D^b(**Intro**)

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These are the lecture notes for the introductory school on derived categories in Warwick, September 2014. They cover some basic facts about derived categories of coherent sheaves on smooth projective varieties, assuming some kind of familiarity with the definition of a derived category. There are bound to be some mistakes that I haven't found yet: please feel free to let me know about them.

1. The derived category of an abelian category

In this section we summarize the most important properties of the derived category of an abelian category. We illustrate some of these by considering the duality functor for coherent sheaves on $\mathbb{A}^2$.

1.1. **Basics.** Let $\mathcal{A}$ be an abelian category, e.g. $\mathrm{Mod}(R)$ or $\mathrm{Coh}(X)$. Let $C(\mathcal{A})$ denote the category of cochain complexes in $\mathcal{A}$. A typical morphism $f^\bullet\colon M^\bullet \to N^\bullet$ in this category looks as follows

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M^{i-1} & \xrightarrow{d^{i-1}} & M^i & \xrightarrow{d^i} & M^{i+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ \cdots & \longrightarrow & N^{i-1} & \xrightarrow{d^{i-1}} & N^i & \xrightarrow{d^i} & N^{i+1} & \longrightarrow & \cdots \end{array}$$

Recall that such a morphism $f^\bullet\colon M^\bullet \to N^\bullet$ is called a *quasi-isomorphism* if the induced maps on cohomology objects

$$H^i(f^\bullet)\colon H^i(M^\bullet) \to H^i(N^\bullet)$$

are all isomorphisms. The derived category $D(\mathcal{A})$ is obtained from $C(\mathcal{A})$ by formally inverting quasi-isomorphisms. Thus there is a localisation functor

$$Q\colon C(\mathcal{A}) \to D(\mathcal{A})$$

which is universal with the property that it takes quasi-isomorphisms to isomorphisms. The objects of $D(\mathcal{A})$ can be taken to be the same as the objects of $C(\mathcal{A})$.

Date: September 12, 2014.

It is immediate from the universal property that there are well-defined functors $H^i\colon D(\mathcal{A}) \to \mathcal{A}$ sending a complex to its cohomology objects. The bounded derived category is defined to be the full subcategory

$$D^b(\mathcal{A}) = \{E \in D(\mathcal{A}) : H^i(E) = 0 \text{ for } |i| \gg 0\} \subset D(\mathcal{A}).$$

There is an obvious functor $\mathcal{A} \to D(\mathcal{A})$ which sends an object $E \in \mathcal{A}$ to the corresponding trivial complex with E in position 0:

$$E \in \mathcal{A} \longmapsto (\cdots \longrightarrow 0 \longrightarrow E \longrightarrow 0 \longrightarrow \cdots) \in D(\mathcal{A}).$$

This functor is full and faithful. Its essential image is the full subcategory

$$\{E \in D(\mathcal{A}) : H^i(E) = 0 \text{ unless } i = 0\}.$$

Objects of this subcategory are said to be *concentrated in degree 0.* We shall always identify the category $\mathcal{A}$ with its image under this functor.

Two objects in $D(\mathcal{A})$ with the same cohomology objects need not be isomorphic (in much the same way as two modules with the same composition series need not be isomorphic). The extra information determining an object can be thought of as a 'cohomological glue' holding the cohomology objects together. If this glue vanishes then

$$E \cong \bigoplus_{i\in\mathbb{Z}} H^i(E)[-i]$$

$$\cong (\cdots \longrightarrow H^{i-1}(E) \xrightarrow{0} H^i(E) \xrightarrow{0} H^{i+1}(E) \longrightarrow \cdots).$$

Well-behaved functors between abelian categories $F\colon \mathcal{A} \to \mathcal{B}$ induce derived functors $\mathbf{F}\colon D(\mathcal{A}) \to D(\mathcal{B})$. The composite functors

$$A \in \mathcal{A} \mapsto H^i(\mathbf{F}(\mathcal{A})) \in \mathcal{A}$$

are the classical derived functors of F.

1.2. **Example: Duality for modules.** When $R = \mathbb{C}$ the dualizing functor

$$\mathbb{D}(M) = \mathrm{Hom}_R(M, R)$$

defines an anti-equivalence

$$\mathbb{D}\colon \mathrm{Mod}_{fg}(R) \longrightarrow \mathrm{Mod}_{fg}(R)$$

satsifying $\mathbb{D}^2 \cong \mathrm{id}$. What happens when R is a more interesting ring?

Consider the case $R = \mathbb{C}[x, y]$. Of course $\mathrm{Mod}_{fg}(R) = \mathrm{Coh}(\mathbb{A}^2_{\mathbb{C}})$. Defining a dualizing functor exactly as above we get an anti-equivalence

$$\mathbb{D}\colon \mathrm{Proj}_{fg}(R) \to \mathrm{Proj}_{fg}(R)$$

satisfying $\mathbb{D}^2 \cong \mathrm{id}$. (You may find it comforting to note that by the Quillen-Suslin theorem, any finitely-generated projective R-module is in fact free). But this functor is not an anti-equivalence on the full category $\mathrm{Mod}_{fg}(R)$ since, for example, if $M = R/(x)$ then

$$\mathbb{D}(M) = \mathrm{Hom}_R(R/(x), R) = (0).$$

To try to remedy this, let us consider also the classical derived functors

$$\mathbb{D}^i(M) = \mathrm{Ext}^i_R(M, R), \quad i \geqslant 0.$$

To compute these we replace $M = R/(x)$ by a free resolution

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

and apply $\mathbb{D}(-) = \mathrm{Hom}_R(-, R)$ to get

$$0 \longleftarrow R \xleftarrow{x} R \longleftarrow 0.$$

Taking cohomology gives

$$\mathbb{D}^i(M) = \begin{cases} M & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

so we have $\mathbb{D}^1(\mathbb{D}^1(M)) \cong M$.

Similarly, if we take the module $M = R/(x, y)$ then

$$\mathbb{D}^i(M) = \begin{cases} M & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and once again we have $\mathbb{D}^2(\mathbb{D}^2(M)) \cong M$.

But suppose now that we consider modules M fitting into a short exact sequence

$$0 \longrightarrow R/(x, y) \longrightarrow M \longrightarrow R/(x) \longrightarrow 0. \tag{1}$$

Then from the long exact sequence in Ext-groups

$$\mathbb{D}^i(M) = \begin{cases} R/(x) & i = 1, \\ R/(x, y) & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

However M is not uniquely determined by the sequence (1), since

$$\mathrm{Ext}^1_R(R/(x), R/(x, y)) = \mathbb{C}.$$

We conclude that we cannot recover M from the objects $\mathbb{D}^i(M)$.

