Lecture 1

Rem: In these series of lectures we show how a geometric machine called Steinberg variety leads to various versions of Hecke algebra and Hecke category. We start from the classical construction called Springer correspondence.

Fix a reductive algebraic group G over $\mathbb{C}$. Choose a maximal torus $T \subset G$ and a Borel (maximal solvable) subgroup $T \subset B \subset G$. Denote the unipotent radical of B by N.

Let $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$, $\mathfrak{n} \subset \mathfrak{b}$ be the corresponding Lie algebras.

Example:

Our standard example is

- $G = SL(n) = \{ A \in Mat(n \times n) \mid \det(A) = 1$
- $B = \{ A \in Mat(n \times n) \mid A$ - non-strictly upper triang, $\det(A) = 1 \}$
- $N = \{ A \in Mat(n \times n) \mid A$ - strictly upper triang, $\det(A) = 1 \}$
- $T = \{ A \in Mat(n \times n) \mid A$ - diagonal, $\det(A) = 1$

Facts about Borel subgroups

i) $Norm_G(B) = B$

ii) $\forall\ B_1, B_2$ - Borel (maximal solvable subgroups in G $\exists\, g \in G : gB_1g^{-1} = B_2$

(iii) It follows that the set of all Borel subgroups $Fl_G \cong G/B$

(iv) Fl_G is a projective algebraic variety over $\mathbb{C}$ called the Flag variety for G.

Example:

$G = SL(n)$, G acts on $\mathbb{C}^n$.

Consider $\widetilde{Fl} = \{ V_\cdot \mid V_\cdot = \{0 \subset V_1 \subset \dots \subset V_n = \mathbb{C}^n\}, \dim V_i = i \}$

Ex: (i) G acts on $\widetilde{Fl}$

(ii) Let $V_\cdot^0$ be the coordinate flag then $\mathrm{Stab}_G(V_\cdot^0) = B$

(iii) $\forall\ V_\cdot^1, V_\cdot^2\ \exists\ g \in G : g(V_\cdot^1) = V_\cdot^2$

(iv) It follows that $Fl_G \cong \widetilde{Fl}$

Rem: this justifies the name.

Example: $G = SL(2)$

then $\widetilde{Fl} = \mathbb{C}P^1$ by definition.

Bruhat decomposition and the Weyl group

Def: $W := \mathrm{Norm}_G(T)/T$

Ex: $G = SL(n)$ with the standard T.

then $\mathrm{Norm}_G(T) = \{ d(\sigma, d_1, \dots, d_n) \mid \sigma \in S_n, d_1, \dots d_n \in \mathbb{C}^*,$
$d(\sigma, d_1, \dots d_n)_{ij} = \begin{cases} 0, & j \neq \sigma(i) \\ d_i, & j = \sigma(i), \end{cases}$
$\det(d(\sigma, d_1, \dots d_n)) = 1 \}$

<u>Corollary</u>: $W = S_n$ for $G = SL(n)$

Choose a set-theoretic lift $\dot{w} \in \mathrm{Norm}_G(T)$ for every $w \in W$

<u>Theorem</u>: $G = \bigsqcup_{w \in W} B \dot{w} B$

<u>Example</u>: $G = SL(n)$. Then every $A \in G$ can be transformed into ~~a~~ a permutation matr. $A(\sigma)$, $\sigma \in S_n$, by elementary row and column operations.

Springer resolution of the nilpotent cone

Def: the nilpotent cone
$\mathcal{N} \subset \mathfrak{g}$: $\mathcal{N} := \{A \in \mathfrak{g} \mid A$ is nilpotent$\}$

Example: $\mathfrak{g} = sl(n) \Rightarrow \mathcal{N}$ consists of $\{n \times n$ matr $A : \exists\, m = m(A) : A^m = 0\}$

Rem: Jordan normal form theorem states that for $\mathfrak{g} = sl(n)$
$\mathcal{N} = \bigsqcup_{G\text{-orb}} \mathbb{O}$, and $\{\mathbb{O}\} \xleftrightarrow{1:1} \{$ Young diagrams of total size $n\}$

Claim: Let $e \in Fl_G$ be the image of $e \in G$ in G/B (corresponding to the point $\{B\}$). Then
$T_e Fl_G \cong \mathfrak{g}/\mathfrak{b}$

Corollary: Identify $\mathfrak{g} \cong \mathfrak{g}^*$ via Killing form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$
$A, B \longmapsto Tr(AB)$ (in the adjoint rep.)
then $(\mathfrak{g}/\mathfrak{b})^* \cong \mathfrak{n} \subset \mathfrak{g}$

