Generalised Dehn twists and Homological Mirror Symmetry $\boxed{L1}$

Rough sketch of the Physics background:

wiki: **QFT** = Quantum Field Theory is a theoretical framework which attempts to describe/model the subatomic particles.
A QFT treats particles as excited states of an underlying field
↳ Physicists call them field quanta.

⇝ **CFT** = Conformal Field Theory is a QFT which is invt under conformal transformations, i.e. f holo $\&$ $df \neq 0$ everywhere

scale invariance is important in the applications of CFTs in string theory. (stretching/distortion of spacetime etc).

⇝ **SCFT** = Super CFT = CFT + supersymmetry (SUSY) $(N=2)$

$N=2$ SCFT ⇝ $N=(2,2)$ SCFT
chiral half
two left/right moving supsymts

proposed ext^n of spacetime that relates bosons (with $\mathbb{Z}$-valued spin) with fermions (which have $\mathbb{Z}/2\mathbb{Z}$-valued spin) e.g. an electron is a fermion and supersymm predicts the existence of a "boson version" of it called the selectron with same mass energy and internal quantum no's except spin.

↳ However, no "super-partners" have been observed. Physicists say this is because the symmetry is "spontaneously broken".

- If supersymm is a true symm of nature then it would explain many mysterious properties of particle physics and help solve the cosmological constant problem, i.e. the energy density of the vacuum of space ↳ dark energy etc.

Mathematically, we should think of a <u>SCFT</u> as an infinite dimensional Lie algebra equipped with a $\mathbb{Z}_2$-grading.

e.g. super Virasoro alg with gens L_m & central elt c.

The data required for a SCFT are:
- a CY mfld X, i.e. $\omega_X \simeq \mathcal{O}_X$.
- a cx str $I \in \mathrm{End}(TX)$
- a complexified Kähler class $\beta + i\omega \in H^2(X, \mathbb{C})$.

really we should think of X as a real mfld equipped with a cx str. which makes it into a CY var i.e. $\exists$ a sympl. form $\omega \in H^0(X, \Omega^2)$ which is compatible with I.
i.e. $g(u,v) = \omega(u, Iv)$ where g is the Riemannian metric.

that is, the cx str. gives ω and so we only need to specify β which is called the <u>B-field</u>.

$\Rightarrow$ <u>SCFT</u> determined by $(X, I, \beta + i\omega)$ or equivalently by (X, ω, B) "β.

<u>Note</u>: Usually, the construction of a SCFT goes via a <u>non-linear σ-model</u> which we should think of as a locally defined function $\Sigma: M \to X$ from Minkowski space to our target mfld X. diff^ble map

In ptic, a SCFT depends on <u>both</u> the cx. and sympl. structures.

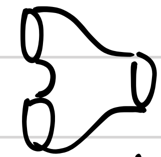
$\rightsquigarrow$ isolating those parts of the SCFT which only depend on the sympl. str. (A-side) or the cx. str. (B-side) is known as "<u>topological twisting</u>".

$$\text{SCFT} \begin{cases} \to \text{TCFT} \ \text{(A)} \\ \to \text{TCFT} \ \text{(B)} \end{cases}$$

where (A) ← sympl. data, (B) ← cx. data

TCFT = topological CFT = differential graded version of the Atiyah–Segal axiomatic def$^{\underline{n}}$ of a TFT ($=$ TQFT), i.e. fnctr $Z: \mathrm{Bord} \longrightarrow \mathrm{Vec}$

theory is topological on a derived level – its outputs are topologically invt up to higher htpies.

 $\longmapsto (V \otimes V \to V)$

s.t. $Z(\Sigma)$ = fin. gen$^{\underline{d}}$ $\mathbb{C}$-module

cl. orient$^{\underline{d}}$ sm. mfld. $Z(\Sigma) \in Z(\partial\Sigma)$

satisfy certain axioms.

Precisely, we should think of a TCFT as an A_∞-category, that is, a category whose associativity cond$^{\underline{n}}$ on morphisms is relaxed "up to higher coherent htpy".

Moreover, Kontsevich proposes that:

10 dim$^{\underline{l}}$ String thy
$= 3(\text{space}) + 1(\text{time})$
$+ 6(CY3)$ ← Quantum thy happens here

TCFT Ⓐ = derived Fukaya cat. of Lag. submflds $D\mathcal{F}uk(X,\omega)$

not the der. cat of an abelian cat.
& we have suppressed the twist by the B-field β.