The solution (of course) is to consider the derived functor of $\mathbb{D}$, which defines an anti-equivalence

$$\underline{\mathbb{D}}\colon D^b\,\mathrm{Mod}_{fg}(R) \longrightarrow D^b\,\mathrm{Mod}_{fg}(R)$$

On the level of objects this means 'replace a complex by a quasi-isomorphic complex of projective modules and then dualize'. It is immediate that $\underline{\mathbb{D}}^2 \cong \mathrm{id}$ because we already know that duality works well for projective modules. If $M \in \mathrm{Mod}_{fg}(R)$ then we have

$$\mathbb{D}^i(M) = \mathrm{Ext}^i_R(M, R) = H^i(\underline{\mathbb{D}}(M)),$$

but as we saw above, these cohomology modules are not in general enough to determine the object $\underline{\mathbb{D}}(M)$, nor to recover the module M.

1.3. **Structure of $D(\mathcal{A})$.** The category $D(\mathcal{A})$ has two important structures which it is important to keep separate in one's mind.

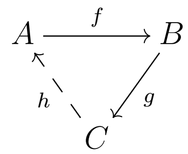
(a) The category $D(\mathcal{A})$ is *triangulated*: it has a shift functors

$$[n]\colon D(\mathcal{A}) \to D(\mathcal{A}),$$

$$M^\bullet[n]^i = M^{i+n}, \qquad d^i_{M^\bullet[n]} = (-1)^n d^{i+n}_{M^\bullet},$$

and a collection of distinguished triangles

(2)



obtained from the mapping cone construction. Any such triangle is a sequence of maps

$$\cdots \longrightarrow C[-1] \xrightarrow{h[-1]} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \longrightarrow \cdots .$$

Distinguished triangles in a triangulated category play a very similar role to short exact sequences in an abelian category. All derived functors are triangulated: they commute with the shift functors and takes distinguished triangles to distinguished triangles.

Given objects $E, F \in D(\mathcal{A})$ we define

$$\mathrm{Hom}^i_{D(\mathcal{A})}(E, F) := \mathrm{Hom}_{D(\mathcal{A})}(E, F[i]).$$

If $E, F \in \mathcal{A}$ then these agree with the usual Ext-groups:

$$\mathrm{Ext}^i_{\mathcal{A}}(E, F) = \mathrm{Hom}_{D(\mathcal{A})}(E, F).$$

It follows from the axioms of a triangulated category that if E is a fixed object and (2) is a distinguished triangle then there is a long exact sequences of abelian groups

(3) $$\cdots \to \mathrm{Hom}^i_{D(\mathcal{A})}(E, A) \to \mathrm{Hom}^i_{D(\mathcal{A})}(E, B) \to \mathrm{Hom}^i_{D(\mathcal{A})}(E, C) \to$$

$$\to \mathrm{Hom}^{i+1}_{D(\mathcal{A})}(E, A) \to \mathrm{Hom}^{i+1}_{D(\mathcal{A})}(E, B) \to \cdots .$$

There is a similar long exact sequence involving Hom groups into E.

(b) The category $D(\mathcal{A})$ comes equipped with the *standard t-structure*. In particular, there is a full and faithful embedding $\mathcal{A} \hookrightarrow D(\mathcal{A})$ and cohomology functors $H^i \colon D(\mathcal{A}) \to \mathcal{A}$ as discussed above.

A short exact sequence

(4) $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{A}$ becomes a distinguished triangle of the form (2) in $D(\mathcal{A})$. The extra morphism $h \in \mathrm{Ext}^1_{\mathcal{A}}(C, A)$ is the extension-class defined by the sequence. Conversely, any distinguished triangle (2) induces a long exact sequence in cohomology objects

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots .$$

Also important are the *truncation functors* $\tau_{\leqslant i} \colon D(\mathcal{A}) \to D(\mathcal{A})$ defined by

$$\tau_{\leqslant i}(M^\bullet) = \big(\cdots \to M^{i-1} \to \ker(d^i) \to 0 \to \cdots\big).$$

Note that

$$H^j(\tau_{\leqslant i}(M^\bullet)) = \begin{cases} H^j(M^\bullet) & j \leqslant i, \\ 0 & \text{otherwise.} \end{cases}$$

There is an obvious natural map of complexes $\tau_{\leqslant i-1}(M^\bullet) \to \tau_{\leqslant i}(M^\bullet)$ which induces isomorphisms in cohomology in degree $\leqslant i-1$. Taking the cone C on this map, and applying the long exact sequence in cohomology, we see that C is concentrated in degree i. We thus have distinguished triangles

(5) $$\begin{array}{ccc} \tau_{\leqslant i-1}(M^\bullet) & \longrightarrow & \tau_{\leqslant i}(M^\bullet) \\ & \nwarrow \,\text{-}\,\text{-} \quad \swarrow & \\ & H^i(M^\bullet)[-i] & \end{array}$$

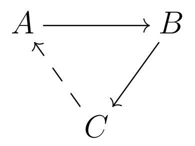
This is to be interpreted as saying that every object of $D(\mathcal{A})$ has a canonical 'filtration' whose 'factors' are shifts of objects of $\mathcal{A}$.

Note that genuinely derived functors do not preserve the standard t-structures. In fact, a triangulated functor $D(\mathcal{A}) \to D(\mathcal{B})$ that preserves the standard t-structures induces an exact functor $\mathcal{A} \to \mathcal{B}$, and conversely, an exact functor $\mathcal{A} \to \mathcal{B}$ induces a functor $D(\mathcal{A}) \to D(\mathcal{B})$ in a trivial way. We often say that such functors are exact and hence 'do not need to be derived'.

1.4. **Grothendieck groups.** The Grothendieck group $K_0(D)$ of a triangulated category D is the free abelian group on isomorphism classes of objects modulo relations

$$[B] = [A] + [C]$$

for distinguished triangles



It follows from the 'rotating triangle' axiom that $[E[n]] = (-1)^n[E]$.

Suppose that $D = D^b(\mathcal{A})$. The inclusion $\mathcal{A} \hookrightarrow D$ clearly induces a group homomorphism

$$I\colon K_0(\mathcal{A}) \to K_0(\mathcal{B}).$$

It follows immediately from the existence of the filtration (5) that I is in fact an isomorphism, with inverse map P given by

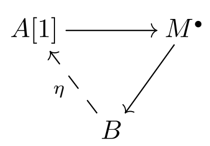
$$P([E]) = \sum_{i\in\mathbb{Z}} [H^i(E)[i]] = \sum_{i\in\mathbb{Z}} (-1)^i [H^i(E)]$$

1.5. **Problems.**

1.5.1. *Two-step complexes.* Fix objects $A, B \in \mathcal{A}$ and consider objects $E \in D(\mathcal{A})$ such that

$$H^j(E) = \begin{cases} A & \text{if } j = -1 \\ B & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that any such object fits into a distinguished triangle



Use this to give a complete classification of the isomorphism classes of such objects in terms of the group $\mathrm{Ext}^2_{\mathcal{A}}(B, A)$.

1.5.2. Consider the two-step complexes obtained by applying the functor $\mathbb{D}$ to the modules M which fit into a short exact sequence of the form (1). How is the extension class defining this short exact sequence reflected in the structure of $\mathbb{D}(M)$?

