SPHERICAL OBJECTS

X - SMOOTH PROJ. VARIETY $/\mathbb{C}$

$D(X) := D^b_{Coh}(X)$, THE BOUNDED DERIVED CATEGORY OF COHERENT SHEAVES ON X.

DEFINITION: AN OBJECT $E \in D(X)$ IS SPHERICAL IF:

1) $\mathrm{Ext}^i_X(E,E) = \begin{cases} \mathbb{C} & i = 0 \text{ OR } \dim X \\ 0 & \text{OTHERWISE} \end{cases}$

2) $E \otimes \omega_X \simeq E$

EXAMPLES:

1). X - ELLIPTIC CURVE, $E = \mathcal{L} \in \mathrm{Pic}X$ ← SERRE DUALITY

$\mathrm{Ext}^1_X(\mathcal{L},\mathcal{L}) = \mathrm{Ext}^1_X(\mathcal{O}_X,\mathcal{O}_X) = H^1(\mathcal{O}_X)$ AND $H^0(\mathcal{O}_X) = H^1(\mathcal{O}_X) = \mathbb{C}$

$E = \mathcal{O}_p$, THE SKYSCRAPER SHEAF OF A POINT $p \in X$

GENERAL FACT: S SCHEME, $p \in S$ REGULAR POINT $\Rightarrow \mathrm{Ext}^i_S(\mathcal{O}_p,\mathcal{O}_p) = \Lambda^i T_{S,p}$ (COHOMOLOGIES OF KOSZUL CX.)

$\Rightarrow \mathrm{Ext}^0_X(\mathcal{O}_p,\mathcal{O}_p) = \mathrm{Ext}^1_X(\mathcal{O}_p,\mathcal{O}_p) = \mathbb{C}$

2). X - K3 SURFACE, $E = \mathcal{L} \in \mathrm{Pic}X$

$H^0_X(\mathcal{O}_X) = H^2_X(\mathcal{O}_X) = \mathbb{C}$, $H^1_X(\mathcal{O}_X) = 0$ (CY CONDITION) $\xrightarrow{\text{AS ABOVE}}$ $\mathcal{L}$ IS SPHERICAL

$E = \mathcal{O}_C$, WHERE C IS A CURVE WITH $C.C = -2$.

COMPUTATION $\Rightarrow \mathcal{O}_C$ IS SPHERICAL

3). X - CY 3-FOLD $\omega_X \simeq \mathcal{O}_X$, $H^i_X(\mathcal{O}_X) = 0$ $i \neq 0,3$

$E = \mathcal{L} \in \mathrm{Pic}X \xrightarrow{\text{AS ABOVE}} \mathcal{L}$ IS SPHERICAL

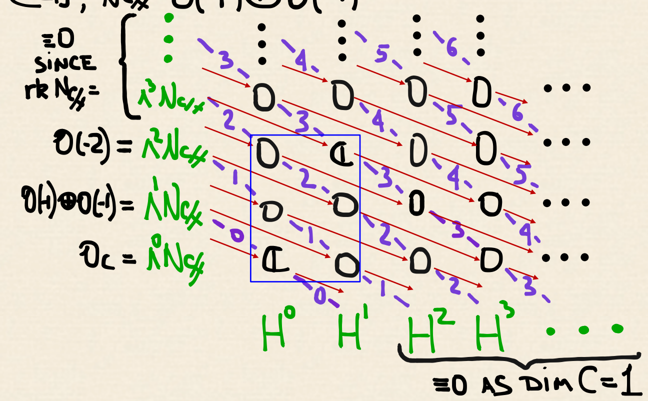
$E = \mathcal{O}_C$ OR $\mathcal{O}_D$, $C \simeq \mathbb{P}^1 \subset X$ WITH $N_{C/X} = \mathcal{O}(-1,-1)$, $D \simeq \mathbb{P}^2 \subset X$ WITH $N_X \simeq \mathcal{O}(-2)$

GENERAL COMPUTATION: S - SCHEME, $Z \xhookrightarrow{\iota} X$ A REGULAR IMMERSION ← X SMOOTH, Z SMOOTH $\Rightarrow Z \hookrightarrow X$ REGULAR

SPECTRAL SEQUENCE $E_2^{i,j} = (\Lambda^j N_{Z/X}) \Rightarrow \mathrm{Ext}^{i+j}_X(\mathcal{O}_Z,\mathcal{O}_Z)$

REASON: $R\mathrm{Hom}_X(\iota_*\mathcal{O}_Z, \iota_*\mathcal{O}_Z) = R\mathrm{Hom}_Z(L\iota^*\iota_*\mathcal{O}_Z,\mathcal{O}_Z) = R\Gamma(R\mathcal{E}xt_Z(L\iota^*\iota_*\mathcal{O}_Z,\mathcal{O}_Z))$ (J^{TH} COHOMOLOGY SHEAF $= \Lambda^J N_{Z/X}$)