& TCFT Ⓑ = derived cat. of coh. Sh $D(X)$.

we said these TCFTs were A_∞-cats so we should really say a DG enhancement of $D^b(\mathrm{Coh})$.
i.e. $D^b(\mathrm{Coh}) = H^0(D(X))$.

- $\mathrm{ob}(D\mathcal{F}uk(X,\omega))$ = triples (L, E, ∇) where $L \subset X$ is Lag. submfld i.e. $\omega|_L = 0$ and E is a v.b. on L with unitary conn. ∇.
- $\mathrm{mor}(D\mathcal{F}uk(X,\omega))$ "=" Floer cohomology gps.

- $\mathrm{ob}(D(X))$ = cx's of coh Sh.
  "$\mathrm{ob}(K(X)) = \mathrm{ob}(\mathrm{Kom}(X))$"
  htpy cat ch. cx's
- $\mathrm{mor}(D(X))$ = equiv classes of diagrams

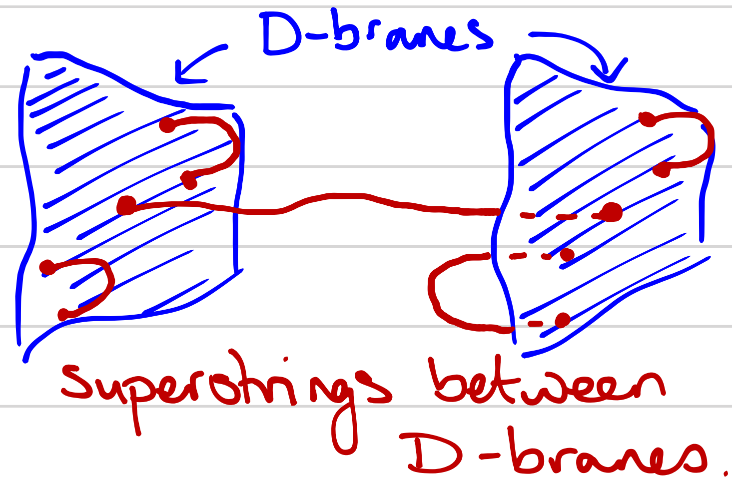
$$\begin{tikzcd} & G^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & \\ E^\bullet & & F^\bullet \end{tikzcd}$$

A physicist would say that "$D(X)$ is the cat. of branes in a topological twist of the sigma model" (Σ i.e. the B-twist)

objects of TCFTs are called <u>branes</u>
A-branes / B-branes.

Sometimes also D-branes for Dirichlet, i.e. bdary conditions for the eqn's i.e. pdes, governing the propagation of strings.

<u>Heuristic picture</u>:



<u>Homological Mirror Symmetry</u>: We say that two SCFTs (X, ω, B) & (X', ω', B') are <u>mirror</u> to each other if the associated Lie algebras are isomorphic in an appropriately weak sense, i.e. certain generators are respected and others (cx & sympl. ones) are <u>swapped</u>.

<u>Conj</u> [Kontsevich] If two CY mflds (X, ω, B) & (X', ω', B') define mirror symm. SCFTs then there exist equivalences:

$$D(X) \simeq DFuk(X', \omega') \quad \& \quad DFuk(X, \omega) \simeq D(X').$$

This suggests the existence of a fund. bridge between alg. geom. & sympl. geom.

<u>N.B.</u> If X, X' are CY3s then the "stringy Kähler moduli space" $\mathcal{M}_K(X)$ is identified with the moduli space $\mathcal{M}_{\mathbb{C}}(X')$ of cx. str's on X' up to diffeo, i.e. the mirror map gives isoms:

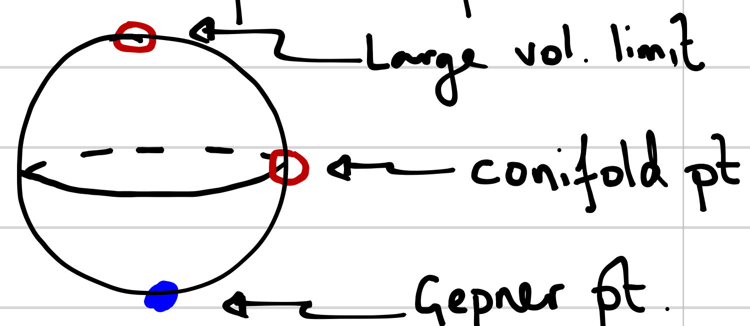
$$\mathcal{M}_K(X) \simeq \mathcal{M}_{\mathbb{C}}(X') \quad \& \quad \mathcal{M}_K(X') \simeq \mathcal{M}_{\mathbb{C}}(X).$$

One of the motivations for introducing stability conditions was to provide a precise mathematical def^{n} for the Kähler moduli sp.

↳ it is expected that one should be able to realise it as a submfld of the double quotient:

$$\mathcal{M}_K(X) \hookrightarrow {}^{\mathrm{Aut}\,D(X)}\backslash^{\mathrm{Stab}(X)}/_{\mathbb{C}}$$

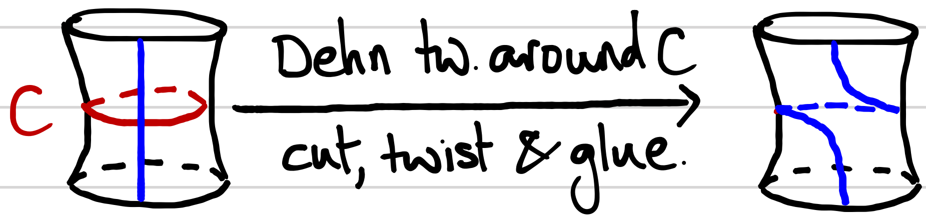
If $X =$ quintic 3fold then $\mathcal{M}_K(X) =$ twice-punctured two-sphere with a special point.



HMS has only been proven in a few cases (ell. curves, tori, K3s) but even the conjectural relation can be very illuminating.

↳ We illustrate this with Dehn twists & sph. objects.

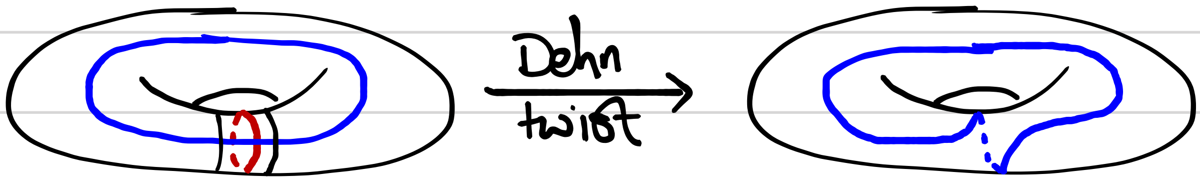
Consider a cylinder with an embedded curve C then roughly speaking, we have:



More precisely, given any simple closed curve inside an orientable surface, we can consider the tubular nbhd (or annulus) $S^1 \times I$ and define $f: S \times I \longrightarrow S \times I$; $(s,t) \longmapsto (se^{i2\pi t}, t)$ as the Dehn twist around C.

↳ this self-homeo^{m} is then extended to act as the identity on the rest of the surf.