1.5.3. Let $\mathcal{A}$ be an abelian category of global dimension 1, i.e.

$$\mathrm{Ext}^p_{\mathcal{A}}(M, N) = 0 \text{ for all } p > 1 \text{ and all } M, N \in \mathcal{A}.$$

Prove that every $E \in D^b(\mathcal{A})$ satisfies $E \cong \bigoplus_{i\in\mathbb{Z}} H^i(E)[-i]$.

2. Derived categories of coherent sheaves

This lecture focuses on the derived category of coherent sheaves on a smooth projective variety. We introduce the basic abstract properties of this category and consider the example of the projective line.

2.1. **Basic properties.**

Let X be a smooth complex projective variety of dimension d. We set $D(X) = D^b \operatorname{Coh}(X)$. Note that this is a $\mathbb{C}$-linear category: the Hom sets are all vector spaces over $\mathbb{C}$, and the composition maps are bilinear. From Section 1.4 we know that

$$K_0(D(X)) = K_0(\operatorname{Coh}(X)) = K_0(X)$$

is the usual Grothendieck group of X. Since X is smooth and projective this also coincides with the Grothendieck group of locally-free sheaves $K^0(X)$. This is a commutative ring, with multiplication induced by tensor product of vector bundles.

The category $D(X)$ has three very important properties

(a) **Finiteness**. $D(X)$ is of finite type: for all objects $E, F \in D(X)$

$$\dim_{\mathbb{C}} \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}^i_{D(X)}(E, F) < \infty.$$

This enables us to define

$$\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}} \operatorname{Hom}^i_{D(X)}(E, F).$$

Note that the long exact sequence (3) shows that this expression is additive: given a distinguished triangle (2) we have

$$\chi(E, B) = \chi(E, A) + \chi(E, C).$$

It follows that it descends to give a bilinear form

$$K_0(X) \times K_0(X) \to \mathbb{Z}$$

which is known as the *Euler form.*

(b) **Riemann-Roch.** The Chern character defines a ring homomorphism

$$\operatorname{ch}\colon K_0(X) \to H^*(X, \mathbb{Q})$$

$$\operatorname{ch}(E) = \big(c_0(E), c_1(E), \tfrac{1}{2} c_1(E)^2 - c_2(E), \cdots\big).$$

The Riemann-Roch theorem states that for all $E, F \in D(X)$

$$\chi(E, F) = [\operatorname{ch}(E)^\vee \cdot \operatorname{ch}(F) \cdot \operatorname{td}(X)]_{2d}.$$

In this formula $\operatorname{ch}(E)^\vee$ denotes the sum $\sum_i (-1)^i \operatorname{ch}_i(E)$,

$$\operatorname{td}(X) = 1 + \frac{1}{2}\, c_1(X) + \frac{1}{12}\,(c_1(X)^2 + c_2(X)) + \cdots$$

is the Todd class of X, and $[\cdots\cdots]_{2d}$ means take the projection to the top degree component $H^{2d}(X,\mathbb{Q}) = \mathbb{Q}$.

(c) **Serre duality.** There are functorial isomorphisms

$$\operatorname{Hom}^i_{D(X)}(E,F) \cong \operatorname{Hom}^{d-i}_{D(X)}(F, E \otimes \omega_X)^*$$

for all objects $E, F \in D(X)$. Here ω_X denotes the canonical line bundle of X, and $d = \dim_{\mathbb{C}}(X)$. If $E, F \in \operatorname{Coh}(X)$ this implies in particular that

$$\operatorname{Ext}^i_X(E,F) = 0 \text{ for } i > d.$$

Note that if X is Calabi-Yau (meaning that $\omega_X \cong \mathcal{O}_X$ is trivial) then the category $D(X)$ has the CY_d property:

$$\operatorname{Hom}^i_{D(X)}(E,F) \cong \operatorname{Hom}^{d-i}_{D(X)}(F,E)^*.$$

The Euler form $\chi(-,-)$ is then $(-1)^d$–symmetric.

Numerical Grothendieck group. Serre duality shows that the left- and right-kernels of the Euler form are the same: for a given class $\gamma \in K_0(X)$ we have

$$\chi(\alpha,\gamma) = 0 \quad \forall \alpha \in K_0(X) \iff \chi(\gamma,\beta) = 0 \quad \forall \beta \in K_0(X).$$

The numerical Grothendieck group is defined to be the quotient

$$\mathcal{N}(X) = K_0(X)/\ker \chi(-,-).$$

It is a finitely-generated free abelian group. Note that it is not clear that the Chern character descends to $\mathcal{N}(X)$ (this has to do with the standard conjectures), but this is certainly true for example when $\dim_{\mathbb{C}}(X) \leqslant 2$.

Serre functor. The functor $S_X \colon D(X) \to D(X)$ defined by

$$S_X(-) = (- \otimes \omega_X)[d]$$

is called the Serre functor. Serre duality may be trivially restated as the property that there are bifunctorial isomorphisms

$$\operatorname{Hom}_{D(X)}(E,F) \cong \operatorname{Hom}_{D(X)}(F, S_X(E))^*$$

for all objects $E, F \in D(X)$. It is easy to see using the Yoneda Lemma that this property determines S_X uniquely up to isomorphism of functors.

2.2. **Coherent sheaves.** Objects of Coh(X) can be thought of as 'vector bundles with varying fibres'. The fibre of $E \in \mathrm{Coh}(X)$ at a closed point $x \in X$ is

$$E_{(x)} = E_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C}.$$

It is a simple consequence of Nakayama's Lemma that the subsets

$$S_i(E) = \{x \in X : \dim_{\mathbb{C}} E_{(x)} \geqslant i\} \subset X$$

are closed. Setting $V_i(E) = S_i(E) \setminus S_{i+1}(E)$ we get a stratification of X into disjoint, locally-closed subvarieties $V_i(E)$, such that each restriction $E|_{V_i(E)}$ is locally-free. In particular, given $E \in \mathrm{Coh}(X)$ the *support of* E is the closed subset

$$\mathrm{supp}(E) = S_0(E) \subset X$$

consisting of points where E has nonzero fibre. A sheaf E is *torsion-free* if $\mathrm{supp}(A) = X$ for all $0 \neq A \subset E$. Note that a subsheaf of a torsion-free sheaf is automatically torsion-free.

To form non-stacky moduli spaces of coherent sheaves we must first restrict attention to a class of stable sheaves. There are several notions of stability, but for simplicity in what follows we will only consider μ-stability. To define this we must first fix a *polarization* of X: a class $\omega \in H^2(X, \mathbb{Z})$ which is the first Chern class of an ample line bundle. The *degree* of a sheaf E is then defined to be

$$d(E) = c_1(E) \cdot \omega^{d-1},$$

and the *slope* of a torsion-free sheaf is $\mu(E) = d(E)/r(E)$. A torsion-free sheaf is said to be *μ-semistable* if

$$0 \neq A \subsetneq E \implies \mu(A) \leqslant \mu(E).$$

Replacing the inequality with strict inequality gives the notion of *μ-stability.*

Theorem 2.1. (a) *Fix a Chern character v such that sheaves of this class have $r(E)$ and $d(E)$ coprime. Then there is a fine projective moduli scheme $\mathcal{M}_{X,\omega}(v)$ for μ-stable torsion-free sheaves of this class.*

(b) *Every torsion-free sheaf E has a unique Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E \tag{6}$$

whose factors $F_i = E_i/E_{i-1}$ are μ-semistable with descending slopes:

$$\mu(F_1) > \mu(F_2) > \cdots > \mu(F_n).$$

(c) *If E and F are μ-semistable and $\mu(E) > \mu(F)$ then* $\mathrm{Hom}_X(E, F) = 0$.