$C \simeq \mathbb{P}^1$, $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$



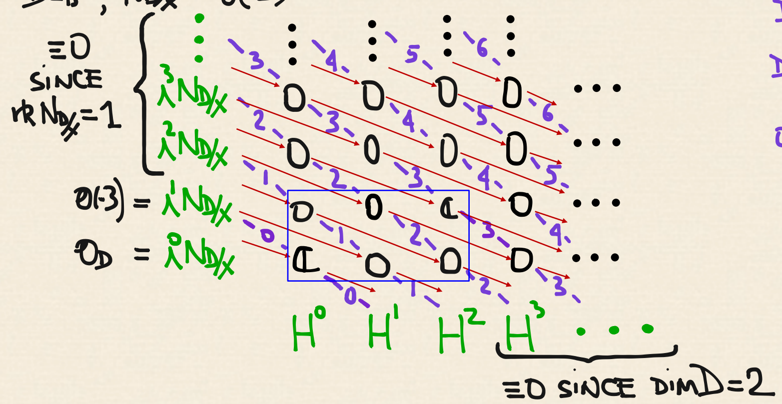
DIAGONAL 0 $\Rightarrow \mathrm{Ext}^0_X(\mathcal{O}_C,\mathcal{O}_C) = \mathbb{C}$

DIAGONAL 3 $\Rightarrow \mathrm{Ext}^3_X(\mathcal{O}_C,\mathcal{O}_C) = \mathbb{C}$

OTHER DIAGONALS $\Rightarrow \mathrm{Ext}^i_X(\mathcal{O}_C,\mathcal{O}_C) = 0$ $i \neq 0,3$

$\therefore \mathcal{O}_C$ IS SPHERICAL

$D = \mathbb{P}^2$, $N_{D/X} \simeq \mathcal{O}(-3)$



DIAGONAL 0 $\Rightarrow \mathrm{Ext}^0_X(\mathcal{O}_D,\mathcal{O}_D) = \mathbb{C}$

DIAGONAL 3 $\Rightarrow \mathrm{Ext}^3_X(\mathcal{O}_D,\mathcal{O}_D) = \mathbb{C}$

OTHER DIAGONALS $\Rightarrow \mathrm{Ext}^i_X(\mathcal{O}_D,\mathcal{O}_D) = 0$ $i \neq 0,3$

$\therefore \mathcal{O}_D$ IS SPHERICAL

PROPOSITION: Let $E \in D(X)$. There exists a natural functor $T_E : D(X) \to D(X)$ which fits into

$$RHom_X(E,-)\otimes_{\mathbb{C}} E \xrightarrow{\text{EVAL}} \mathrm{Id}_{D(X)} \longrightarrow T_E \xrightarrow{[1]} \qquad (\dagger)$$

FUNCTORIAL EXACT TRIANGLE

This functor is called **the twist** of $D(X)$ around E.

REMARKS:

1) The map EVAL in $(\dagger)$ is a derived version of the evaluation map $Hom(V,W)\otimes V \xrightarrow{d\otimes v \mapsto d(v)} W$

2) The exact triangle $(\dagger)$ implies that $\forall \mathcal{E}, \mathcal{F} \in Coh X$ we have a long exact sequence

$$\underset{\mathcal{H}^{-1}(\mathcal{F})}{0} \to T_{\mathcal{E}}^{-1}(\mathcal{F}) \to Hom_X(\mathcal{E},\mathcal{F})\otimes_{\mathbb{C}}\mathcal{E} \xrightarrow{ev} \mathcal{F} \to T_{\mathcal{E}}^{0}(\mathcal{F}) \to Ext_X^1(\mathcal{E},\mathcal{F})\otimes_{\mathbb{C}}\mathcal{E} \to \underset{\mathcal{H}^{1}(\mathcal{F})}{0}$$

and moreover

$$\forall i \geq 1 \quad T_{\mathcal{E}}^{i}(\mathcal{F}) = Ext_X^{i+1}(\mathcal{E},\mathcal{F})\otimes_{\mathbb{C}}\mathcal{E}$$

where $T_{\mathcal{E}}^{i}(\mathcal{F})$ = ith cohomology sheaf of $T_{\mathcal{E}}(\mathcal{F})$.

3) The functorial exact triangle $(\dagger)$ is constructed on the level of Fourier-Mukai kernels. That is, we write down a morphism of kernels which induces

$$RHom_X(E,-)\otimes_{\mathbb{C}} E \xrightarrow{\text{EVAL}} \mathrm{Id}_{D(X)}$$

and take its cone.