Corollary:

$$T^*Fl_G \simeq \frac{G \times n}{B}$$

(quotient by the diagonal B-action)

Corollary: $T^*Fl_G \simeq \{(B_x, n_x) \mid B_x \in Fl_G,\ n_x \in \operatorname{rad} \operatorname{Lie}(B_x)\}$

Def: the latter space is called the Springer variety for G and is denoted by $\tilde{N}$

We have the canonical maps

$$\begin{tikzcd} & \tilde{N} \arrow[dl, "p"'] \arrow[dr, "\mu"] & \\ Fl_G & & N \end{tikzcd}$$

Example: For $G = SL(n)$

$$N = \{A \in SL(n) \mid A(V_i^0) \subset V_{i-1}^0 \ \forall i\}$$

It follows that

$$\tilde{N} = \left\{(V_\cdot, A) \;\middle|\; \begin{array}{l} V_\cdot \in Fl \\ A(V_i) \subset V_{i-1}\ \forall i \end{array}\right\}$$

Claim: $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ is generically $1-1$

Example: For $G = SL(n)$ a generic $A \in \mathcal{N}$ is conjugate to $\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} =: J_n$. $\exists!$ Borel subgroup, the standard B s.t. $J_n \in \mathrm{Lie}(B)$

Def: $A \in \mathcal{N}$, $\mu^{-1}(A) =: \tilde{\mathcal{N}}_A$ is called the Springer fiber at A.

Def: $St_G := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ is called the Steinberg variety for G.

Ex: $H \subset K$ - a group and a subgroup. then
$\{H\text{-orb in } K/H\} \overset{1:1}{\longleftrightarrow}$

$\{K\text{-orb in } K/H \times K/H\}$

Example: Pairs of flags $(V^1_\cdot, V^2_\cdot)$ up to conjugation are enumerated by $\sigma \in S_n$.

- $\sigma = e \Rightarrow V_\cdot^1 = V_\cdot^2$, and the corresponding orbit is the diagonal $Fl_\Delta \subset Fl \times Fl$

- $\sigma = (i, i+1) \in S_n \Rightarrow$ $(V_\cdot^1, V_\cdot^2)$ are in the corresp. orbit iff $V_j^1 = V_j^2$, $j \neq i$, $V_i^1 \neq V_i^2$

<u>Notation</u>: For $w \in W$ the corresp. G-orbit in $Fl_G \times Fl_G$ is denoted by X_w

<u>Th</u>: $St_G \subset \tilde{N} \times \tilde{N}$ is identified with $\bigsqcup_{w \in W} T^*_{X_w}(Fl_G \times Fl_G)$ - the union of conormal bundles to G-orbits

<u>Corollary</u>: Components of St_G are

$\overline{T^*_{X_w}(Fl_G \times Fl_G)}$

Rem.: We have described the geometry of Springer correspondence the homological bridge to algebra is given by

Borel-Moore homology

Rem.: We need a homology theory which

- allows inverse images along smooth (or flat) maps of alg. varieties $/\mathbb{C}$
- allows direct images along proper maps of top spaces

Rem.: all topological spaces in the story are good enough: locally compact and homotopically equivalent to a finite CW-complex.

Def 1: $H^{BM}_\cdot(X) = H_\cdot(\hat{X}, \infty)$.
Here $\hat{X} = X \cup \infty$ – one point compactification of X

Def 2 : $\overline{X}$ - any compactification of X. $H_\cdot^{BM}(X) := H_\cdot(\overline{X}, \overline{X}\setminus X)$

Def 3 : M - smooth oriented mfld / $\mathbb{R}$, $\dim M = m$, $X \hookrightarrow M$ - closed

$$H_i^{BM}(X) = H^{m-i}(M, M\setminus X)$$

Rem : Def 1 $\Leftrightarrow$ Def 2 $\Leftrightarrow$ Def 3

Basic properties :

(i) $f : X \to Y$ - proper $\Rightarrow$
$f_* : H_\cdot^{BM}(X) \to H_\cdot^{BM}(Y)$

(ii) fundamental class :

X - alg. var / $\mathbb{C}$ (probably singular)
$X = X_1 \cup \dots \cup X_m$ - components, all of complex dimension n.

$X_i^{reg} \subset X_i$ - regular parts

$\Rightarrow H_{2n}^{BM}(X)$ has a basis of $[X_i] = [X_i^{reg}]$ (fund classes

(iii) Intersection pairing

Z_1, Z_2 - closed in a smooth oriented mfld M, $\dim_{\mathbb{R}} M = m$

$$\cap : H_i^{BM}(Z_1) \times H_j^{BM}(Z_2) \to H_{i+j-m}^{BM}(Z_1 \cap Z_2)$$

<u>Rem</u>:

(i), (ii) and (iii) give rise to <u>convolution</u> in BM homology.