(d) *If E and F are μ-stable of the same slope then any nonzero map $E \to F$ is an isomorphism.*

(e) *If* E *is* μ*-stable then* $\mathrm{End}_X(E) = \mathbb{C}$.

Proof. Part (a) comes from geometric invariant theory. The given assumptions ensure that for torsion-free sheaves of class v the notions of μ-stability and μ-semistability coincide, leading to a projective moduli space. They also ensure that this moduli space is fine. Part (b) is fairly easy. For (c) consider a nonzero map $f\colon E \to F$ and factor it via its image

$$0 \to K \hookrightarrow E \twoheadrightarrow I \hookrightarrow F \twoheadrightarrow Q \to 0.$$

Then $K = \ker(f)$ satisfies $\mu(K) < \mu(E)$, which by the additivity of rank and degree implies that $\mu(E) < \mu(I)$. On the other hand, $I = \mathrm{Im}(f)$ is a subsheaf of F and hence satisfies $\mu(I) \leqslant \mu(F)$. This implies that $\mu(E) < \mu(F)$, a contradiction. The same argument works for part (d). Part (e) then holds because $\mathrm{End}_X(E)$ is a finite-dimensional division algebra over $\mathbb{C}$. □

2.3. **Derived category of** $\mathbb{P}^1$**.** By Exercise 1.5.3 every object in $D(\mathbb{P}^1)$ is a sum of its cohomology sheaves. Exercise 2.4.1 shows that any indecomposable sheaf is either a vector bundle or a fattened skyscraper. A well-known result (see Exercise 2.4.2) states that the only indecomposable vector bundles on $X = \mathbb{P}^1$ are the line bundles $\mathcal{O}(i)$ for $i \in \mathbb{Z}$.

We can represent the category $D(\mathbb{P}^1)$ graphically by drawing its Auslander-Reiten quiver: this has a vertex for each indecomposable object of $D(\mathbb{P}^1)$, and an arrow for each irreducible morphism (a morphism is called irreducible if it cannot be written as a composition $g \circ h$ with neither g nor h an isomorphism).

In fact the same category can be described in a different way. Consider the Kronecker quiver Q and the abelian category $\mathrm{Rep}(Q)$ of its finite-dimensional representations. It is easy enough to show that for all $n \geqslant 1$ there is a unique (up to isomorphism) indecomposable representation of Q of dimension vector $(n, n-1)$ and $(n-1, n)$, and a $\mathbb{P}^1$ worth of indecomposable representations of dimension vector (n, n). Categories of representations of quivers (without relations) always have global dimension 1, so Exercise 1.5.3 applies again, and we can draw the Auslander-Reiten quiver as before.

The pictures suggest that the categories $D(\mathbb{P}^1)$ and $D(Q)$ are equivalent. In fact, if we choose a basis for $\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}(1)) \cong \mathbb{C}^2$ we can define a functor $F\colon \mathrm{Coh}(\mathbb{P}^1) \to \mathrm{Rep}(Q)$ by the rule

$$E \mapsto \big(\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(1), E) \Longrightarrow \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}, E)\big).$$

The associated derived functor is then an equivalence $D(\mathbb{P}^1) \to D(Q)$ which matches up the two pictures as in the diagram.

Note that the underived functor F is definitely not an equivalence since it kills the objects $\mathcal{O}(i)$ for $i < 0$. In fact, the abelian categories $\mathrm{Coh}(\mathbb{P}^1)$ and $\mathrm{Rep}(Q)$ are not equivalent: to see this note that the simple objects in $\mathrm{Coh}(\mathbb{P}^1)$ are the skyscraper sheaves $\mathcal{O}_x$ and the only finite-length objects are sheaves supported in dimension 0, whereas the category $\mathrm{Rep}(Q)$ is a finite-length category with only two simple objects $(0,1)$ and $(1,0)$ up to isomorphism.

We can use the above equivalence to identify the two derived categories and think of a single triangulated category D. But we then have two different abelian subcategories $\mathrm{Coh}(\mathbb{P}^1), \mathrm{Rep}(Q) \subset D$. There is an interesting auto-equivalence of D which corresponds to tensoring with $\mathcal{O}(1)$ in $D(\mathbb{P}^1)$. This auto-equivalence preserves the subcategory $\mathrm{Coh}(\mathbb{P}^1) \subset D$ but not $\mathrm{Rep}(Q) \subset D$, illustrating the fact that the derived category of an abelian category can have extra symmetries not visible at the underived level.

Tilting objects. Let X be a smooth projective variety. An object $T \in D(X)$ is called a *tilting object* if

$$\mathrm{Ext}^i_X(T,T) = 0 \text{ unless } i = 0 \text{ and } \mathrm{Hom}^\bullet_X(T,E) = 0 \implies E \cong 0.$$

It follows that the (usually non-commutative) finite-dimensonal $\mathbb{C}$-algebra $A = \mathrm{End}_X(E)$ is of finite global dimension, and the derived functor

$$\mathbf{R}\,\mathrm{Hom}_X(T,-)\colon D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Mod}_{fg}(A))$$

is an equivalence. In the above example $T = \mathcal{O} \oplus \mathcal{O}(1)$, and A is the path algebra of Q.

2.4. **Problems.**

2.4.1. Let X be a curve. Prove that any indecomposable object $E \in \mathrm{Coh}(X)$ is either locally-free, or is of the form $\mathcal{O}_{nx}$ for some $x \in X$ and $n \geqslant 1$.

2.4.2. Prove that every indecomposable vector bundle on $X = \mathbb{P}^1$ is a line bundle as follows. First prove using the Harder-Narasimhan filtration and Serre duality that every indecomposable vector bundle is stable. Next use Serre duality to show that any stable vector bundle E is rigid, i.e. satisfies $\mathrm{Ext}^1_X(E,E) = 0$. Finally use Riemann-Roch to get the result.

2.4.3. Suppose that X is an elliptic curve and $E \in \mathrm{Coh}(X)$ is locally-free. Prove that

$$E \ \mu\text{-stable} \implies E \text{ indecomposable} \implies E \ \mu\text{-semistable}.$$

Conclude that if $\mathrm{ch}(E) = (r,d)$ with $\gcd(r,d) = 1$ then all three notions coincide.