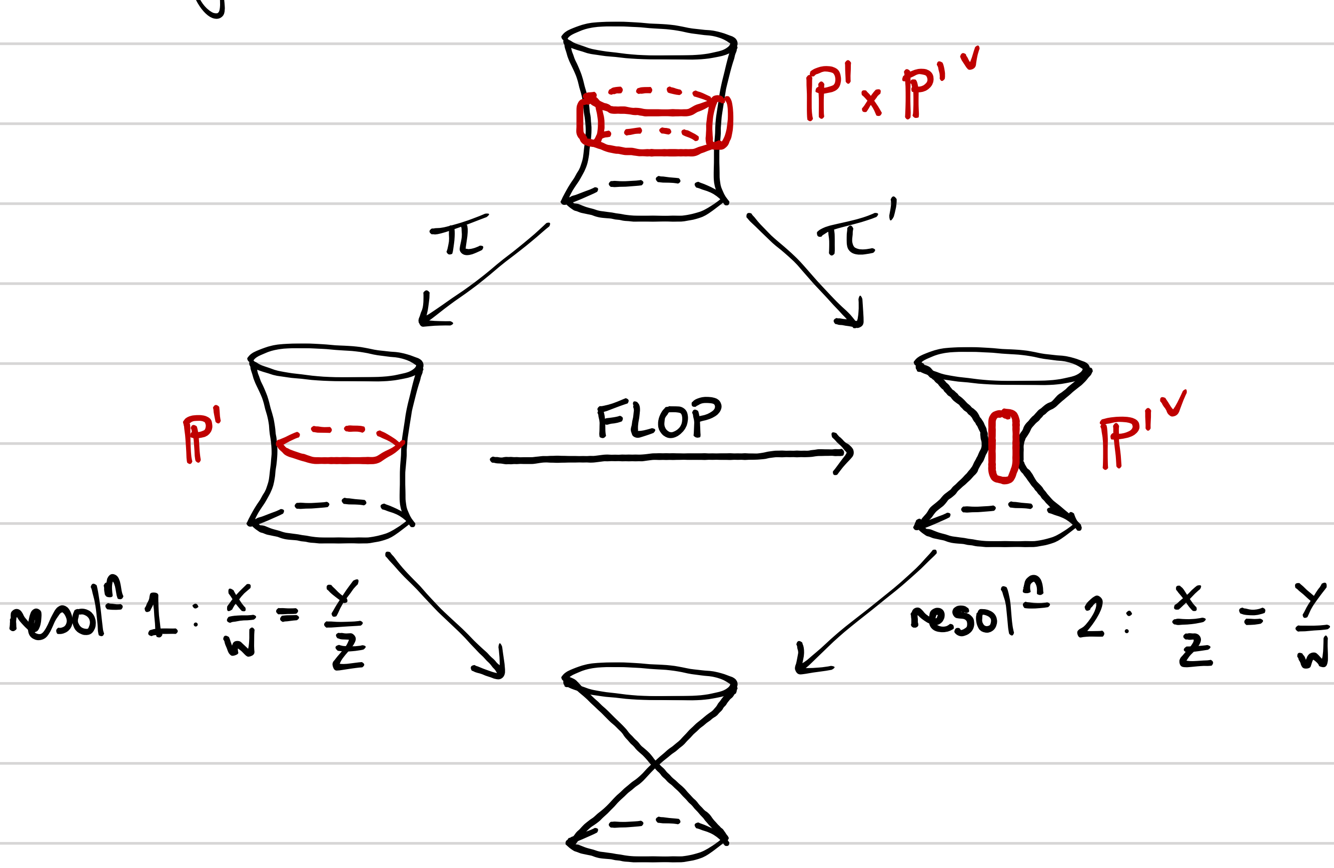
Example: Torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$



