r, two heads, blue] \arrow[d, "\pi"] & \textcolor{blue}{Q} \arrow[d, equal, blue, "?"] \\
T_E^H \arrow[r, squiggly] & W_m \arrow[r, hook] & \mathrm{Aut}^+ H^*(X) \arrow[r, two heads, blue] & \textcolor{blue}{Q'}
\end{tikzcd}$$

Example:

$X =$ 


$C_i \simeq \mathbb{P}^1$ $C_1 \cap C_2 = pt.$ $\mathrm{Pic}(X) \geq 3$.

Define $E_i := \mathcal{O}_{C_i}(-1)$.

locally resol$^n$ of Kleinian sing. $G = \mathbb{Z}/3\mathbb{Z}$. $Y \xrightarrow{BKR} \mathbb{C}^2/G \leftarrow \mathbb{C}^2$ $\Rightarrow$ $A_2$-config.

$B_3 \to \mathrm{Aut}D(X)$ ; $\beta_i \mapsto T_i = T_{E_i}$

Need to check $T_1 T_2 T_1 \simeq T_2 T_1 T_2$

$\Phi T_E \simeq T_{\Phi(E)} \Phi$ $\Rightarrow$ $\wr ?$

$$T_1 T_{T_2(E_1)} T_2 \simeq T_{T_1(T_2(E_1))} T_1 T_2$$

$\boxed{\underline{\text{Exc}}\text{: Show that } T_E \simeq T_{E[1]}.}$

$\Leftarrow$ enough to show $T_1 T_2 (E_1) \simeq E_2[k]$

Apply $T_1$ to this: $\overbrace{\mathrm{Hom}(E_2, E_1)}^{\mathbb{C}} \otimes E_2 \longrightarrow E_1 \longrightarrow T_2(E_1)$

to get $T_1(E_2) \xrightarrow{\varphi_1} T_1(E_1) \simeq E_1[-1] \longrightarrow T_1 T_2 (E_1)$.

By def$^n$, $T_1(E_2)$ fits into: $E_1[-2] \longrightarrow E_2 \longrightarrow T_1(E_2)$

$\Leftrightarrow$ $T_1(E_2) \xrightarrow{\varphi_2} E_1[-1] \longrightarrow E_2[1]$ $\quad \underset{\wr ?}{\mathbb{C}}$ $\mathrm{Hom}(E_2, E_1) \simeq \mathrm{Hom}(T_1(E_2), T_1(E_1))$

$\Rightarrow \varphi_1 = \varphi_2$ up to scalars.

Hence $T_1 T_2 (E_1) \simeq E_2[1]$ by TR3 (five lemma).

Pure braid gp: $PB_3 = \langle \beta_1^2, \beta_2^2, \beta_1\beta_2^2\beta_1^{-1} = (\beta_1\beta_2\beta_1^{-1})^2 \rangle$

beginning & end of each strand are in the same position.

$\beta_1\beta_2\beta_1^{-1} \leftrightarrow T_{T_1(\mathcal{O}_{C_2}(-1))} = T_{\mathcal{O}_{C_1 \cup C_2}(-1)}$

Indeed, $T_1 T_2 T_1^{-1} \simeq T_{T_1(E_2)} T_1 T_1^{-1}$

and $\mathcal{O}_{C_1}(-1)[-2] \longrightarrow \mathcal{O}_{C_2}(-1) \longrightarrow T_1(\mathcal{O}_{C_2}(-1)) = T_1(E_2)$.

$$\begin{tikzcd}
\mathcal{O}_{C_2}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{C_2} \arrow[r] \arrow[d] & \mathcal{O}_{C_1 \cap C_2} \arrow[d, equal] \\
\mathcal{O}_{C_1 \cup C_2} \arrow[r] \arrow[d] & \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \arrow[r] \arrow[d] & \mathcal{O}_{C_1 \cap C_2} \\
\mathcal{O}_{C_1} \arrow[r, equal] & \mathcal{O}_{C_1} &
\end{tikzcd}$$

Now apply $T_1$ to this diagram or just observe that the left column gives:

$\mathcal{O}_{C_1}(-1)[-1] \to \mathcal{O}_{C_2}(-1) \to \mathcal{O}_{C_1 \cup C_2}(-1)$.

$\Rightarrow T_1(E_2) \simeq T_{\mathcal{O}_{C_1}(-1)}(\mathcal{O}_{C_2}(-1)) \simeq \mathcal{O}_{C_1 \cup C_2}(-1)$.

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In general, $\mathrm{Hom}^*(E,F) = \begin{cases} 0 \iff T_E T_F = T_F T_E & \underline{\text{commute}} \\ 1 \iff T_E T_F T_E = T_F T_E T_F & \underline{\text{braid}} \\ \geq 2 \iff \langle T_E, T_F \rangle = \text{Free gp on 2 gens } \mathbb{Z}[E] * \mathbb{Z}[F]. \end{cases}$

Notice that the str seq for a sm. rat. curve consists of sph. objs

$\mathcal{O}(-c) \to \mathcal{O}_X \to \mathcal{O}_c \rightsquigarrow T_{\mathcal{O}(-c)} \to T_{\mathcal{O}_X} \to T_{\mathcal{O}_c}$

but $\mathcal{O}_c = T_{\mathcal{O}(-c)}\mathcal{O}_X$ and so $T_{\mathcal{O}_c} = T_{\mathcal{O}(-c)} T_{\mathcal{O}_X} T_{\mathcal{O}(-c)}^{-1}$

i.e. $T_{\mathcal{O}_c} \in \langle T_{\mathcal{O}(-c)}, T_{\mathcal{O}_X} \rangle$. That is, only two out of the three objs are needed to gen the same gp.