2.4.4. Let $\mathcal{M}_X(2,1)$ be the moduli space of μ-stable vector bundles on an elliptic curve X of rank 2 and degree 1. Prove that $\mathcal{M}_X(2,1) \cong X$ by showing that every such bundle is an extension of line bundles of degrees 0 and 1 respectively.

3. Fourier-Mukai transforms

In this lecture we introduce integral functors and state the famous Bondal-Orlov theorem, which gives a criterion for such a functor to be an equivalence.

3.1. **Integral functors.** Let X, Y be smooth projective varitieties. For each $y \in Y$ we denote by $i_y \colon X \hookrightarrow Y \times X$ the closed embedding $x \mapsto (y, x)$. We can view an object $\mathcal{P} \in D(Y \times X)$ as defining a family of objects

$$\mathcal{P}_y = \mathbf{L}i_y^*(\mathcal{P}) \in D(X)$$

parameterised by $y \in Y$. Here $\mathbf{L}i_y^*$ denotes the left derived functor of the right exact functor i_y^*.

Lemma 3.1. *The objects $\mathcal{P}_y \in D(X)$ are all sheaves (i.e. they are all concentrated in degree 0), precisely if $\mathcal{P} \in D(Y \times X)$ is a sheaf, flat over Y.*

Proof. One implication is easy: to define the derived restriction $\mathbf{L}i_y^*(\mathcal{P})$ one first replaces $\mathcal{P}$ by a quasi-isomorphic complex of Y-flat sheaves; if $\mathcal{P}$ is Y-flat itself then $\mathcal{P}_y$ is just the usual restricted sheaf $\mathcal{P}|_{\{y\} \times X}$.

In the other direction let us assume that all the objects $\mathcal{P}_y \in D(X)$ are concentrated in degree 0. Let n be the maximum integer such that $H^n(\mathcal{P}) \neq 0$. Consider the triangle in $D(Y \times X)$

$$\tau_{\leqslant n-1}(\mathcal{P}) \longrightarrow \mathcal{P} \longrightarrow H^n(\mathcal{P})[-n]. \tag{7}$$

Applying the derived functor $\mathbf{L}i_y^*(-)$ gives a triangle

$$\mathbf{L}i_y^*(\tau_{\leqslant n-1}(\mathcal{P})) \longrightarrow \mathcal{P}_y \longrightarrow \mathbf{L}i_y^*(H^n(\mathcal{P}))[-n] \tag{8}$$

in $D(X)$. Since $\mathbf{L}i_y^*$ is a left derived functor, it 'spreads things out to the left', so all terms are concentrated in degrees $\leqslant n$ and the first term is concentrated in degrees $\leqslant n-1$. Taking the long exact sequence in cohomology we see that $H^n(\mathcal{P}_y) = i_y^*(H^n(\mathcal{P}))$, which by assumption on n is nonzero for some $y \in Y$. Since $\mathcal{P}_y$ is assumed to be concentrated in degree 0 we conclude that $n = 0$.

Taking cohomology of (8) again we see that $H^{-1}(\mathbf{L}i_y^*(H^0(\mathcal{P})) = 0$ for all $y \in Y$, which by the local criterion of flatness tells us that $H^0(\mathcal{P})$ is flat over Y. We now know that the last two terms in (8) are concentrated in degree 0. Since the first one is concentrated in degrees $\leqslant -1$ it must be zero. It follows that $\tau_{\leqslant -1}(\mathcal{P}) = 0$ which shows that $\mathcal{P}$ is concentrated in degree 0. □

Remark 3.2. The same argument gives a local version of this statement: if some particular $\mathcal{P}_y$ is concentrated in degree 0, then for all $x \in X$, the stalk of $H^i(\mathcal{P})$ at (x, y) is zero for $i \neq 0$, and flat over $\mathcal{O}_{Y,y}$ for $i = 0$.

Define projection maps

$$Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$$

and consider the functor

$$\Phi^{\mathcal{P}}_{Y\to X}(-) = \mathbf{R}\pi_{X,*}(\mathcal{P} \otimes \pi_Y^*(-)).$$

Note that we do not need to derive π_Y^* because it is an exact functor (the projection map is flat). The tensor product appearing is the one in $D(X)$, which is computed by first replacing objects by quasi-isomorphic complexes of locally-free sheaves.

Lemma 3.3. *We have the relation*

$$\Phi^{\mathcal{P}}_{Y\to X}(\mathcal{O}_y) = \mathcal{P}_y.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{i_y} & X \times Y & \xrightarrow{\pi_X} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_Y} & & \\ \{y\} & \xrightarrow{j_y} & Y & & \end{array}$$

First use base-change around the Cartesian square

$$\pi_Y^*(\mathcal{O}_y) = \pi_Y^*(j_{y,*}(\mathcal{O})) \cong i_{y,*}(p^*(\mathcal{O})) = i_{y,*}(\mathcal{O}_X).$$

Note that these functors are all exact. Now use the projection formula

$$\mathcal{P} \otimes i_{y,*}(\mathcal{O}_X) \cong i_{y,*}(\mathbf{L}i_y^*(\mathcal{P}) \otimes \mathcal{O}_X) \cong i_{y,*}(\mathcal{P}_y).$$

Finally, using the fact that $\pi_X \circ i_y \cong \mathrm{id}_X$, we get

$$\Phi^{\mathcal{P}}_{Y\to X}(\mathcal{O}_y) = \mathbf{R}\pi_{X,*}(i_{y,*}(\mathcal{P}_y)) \cong \mathcal{P}_y,$$

which completes the proof. □

A functor $\Phi\colon D(Y) \to D(X)$ which is isomorphic to one of the form $\Phi^{\mathcal{P}}_{Y\to X}$ is called an *integral functor*. Such functors are very important due to

Theorem 3.4 (Orlov)**.** *If X and Y are smooth projective varieties then any triangulated equivalence $\Phi\colon D(Y) \to D(X)$ is an integral functor.* □

3.2. **The Bondal-Orlov theorem.** The following very useful result allows us to write down many examples of varieties with equivalent derived categories.

Theorem 3.5 (Bondal, Orlov)**.** *Let X and Y be smooth projective varieties. An integral functor $\Phi\colon D(Y) \to D(X)$ is an equivalence if and only if*

(a) $\mathrm{Hom}^i_{D(X)}(\Phi(\mathcal{O}_{y_1}), \Phi(\mathcal{O}_{y_2})) = 0$ *unless* $y_1 = y_2$ *and* $0 \leqslant i \leqslant \dim(Y)$,

(b) $\mathrm{Hom}_{D(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) = \mathbb{C}$,

(c) $\Phi(\mathcal{O}_y) \otimes \omega_X \cong \Phi(\mathcal{O}_y)$.