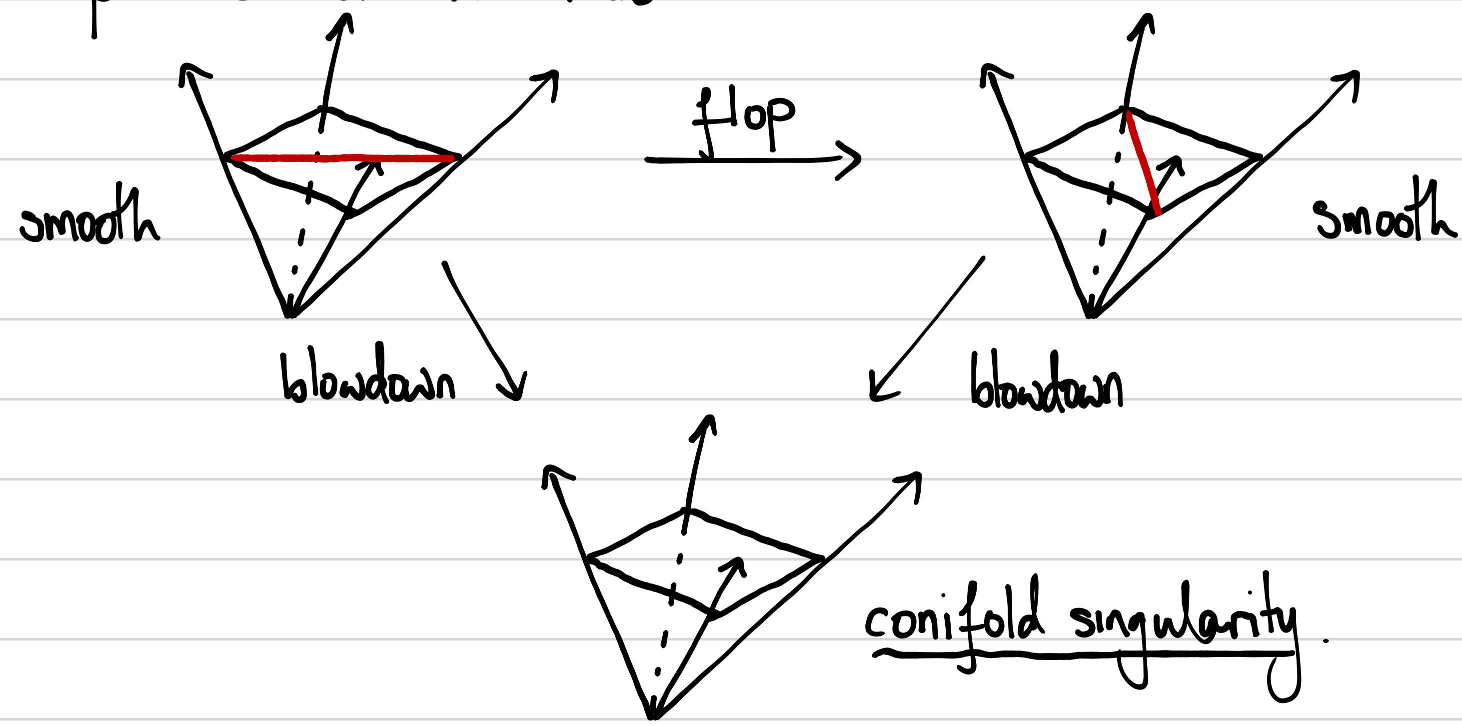
If we consider the induced action on the fundamental group $\pi_1(\mathbb{T})$ and let $[a]$ be the htpy class of the red curve and $[b]$ the htpy class of the blue curve then the Dehn twist sends $[a] \mapsto [a]$
and $[b] \mapsto [b * a]$

In his thesis, Paul Seidel generalised this operation to $\mathbb{S}^2 \subset T^*\mathbb{S}^2$ and then observed that his construction worked for arb^y $\mathbb{S}^n$ or indeed a projective space $\mathbb{P}^n \subset T^*\mathbb{P}^n$. This was the principal motivation behind the seminal papers of Seidel-Thomas & Huybrechts-Thomas.