M_1, M_2, M_3 - smooth oriented mflds $/\mathbb{R}$

$Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_2 \times M_3$

$Z_{13} = Z_{12} \circ Z_{23} \subset M_1 \times M_3$

$$\begin{tikzcd}
 & M_1 \times M_2 \times M_3 \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\
M_1 \times M_2 & M_1 \times M_3 & M_2 \times M_3
\end{tikzcd}$$

$$\pi_{13} : \pi_{12}^{-1}(Z_{12}) \cap \pi_{23}^{-1}(Z_{23}) \to Z_{12} \circ Z_{23}$$

- proper

Claim: We have the convolution pairing in BM Homology

$$*: H_i^{BM}(Z_{12}) \times H_j^{BM}(Z_{23}) \longrightarrow H^{BM}_{i+j-m}(Z_{12} \circ Z_{23})$$

$m = \dim M_2$

given by

$$c_{12} * c_{23} = p_{13*}\left(p_{12}^*(c_{12}) \cap p_{23}^*(c_{23})\right)$$

Here $p_{12}^*(\alpha) = \alpha \boxtimes [M_3]$
$p_{23}^*(\beta) = [M_1] \boxtimes \beta$.

Application: $M_1 = M_2 = M_3 = M$

- smooth oriented

$f: M \longrightarrow N$ (probably a singular alg. var.

$Z = M \times_N M \subset M \times M$, $\dim_{\mathbb{R}} M = m$

Get $H_i^{BM}(Z) \times H_j^{BM}(Z) \to H^{BM}_{i+j-m}(Z)$

Claim: $H_\cdot^{BM}(Z)$ is an algebra

$\mathcal{H}(Z) := H_m^{BM}(Z)$ - a subalgebra.

Application:

$M = \tilde{\mathcal{N}}$, $N = \mathcal{N}$

$\mu : \tilde{\mathcal{N}} \longrightarrow \mathcal{N}$

$Z = St_G = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \overline{\bigcup_w T^*_{X_w}(Fl_G \times Fl_G)}$

We get that

Corollary: $H(St_G)$ is an associative algebra, it has a basis $[T^*_{X_w}(Fl_G \times Fl_G)]$

Rem: the size of the algebra hints on $\mathbb{C}[W]$, but in the basis above the multiplication is different from $e_{s_1} \cdot e_{s_2} = e_{s_1 s_2}$

Yet we have the theorem:

Theorem: $H(St_G) \simeq \mathbb{C}[W]$, the group algebra of W.

Rem: Each Springer fiber $\widetilde{\mathcal{N}}_A = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \{A\}$ is a correspondence

$\Rightarrow$ have the action

$$\mathcal{H}(St_G) \times H^{BM}_{top}(\widetilde{\mathcal{N}}_A) \to$$
$$\to H^{BM}_{top}(\widetilde{\mathcal{N}}_A)$$

Examples of Springer fibers

(i) A - regular $\Rightarrow$ $\widetilde{\mathcal{N}}_A = \{pt\}$

(ii) $A = 0$ $\Rightarrow$ $\widetilde{\mathcal{N}}_A = Fl_G$

Rem: $H^{BM}(\widetilde{\mathcal{N}}_A)$ is acted on by $\pi_1(\mathbb{O}_A, \{A\})$, thus it decomposes into $\oplus$ of isotypic components $H^{BM}(\widetilde{\mathcal{N}}_A)_\phi$

ϕ - an irrep of $\pi_1(\mathbb{O}_A, \{A\})$ (or an irreducible local system on $\mathbb{O}_A$.

Th: $\exists$ a 1-1 correspondence between irreps of W and (G-conj. classes of) pairs (A, ϕ), $A \in \mathcal{N}$ - nilp. element, ϕ - an irrep of $\pi_1(\mathbb{O}_A, \{A\})$. Namely, the corresponding rep of W is given by $\mathcal{H}(St_G)$-module $H^{BM}_{top}(\tilde{\mathcal{N}}_A)_\phi$

Example: $G = SL(n)$,

$\mathbb{O}_A \xleftrightarrow{(1,1)} \{$ Young diagrams of size $n \}$, all orbits are simply connected.

$\Rightarrow$ Irreps of $W = S_n \xleftrightarrow{(1,1)}$ $\{$ Young diagrams of size $n \}$

E.g. $A = 0 \Rightarrow \tilde{\mathcal{N}}_A = \mathcal{B} \Rightarrow$ the rep. is sign.

A - regular $\Rightarrow \tilde{\mathcal{N}}_A = pt \Rightarrow$ the rep - trivial