One can easily check that the conditions of Theorem 3.5 are necessary. Indeed, one can easily compute (Exercise 3.4.4) that

$$\mathrm{Ext}^i_Y(\mathcal{O}_{y_1}, \mathcal{O}_{y_2}) = \begin{cases} \bigwedge^i \mathbb{C}^d & \text{if } y_1 = y_2, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

If Φ is an equivalence commuting with the shift functors then it preserves the $\mathrm{Hom}^i(-,-)$ spaces so (a) and (b) must hold. For (c) note that an equivalence must intertwine the Serre functors on $D(Y)$ and $D(X)$ since these are uniquely defined by categorical conditions. Since $\mathcal{O}_y \otimes \omega_X \cong \mathcal{O}_y$ this implies that the objects $\Phi(\mathcal{O}_y)$ must also be invariant under $-\otimes \omega_X$ up to shift. It then follows that they are in fact invariant under $-\otimes \omega_X$, and that moreover X and Y must have the same dimension.

Example 3.6. Let X be an abelian variety, and let $Y = \mathrm{Pic}^0(X)$ be the dual abelian variety. By definition Y parameterizes line bundles L on X with $c_1(L) = 0$. There is a universal object $\mathcal{P}$ on $Y \times X$ called the Poincaré line bundle. The resulting functor $\Phi^{\mathcal{P}}_{Y\to X}$ is called the Fourier-Mukai transform; it was the first non-trivial example of an equivalence betwen derived categories of coherent sheaves.

The conditions (b) and (c) of Theorem 3.5 are immediate in this example. To check (a) one needs to know a non-trivial fact, namely that if $L \in \mathrm{Pic}^0(X)$ is non-trivial then $H^i(X, L) = 0$ for all i. Note that in the dimension one case when X is an elliptic curve this is easy: $H^0(X, L) = 0$ because any nonzero section $\mathcal{O}_X \to L$ would have to be an isomorphism, and Serre duality then implies that also $H^1(X, L) = 0$.

Example 3.7. Take an isomorphism of smooth projective varieties $f\colon Y \to X$, a line bundle $L \in \mathrm{Pic}(Y)$ and an integer $n \in \mathbb{Z}$. Then the functor

$$\Phi(-) = f_*(L \otimes -)[n].$$

is an equivalence $D(Y) \cong D(X)$. Functors of this form are called *standard equivalences.*

The following result gives a useful characterisation of standard equivalences.

Lemma 3.8. *Suppose* $\Phi\colon D(Y) \to D(X)$ *is a triangulated equivalence. Then* Φ *is a standard equivalence precisely if for every point* $y \in Y$ *the object* $\Phi(\mathcal{O}_y) \in D(X)$ *is a shift of a skyscraper sheaf.*

Proof. One implication is immediate, so let us assume that Φ takes skyscrapers to shifts of skyscrapers. We can write $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$ for some object $\mathcal{P}$. By assumption, for each $y \in Y$, the object $\mathcal{P}_y = \mathbf{L}i_y^*(\mathcal{P})$ is concentrated in some fixed degree. By Remark 3.2, this implies that Y is the disjoint union of the supports of the sheaves $H^i(\mathcal{P})$, and since Y is connected it follows that only one of these sheaves is nonzero. Thus composing Φ with a shift we can assume that $\mathcal{P}$ is a Y-flat sheaf such that each $\mathcal{P}_y$ is a skyscraper sheaf on X.

Now, by Exercise 3.4.6, X is a fine moduli space for skyscraper sheaves on X, so there is a morphism $f\colon Y \to X$ and a line bundle $L \in \operatorname{Pic}(Y)$ such that

$$\mathcal{P} \cong (f \times \mathrm{id}_X)^*(\mathcal{O}_\Delta) \otimes \pi_Y^*(L) = \mathcal{O}_{\Gamma(f)} \otimes \pi_Y^*(L),$$

where $\Gamma_f \subset Y \times X$ is the graph of f. It follows (see Exercise 3.4.1) that $\Phi(-) \cong f_*(L \otimes -)$. The fact that Φ is an equivalence then ensures that f is an isomorphism. $\square$

3.3. **Auto-equivalences.** As well as looking for varieties with equivalent derived categories, it is interesting to study self-equivalences of derived categories of coherent sheaves. We denote by $\operatorname{Aut} D(X)$ the group of $\mathbb{C}$-linear, triangulated auto-equivalences of the category $D(X)$, these being considered up to isomorphism of functors.

The *standard auto-equivalences* define a subgroup

$$\operatorname{Aut}_{\mathrm{stand}} D(X) = \mathbb{Z} \times \operatorname{Aut}(X) \ltimes \operatorname{Pic}(X) \subset \operatorname{Aut} D(X).$$

The following result shows that in many interesting cases this is everything:

Lemma 3.9. *Suppose that $\omega_X^{\pm 1}$ is ample and $\Phi\colon D(Y) \to D(X)$ is a triangulated equivalence. Then $Y \cong X$ and Φ is a standard equivalence.*

Proof. Fix $y \in Y$ and set $\mathcal{P}_y = \Phi(\mathcal{O}_y)$. By condition (c) of Theorem 3.5 we have $\mathcal{P}_y \otimes \omega_X \cong \mathcal{P}_y$. Since the functor $- \otimes \omega_X$ is exact, this implies the same identity for each cohomology sheaf $H^i(\mathcal{P}_y)$. But by the ampleness condition, the only sheaves invariant under $-\otimes\omega_X$ have zero-dimensional support. Since $\operatorname{End}_X(\mathcal{P}_y) = \mathbb{C}$, the object $\mathcal{P}_y$ is indecomposable, so we conclude that each cohomology sheaf $H^i(\mathcal{P}_y)$ is supported at the same point $x \in X$.

Note that any sheaf $E \in \operatorname{Coh}(X)$ supported at $x \in X$ has a filtration whose factors are the skyscraper sheaf $\mathcal{O}_x$. It follows that given two such sheaves $E, F \in \operatorname{Coh}(X)$ there always exist nonzero maps $E \to F$.

Consider the spectral sequence

$$(10)\qquad \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^p_X(H^i(\mathcal{P}_y), H^{i+q}(\mathcal{P}_y)) \implies \operatorname{Ext}^{p+q}_X(\mathcal{P}_y, \mathcal{P}_y).$$

Since Φ is an equivalence, the right-hand side is zero unless $0 \leqslant p+q \leqslant d$. If w is the maximum integer such that there is an $i \in \mathbb{Z}$ with $H^i(E)$ and $H^{i+w}(E)$ both nonzero, then we get a nontrivial term $E^{0,-w}$-term in (10) which survives to ∞. Thus $w = 0$ and $\mathcal{P}_y$ is concentrated in a fixed degree. Since the class of $\Phi(\mathcal{O}_y)$ in the numerical Grothendieck group $\mathcal{N}(X)$ must be primitive, it follows that $\mathcal{P}_y$ is a shift of a skyscraper. Applying Lemma 3.8 gives the result. □

3.4. **Problems.**

3.4.1. *Standard functors.* If $\mathcal{P} = \mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset X \times X$ show that $\Phi^{\mathcal{P}}_{X\to X}$ is isomorphic to the identity functor. More generally, let $f\colon Y \to X$ be a morphism of varieties, $L \in \operatorname{Pic}(Y)$ a line bundle, and $n \in \mathbb{Z}$ an integer. Show that if

$$\mathcal{P} = \mathcal{O}_{\Gamma_f}[n] \otimes \pi_Y^*(L) \in D(Y \times X),$$

where $\Gamma_f \subset Y \times X$ is the graph of f, then

$$\Phi^{\mathcal{P}}_{Y\to X}(-) \cong f_*(- \otimes L)[n].$$

3.4.2. *Adjoints of integral functors.* Using standard adjunctions from algebraic geometry calculate the left and right adjoints to the functor $\Phi^{\mathcal{P}}_{Y\to X}$. Use your answer to give another proof that smooth projective varieties with equivalent derived categories have the same dimension.