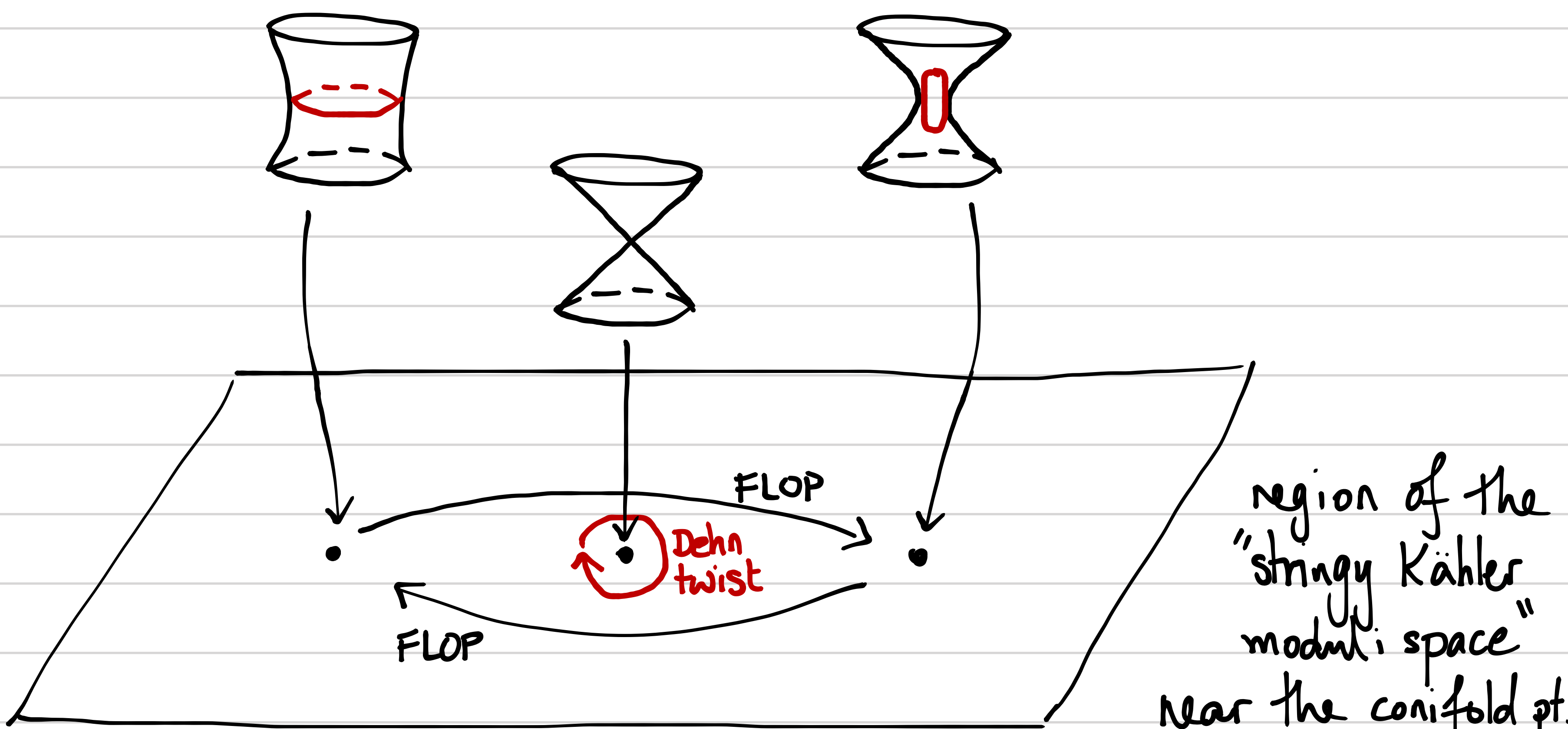
Example: Let $X \subset \mathbb{C}^4$ be given by the eqn $xy - wz = 0$.
Can resolve the singularity in two ways, i.e. can remove the origin and glue in a $\mathbb{P}^1$ as



Or, the toric picture looks like this:



Moreover, Seidel observed that there was a fundamental connection between Dehn twists and monodromy maps. Continuing with the example above, we might illustrate this as follows:



More generally, if (X,ω) is a sympl. mfld and $L \subset X$ is a Lag. submfld then there is a holo^c fibration $E_L \longrightarrow D$ over some disc D s.t. $0 \in D$ is the only crit. val and whose fibre over $S \in D$ is isom to (X,ω) and whose sympl. monodromy is a gen^d Dehn twist along L. [S, Prop 19.1]

Because $D\mathcal{F}uk(X,\omega)$ is defined in terms of sympl. data (and is sufficiently functorial), every (graded) symplectic auto (i.e. $f: X \xrightarrow{\sim} X$ diffeo s.t. $f^*\omega \simeq \omega$) induces an exact auto of $D\mathcal{F}uk(X,\omega)$.

Moreover, an isotopy of (graded) sympl. autos gives rise to an equiv between the induced functors and symplectomorphisms which are isotopic to the identity supposed to act trivially

↳ Recall: two diffeos $\phi_0, \phi_1 : X \to X$ are diffeotopic if they can be connected by a sm. family $(\phi_t)_{0 \le t \le 1}$ of diffeos.

Sim^{ly}, two symplectos are sympl^{ly} isotopic if there is a diffeotopy (ϕ_t) between them s.t. all the ϕ_t are symplectomorphisms, i.e. this is finer than diffeotopy.