THEOREM (SEIDEL-THOMAS, '00):

Let $E \in D(X)$. If E is spherical, then T_E is an auto-equivalence of $D(X)$.

PROOF (A SKETCH):

We use a **general fact**: X, Y - smooth, proj. varieties $/\mathbb{C}$

A functor $\Phi : D(X) \to D(Y)$ is an auto-equivalence if and only if:

(1) $Hom_{D(X)}(A,B) \to Hom_{D(Y)}(\Phi A, \Phi B) \quad \forall A,B \in S$ ⟵ some spanning class of $D(X)$

(2) Φ commutes with Serre functors of $D(X)$ and $D(Y)$.

Here, a **spanning class** is a subset $S \subseteq D(X)$ such that $\forall A \in D(X)$

$$Hom^i_{D(X)}(B,A) = 0 \quad \forall B \in S \implies A \simeq 0$$

To get (2):

The Serre functor on $D(X)$ is $(-)\otimes \omega_X[\dim X]$.

Since an exact functor commutes with shifts, enough to show that T_E commutes with $(-)\otimes\omega_X$.

Take $F \in D(X)$. Apply $(\dagger)$ to F to get

$$RHom_X(E,F)\otimes E \xrightarrow{\text{EVAL}} F \longrightarrow T_E(F) \quad \text{EXACT TRIANGLE}$$

Since $(-)\otimes\omega_X$ is exact

$$RHom_X(E,F)\otimes E\otimes\omega_X \xrightarrow{\text{EVAL}} F\otimes\omega_X \longrightarrow T_E(F)\otimes\omega_X \quad \text{ALSO AN EXACT TRIANGLE}$$

Since $E \simeq E\otimes\omega_X \simeq E\otimes\omega_X^{-1}$

$$RHom_X(E,F\otimes\omega_X)\otimes E \xrightarrow{\text{EVAL}} F\otimes\omega_X \longrightarrow T_E(F)\otimes\omega_X \quad \text{ALSO AN EXACT TRIANGLE}$$

OTOH, APPLYING (†) TO $F \otimes \omega_X$ YIELDS

$$RHom_X(E, F\otimes\omega_X)\otimes E \xrightarrow{eval} F\otimes\omega_X \longrightarrow T_E(F\otimes\omega_X)$$ EXACT TRIANGLE

WE CONCLUDE THAT $T_E(F)\otimes\omega_X \simeq T_E(F\otimes\omega_X)$ AS REQUIRED.

TO GET (i):

USE THE SPANNING CLASS $\{E\}\cup E^{\perp}$ WHERE $E^{\perp}=\{A\in D(X) \mid Hom_{D(X)}(E,A)=0\}$

FOR ANY $F\in E^{\perp}$ WE HAVE $RHom_X(E,F)\simeq 0$ AND SO (†) YIELDS

$$0 \longrightarrow F \longrightarrow T_E(F)$$ EXACT TRIANGLE

AND THUS $F\simeq T_E(F)$.

THIS SHOWS $Hom_{D(X)}(A,B)\to Hom_{D(X)}(T_E(A),T_E(B))$ $\forall A,B\in\{E\}\cup E^{\perp}$ SUCH THAT $A\neq E$ OR $B\neq E$.

REMAINS TO CHECK THAT

$$(*)\quad Hom_{D(X)}(E,E)\longrightarrow Hom_{D(X)}(T_E(E),T_E(E))$$ IS AN ISO.

HERE ASSUME FOR SIMPLICITY THAT E IS A SHEAF $\mathcal{E}\in Coh\,X$.

WE HAVE L.E.S.

$$0\to T^{-1}_{\mathcal{E}}(\mathcal{E})\to \underbrace{Ext^0_X(\mathcal{E},\mathcal{E})}_{\mathbb{C}}\otimes_{\mathbb{C}}\mathcal{E}\xrightarrow[iso]{eval}\mathcal{E}\longrightarrow T^0_{\mathcal{E}}(\mathcal{E})\to \underbrace{Ext^1_X(\mathcal{E},\mathcal{E})}_{0}\to 0$$

AND THUS $T^{-1}_{\mathcal{E}}(\mathcal{E})=T^0_{\mathcal{E}}(\mathcal{E})=0$.

WE HAVE, MOREOVER,

$$T^i_{\mathcal{E}}(\mathcal{E})=Ext^{i+1}(\mathcal{E},\mathcal{E})\otimes_{\mathbb{C}}\mathcal{E}\qquad i\neq -1,0$$

WE CONCLUDE THAT $T_{\mathcal{E}}(\mathcal{E})=\mathcal{E}[\dim X+1]$ AND (*) IMMEDIATELY FOLLOWS.

THE GENERAL CASE IS TREATED SIMILARLY BUT WITH SLIGHTLY MORE DIAGRAM-CHASING INVOLVED.

$\square$