Elementary rep. theory

Work over a field k

From now on G means $SL(n)$

$G \supset B$ means upper triangular matr of size n (with $\det(A) = 1$)

$T \subset B \subset G$ means diagonal matr.

1) Let Fl be the set of all complete flags in k^n:

$$Fl = \{ V_\cdot \mid 0 \subset V_1 \subset \dots \subset V_n = k^n, \ \dim V_i = i \}$$

G acts on $k^n \Rightarrow G \times Fl \longrightarrow Fl$

a) Prove that $\mathrm{Stab}_G(V^0_\cdot) = B$

Here V^0 is the standard coordinate flag $0 \subset k \subset k^2 \subset \dots \subset k^n$

b) Prove that $\mathrm{Orb}_G(V^0_\cdot) = Fl$

<u>Corollary</u>: $Fl \cong G/B$

2) Let $k = \mathbb{F}_q$. Find

a) $\# \{$all lines passing throug 0 in $k^n\}$

b) $\# \{$all m-dim. subspaces in $k^n\}$

c) $\# GL(n,k)$, $\# SL(n,k)$

d) $\# Fl(k)$

Define the Weyl group for G to be

$$W = \mathrm{Norm}_G(T) / T$$

3) a) Prove that $\mathrm{Norm}_G(T)$ consists of matr $A = (a_{ij})$ with $a_{ij} = 0$ unless $\exists\, \sigma \in S_n$ $\sigma = \sigma(A)$ with $j = \sigma(i)$ (and $\det A = 1$)

b) It follows that $W \cong S_n$

4) Using elementary row and column operations on a matrice prove that

$$G = \bigsqcup_{\sigma \in S_n} B \dot{\sigma} B.$$

Here $\dot{\sigma} \in \mathrm{Norm}_G(T)$ is a set-theoretic lift of $\sigma \in S_n$ (e.g. correct one non-zero entry of a permutation matrice by a sign to get $\det(A) = 1$)

<u>Corollary</u> : Fl is a finite union of B-orbits for the standard action

$$B \times Fl \longrightarrow Fl \ ;$$

$$Fl = \bigsqcup_{\sigma \in S_n} B \dot{\sigma} B \big/ B =: \bigsqcup_{\sigma \in S_n} X_\sigma$$

X_σ is called the Schubert cell for $\sigma \in S_n$

5) Consider the example of $SL(2)$:

a) Identify Fl with $\mathbb{P}^1_k$

b) Describe Schubert cells explicitly : $W = \mathbb{Z}/2 = \{e, s\}$

c) Find $\# X_e$, $\# X_s$, $\# \mathbb{P}^1_k$ for $k = \mathbb{F}_q$, write down the identity $\# Fl = \# X_e + \# X_s$ explicitly

6) Prove that in general, X_σ is isomorphic to an affine space of dimension $\ell(\sigma)$

Here $\ell(\sigma)$ is the length of a reduced expr for σ.

Denote the Lie algebra of traceless matrices of size n by $\mathfrak{g} = \mathfrak{sl}(n)$; We have $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ (diagonal and upper triangular traceless matrices).

Denote the point corresponding to the standard flag $V_\cdot^0$ by $e \in Fl$

6) Prove that $T_e^* Fl \simeq (\mathfrak{g}/\mathfrak{b})^*$

Introduce the Killing form
$$\langle\ \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow k$$
$$A, B \longmapsto tr(AB)$$
It is non-degenerate and identifies $\mathfrak{g} \simeq \mathfrak{g}^*$.

Denote strictly upper triangular matrices by $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$

7) Prove that the total space
$$T^* Fl \simeq \frac{G \times \mathfrak{n}}{B}$$
Here B acts on G by right translations and on $\mathfrak{n}$ in the adj. way

Let $\mathcal{N} \subset \mathfrak{g}$ be the set of nilpotent matrices of size n:

$$\mathcal{N} = \{A \in \mathfrak{sl}(n) \mid \exists\, m : A^m = 0\}$$

8) In the case of $\mathfrak{g} = \mathfrak{sl}(2)$ Identify $\mathcal{N}$ with the quadric $\{x^2 + yz = 0\} \subset k^3$

Consider the Springer variety

$$\tilde{\mathcal{N}} = \left\{ (V_\bullet, A) \;\middle|\; \begin{array}{l} V_\bullet \in Fl,\ A \in \mathcal{N} \\ \forall i:\ A(V_i) \subset V_{i-1} \end{array} \right\}$$

$$\begin{tikzcd} & \tilde{\mathcal{N}} \arrow[dl, "p"'] \arrow[dr, "\mu"] & \\ Fl & & \mathcal{N} \end{tikzcd}$$

9) a) Prove that $p^{-1}(e) = \mathfrak{n}$ for $e \in Fl$ corresponding to the standard coordinate flag

b) Identify $\tilde{\mathcal{N}}$ with the total space T^*Fl

10) a) Classify the orbits for the adjoint action of G on $\mathcal{N}$

b) Prove that μ is one to one on the open orbit ⓘ $\subset \mathcal{N}$

$\mu: \tilde{\mathcal{N}} \to \mathcal{N}$ is called the Springer desingularization of the nilpotent cone,

for $A \in \mathcal{N}$, $\mu^{-1}(A) =: \mathcal{N}_A$

is called the Springer fiber for the orbit $\mathrm{Orb}_G(A)$.

c) Find $\# \tilde{\mathcal{N}}$ for $G = SL(2, \mathbb{F}_q)$ describe the Springer fibers and find

$$\# \{x^2 + yz = 0\} \subset \mathbb{F}_q^3$$

11) Prove that for a group K with a subgroup H

$\{H\text{-orbits on } K/H\} \leftrightarrow \{K\text{-orbits on } K/H \times K/H\}$

In particular,

$$Fl \times Fl = \bigsqcup_{b \in S_n} Y_b$$ - disjoint union of G-orbits.

12) a) For $G = Sl(2)$ describe G-orbits in $Fl \times Fl$ explicitly

b) Let $\sigma = (i\ i{+}1) \in S_n$ be the elementary permutation.
Prove that the closure of the orbit $\cancel{X}\ \overline{Y_\sigma} =$

$$= \{ (V^1_\cdot, V^2_\cdot) \mid V^1_j = V^2_j \text{ for } j \neq i \}$$

c) Describe $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ in terms of $T^*Fl \times T^*Fl$ for $G = SL(2)$

The space $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} =$

$$= \left\{ (V^1_\cdot, V^2_\cdot, A) \;\middle|\; \begin{array}{l} A(V^1_i) \subset V^1_{i-1} \\ A(V^2_i) \subset V^2_{i-1} \end{array} \right\}$$

is called the Steinberg variety for $G = SL(n)$