⊛ there are symplectos which are diffeotopic but not sympl^{ly} isotopic.

e.g. gen^{d} Dehn twists live in the kernel of

$$\pi_0(\mathrm{Sympl}(X,\omega)) \longrightarrow \pi_0(\mathrm{Diff}^+(X))$$

So, we obtain a homo^{m}

$$\pi_0(\mathrm{Sympl}(X,\omega)) \longrightarrow \mathrm{Aut}\, D\mathcal{F}uk(X,\omega)$$

If (X,ω) is mirror symm. to (X',ω') then HMS predicts that $D\mathcal{F}uk(X,\omega) \simeq D(X')$, i.e. we have a homo^{m}:

$$\pi_0(\mathrm{Sympl}(X,\omega)) \longrightarrow \mathrm{Aut}\, D(X').$$

Seidel's Study of $\mathrm{Sympl}(X,\omega)$ resulted in a positive answer to the isotopy problem (with gen^{d} Dehn twists) but he also showed that special configurations of Lag. spheres give rise to braid gp actions.

The paper of Seidel & Thomas [2001] was motivated by the belief that it should be possible to detect these special symplectomorphisms of (X,ω) on the B-side as autos of $D(X')$.

The conj is that under $\pi_0(\mathrm{Sympl}(X,\omega)) \longrightarrow \mathrm{Aut}\, D(X')$ the Dehn twist along a Lag. sph. $L \subset X$ corresps to the sph. twist $T_\mathcal{E}$ where $\mathcal{E}$ is the object which S maps to under the equiv:

$$\begin{aligned} D\mathrm{Fuk}(X,\omega) &\xrightarrow{\sim} D(X') \\ S &\longmapsto \mathcal{E} \end{aligned}$$

Main result of [ST01] confirms this belief and shows that an A_n-config. of sph. objs indeed gives rise to a faithful braid gp action.

Timothy will tell us more about the B-side, sph. objs & their twists next time.

For $S^2 \simeq \mathbb{P}^1$ use local model: let T^*S^2 be the ctgt b. of S^2 and ω its canonical sympl. form.

The zero section $S^2 \subset T^*S^2$ is a Lag. submfld. Use the rep$^{\underline{n}}$:

$$T^*S^2 = \{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |u|=1,\ \langle u,v\rangle = 0\}$$

In these coords, we have $\omega = -\sum_j du_j \wedge dv_j$ & $S^2 = \{(u,v) \in T^*S^2 \mid v=0\}$

Let $T^*_\epsilon S^2 = \{(u,v) \in T^*S^2 \mid |v| < \epsilon\}$ be the sub-bundle of ϵ-discs for $\epsilon > 0$ and denote the subgp of autos $\phi \in \mathrm{Aut}(T^*S^2,\omega)$ which are supported inside $T^*_\epsilon S^2$ (that is, $\phi = \mathrm{id}$ outside some cpt subset of $T^*_\epsilon S^2$) by $\mathrm{Aut}^c(T^*_\epsilon S^2,\omega)$.

Consider the Hamiltonian function $\mu(u,v)=|v|$ on $T^*S^2\setminus S^2$. It is well known that $\frac{1}{2}\mu^2$ induces the geodesic flow (true for the corresp. fn on the ctgt b. of any Riemannian mfld).

What is the flow of μ: it transports every ctgt vector along the geodesic emanating from it with unit speed, irrespective of how long the vector is.

On S^2, all geodesics are closed and of period 2π; therefore μ induces a Hamiltonian circle action on $T^*S^2\setminus S^2$.

We can write this action down explicitly:

$$\sigma(e^{it})(u,v)=\left(\cos(t)u+\sin(t)\frac{v}{|v|},\ \cos(t)v-\sin(t)u|v|\right).$$

$\sigma(-1)(u,v)=(-u,-v)$ can be extended to an involution of T^*S^2. We call the involution the antipodal map and denote it by A.

The Hamiltonian flow induced by a (time-indep. or time-dep) Hamilt^n fn H will be denoted by $(\phi_t^H)_{t\in\mathbb{R}}$.

Take a fn $r\in C^\infty(\mathbb{R},\mathbb{R})$. The flow induced by $r(\mu)$ on $T^*S^2\setminus S^2$ is

$$\phi_t^{r(\mu)}(x)=\sigma\left(e^{itr'(\mu(x))}\right)(x)\ ; \qquad (*)$$

this is an elementary fact which holds for any Hamiltonian circle action. If r is even, $r(\mu(u,v))=\sqrt{r}(|v|^2)$ is a sm. fn on all T^*S^2 and every pt in S^2 is a critical pt of it.

As a consequence $(*)$ can be extended to a Hamiltonian flow on T^*S^2 which keeps S^2 pointwise fixed.