3.4.3. Suppose that $\mathcal{P}$ is a Y-flat sheaf on $Y \times X$ and set $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$. Using the cohomology and base-change theorem, show that for any sheaf $E \in \operatorname{Coh}(Y)$ and any ample line bundle L, the image $\Phi(E \otimes L^n)$ is a locally-free sheaf for $n \gg 0$.

3.4.4. Ext-*groups between skyscrapers.* Suppose that X is a smooth variety of dimension d and $x \in X$ is a closed point. Prove that

$$\operatorname{Ext}^i_X(\mathcal{O}_{x_1}, \mathcal{O}_{x_2}) = \begin{cases} \bigwedge^i \mathbb{C}^d & \text{if } y_1 = y_2, \\ 0 & \text{otherwise.} \end{cases}$$

You may wish to use the relationship between local and global Ext-groups, the Cohen structure theorem and the Koszul resolution.

3.4.5. Prove that there is a well-defined functor

$$\mathrm{FM}\colon D(Y \times X) \longrightarrow \operatorname{Fun}(D(Y), D(X)), \quad \mathcal{P} \mapsto \Phi^{\mathcal{P}}_{Y\to X}.$$

Show that this functor is not in general faithful, as follows. Take $Y = X$ an elliptic curve and show using Serre duality that $\operatorname{Ext}^2_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \mathbb{C}$. On the other hand, prove that any morphism of functors $\mathrm{id} \to [2]$ in the category $D(X)$ is zero.

3.4.6. *Moduli of skyscrapers.* Prove that the moduli space of skyscraper sheaves on a smooth variety X is the variety X itself, and that the universal object can be taken to be the structure sheaf of the diagonal in $X \times X$.

3.4.7. *Integral transforms preserve families.* Let S be an arbitrary base variety. An object $E \in D(S \times Y)$ is said to be S-perfect if the derived restrictions $E_s = E|_{\{s\}\times Y}$ all have bounded cohomology objects and hence live in $D(Y)$. Suppose that $\Phi\colon D(Y) \to D(X)$ is an integral functor. By using a relative integral functor defined by the projections $S \times Y \leftarrow S \times Y \times X \to S \times X$, prove that if $E \in D(S \times Y)$ is S-perfect then there is an S-perfect object $F \in D(S \times X)$ such that $F_s = \Phi(E_s)$ for all $s \in S$.

4. Calabi-Yau examples

This lecture is devoted to working out some of the general theory introduced above in the case of low-dimensional Calabi-Yau varieties, namely elliptic curves and K3 surfaces.

4.1. **Elliptic curves.** In this section we shall prove

Theorem 4.1. *Let X be a smooth projective curve of genus 1. Then $D(Y) \cong D(X)$ implies that $Y \cong X$, and moreover there is a short exact sequence*

$$1 \longrightarrow \operatorname{Aut}(X) \ltimes \operatorname{Pic}^0(X) \times \mathbb{Z} \longrightarrow \operatorname{Aut} D(X) \longrightarrow \operatorname{SL}(2,\mathbb{Z}) \longrightarrow 1. \tag{11}$$

Proof. The Chern character map descends to the numerical Grothendieck group and gives an isomorphism

$$\operatorname{ch}\colon \mathcal{N}(X) \to \mathbb{Z} \oplus \mathbb{Z}, \quad [E] \mapsto (r(E), d(E)).$$

Riemann-Roch shows that the Euler form is

$$\chi(E,F) = r(E)\,d(F) - r(F)\,d(E).$$

Any triangulated auto-equivalence of $D(X)$ induces an automorphism of $\mathcal{N}(X)$ preserving the Euler form, so we get a group homomorphism

$$\varpi\colon \operatorname{Aut} D(X) \to \operatorname{SL}(2,\mathbb{Z}).$$

Our first aim is to show that this map is surjective.

The dual abelian variety $Y = \operatorname{Pic}^0(X)$ is non-canonically isomorphic to X, by mapping $x \mapsto \mathcal{O}_X(x - x_0)$ for some base-point $x_0 \in X$. The original Fourier-Mukai transform therefore gives an auto-equivalence $\Phi \in \operatorname{Aut} D(X)$. This satisfies $\Phi(\mathcal{O}_y) = \mathcal{P}_y$. By Exercise 3.4.2, the inverse is given by $\Phi^{\mathcal{P}^*}_{X\to Y}[1]$ and so $\Phi(\mathcal{P}^*_y) = \mathcal{O}_y[1]$. We conclude that

$$\varpi(\Phi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Tensoring with a degree 1 line bundle L gives another auto-equivalence, which clearly satisfies

$$\varpi(- \otimes L) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Since these two matrices generate $\mathrm{SL}(2,\mathbb{Z})$ the map ϖ is indeed surjective.

Consider an auto-equivalence Φ lying in the kernel of ϖ. It must take a skyscraper sheaf $\mathcal{O}_x$ to an indecomposable object of class $(0,1)$. Up to shift such an object is a sheaf. But then it must be a skyscraper. So by Lemma 3.8, any such auto-equivalence is standard. Conversely a standard autoequivalence $f_*(L \otimes -)[n]$ acts trivially on $\mathcal{N}(X)$ precisely if $L \in \mathrm{Pic}^0(X)$ has degree 0, and the integer n is even. This gives the short exact sequence (11).

Finally we prove the first part of the statement. Suppose $\Phi\colon D(Y) \to D(X)$ is an equivalence. Then Φ induces an isomorphism of numerical Grothendieck groups; in particular Φ takes skyscrapers to indecomposable objects having some primitive class $(a,b) \in \mathcal{N}(X)$. Composing with an element of $\mathrm{Aut}\, D(X)$ we can assume that $(a,b) = (0,1)$. But as before, any indecomposable object of this class is a shift of a skyscraper. Thus Φ is standard, and in particular, $Y \cong X$. □

4.2. **K3 surfaces.** Recall that an algebraic K3 surface is a smooth projective surface X which is Calabi-Yau ($\omega_X = \mathcal{O}_X$) and satisfies $H^1(X, \mathcal{O}_X) = 0$. It is an important fact that all such surfaces are deformation equivalent as complex manifolds, and hence have the same cohomology groups. In particular one can calculate that $H^2(X,\mathbb{Z}) \cong \mathbb{Z}^{\oplus 22}$. The Hodge decomposition takes the form

$$H^2(X,\mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X).$$

where $H^{2,0}(X) = H^0(\omega_X) = \mathbb{C}$. The key point is that the isomorphism class of a K3 surface is completely determined by the position of the line $H^{2,0}(X)$ inside the complexification of the lattice $H^2(X,\mathbb{Z})$. This is called the Torelli theorem:

Theorem 4.2. *Two K3 surfaces are isomorphic precisely if they are Hodge isometric, i.e. if there is an isomorphism*

$$\phi\colon H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$$

such that ϕ preserves the intersection form and

$$(\phi \otimes \mathbb{C})\,(H^{2,0}(X_1)) = H^{2,0}(X_2).$$

When considering numerical invariants of coherent sheaves on K3 surfaces it is useful to introduce a minor variant of the Chern character called the *Mukai vector.* This is the map

$$v\colon K_0(X) \to H^*(X,\mathbb{Z})$$

$$v(E) = \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X)} = (r(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E) + r(E)).$$

If we also put a symmetric form on

$$H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^{22} \oplus \mathbb{Z}$$

by setting

$$\langle (r_1, d_1, s_1), (r_2, d_2, s_2) \rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1,$$

then the Riemann-Roch theorem takes the simple form

$$\chi(E, F) = -\langle v(E), v(F) \rangle.$$

The map v descends to the numerical Grothendieck group, and allows us to identify $\mathcal{N}(X)$ with the image of v, which is the subgroup

$$\mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z} \subset H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}). \tag{12}$$

Here $\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ is the Neron-Severi group of the K3 surface X. It is a free abelian group which can have any rank $1 \leqslant \rho \leqslant 19$. The first Chern class defines an isomorphism $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$.

Theorem 4.3. *Fix* $v = (r, D, s) \in \mathcal{N}(X)$ *with* $r > 0$. *Suppose there is a polarization* $\in H^2(X,\mathbb{Z})$ *such that* $\gcd(r, D \cdot \omega) = 1$. *Then the moduli space* $\mathcal{M}_{X,\omega}(v)$ *is a non-empty, smooth, complex symplectic, projective variety of dimension* $2 + \langle v, v \rangle$.

Proof. The non-emptiness statement is tricky: one has to consider a deformation to an elliptic K3. For the rest, recall the very general fact that the tangent space to the moduli space of sheaves at a point $E \in \mathrm{Coh}(X)$ is given by $\mathrm{Ext}^1_X(E, E)$. In our case Riemann-Roch gives

$$\dim_{\mathbb{C}} \mathrm{Ext}^1_X(E, E) - \dim_{\mathbb{C}} \mathrm{Ext}^0_X(E, E) - \dim_{\mathbb{C}} \mathrm{Ext}^2_X(E, E) = \langle v, v \rangle.$$

Since E is stable, $\mathrm{End}_X(E) = \mathbb{C}$ and Serre duality gives $\mathrm{Ext}^2_X(E, E) \cong \mathrm{Hom}_X(E, E)^*$. Thus the tangent space to any point of $\mathcal{M}_{X,\omega}(v)$ has constant dimension $\langle v, v \rangle + 2$, which ensures that it is smooth. The symplectic form is given by the Serre duality pairing $\mathrm{Ext}^1_X(E, E) \cong \mathrm{Ext}^1_X(E, E)^*$. □

These varieties $\mathcal{M}_{X,\omega}(v)$ and deformations of them are basically the only known examples of compact complex symplectic manifolds. More precisely, there is an analogous set of examples coming from moduli spaces of sheaves on abelian surfaces, together with two further sporadic examples related to moduli spaces of sheaves with a very special, non-primitive Mukai vector.

4.3. **Derived Torelli theorem.** Let X be an algebraic K3 surface and choose $v \in \mathcal{N}(X)$ satisfying the conditions of Theorem 4.3. Suppose further that $(v,v) = 0$ so that $Y = \mathcal{M}_{X,\omega}(v)$ is a smooth projective surface.

Lemma 4.4. *The surface Y is a K3 surface and the functor $\Phi\colon D(Y) \to D(X)$ defined by the universal object $\mathcal{P}$ on $Y \times X$ is an equivalence.*

Proof. By the Bondal-Orlov theorem, to check that Φ is an equivalence we just have to check that if $y_1 \neq y_2$ are distinct points of Y then $\operatorname{Ext}^i_X(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = 0$ for all i. By Theorem 2.1(d) there are no maps in degree 0 since the objects $\mathcal{P}_{y_i}$ are distinct stable sheaves of the same slope. Serre duality then shows that there are no maps in degree 2. Since Riemann-Roch gives $\chi(\mathcal{P}_{y_1}, \mathcal{P}_{y_2}) = -\langle v, v\rangle = 0$, this is enough.

The fact that any equivalence commutes with Serre functors implies that $\omega_Y \cong \mathcal{O}_Y$ (this also follows from the fact that Y is complex symplectic). To show that Y is a K3 surface we must check that $H^1(Y, \mathcal{O}_Y) = 0$. If we take a sufficiently ample line bundle L on X then

$$\operatorname{Hom}^i_Y(\Phi^{-1}(L^*), \mathcal{O}_y) = \operatorname{Hom}^i_X(L^*, \mathcal{P}_y) = H^i(X, \mathcal{P}_y \otimes L),$$

which is nonzero only in degree 0. It follows that $\Phi^{-1}(L^*) = M$ is a vector bundle on Y. Now L (and hence also M) is rigid:

$$\operatorname{Ext}^1_Y(M,M) \cong \operatorname{Ext}^1_X(L,L) \cong H^1(X, \mathcal{O}_X) = 0.$$

But for any vector bundle on Y the obvious map $\mathcal{O}_Y \to \mathcal{H}om_{\mathcal{O}_Y}(M,M)$ is split by the trace map, which implies that $H^1(Y, \mathcal{O}_Y)$ is a direct summand of $\operatorname{Ext}^1_Y(M,M)$. $\square$

We have now proved one of the implications in the following famous derived Torelli theorem.

Theorem 4.5 (Mukai, Orlov)**.** *Let X, Y be algebraic K3 surfaces. Then the following statements are equivalent:*

(a) *There is a $\mathbb{C}$-linear triangulated equivalence $\Phi\colon D(Y) \to D(X)$,*

(b) *$Y \cong \mathcal{M}_{X,\omega}(v)$ is a fine moduli space of μ-stable vector bundles on X,*

(c) *There is a Hodge isometry*

$$H^*(Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z}),$$

i.e. an isomorphism of groups preserving the form $\langle -, - \rangle$ whose complexification takes $H^{0,2}(Y) \subset H^*(Y, \mathbb{C})$ to $H^{0,2}(X) \subset H^*(X, \mathbb{C})$.

Proof. We proved (b) $\implies$ (a) above. To get (a) $\implies$ (c) one must show that Φ induces an isomorphism on the full cohomology groups (not just the numerical Grothendieck group (12)). This can be done by hand in a slightly ad hoc way by writing $\Phi = \Phi^{\mathcal{P}}_{Y\to X}$ and using the correspondence on cohomology induced by the Chern character of $\mathcal{P}$.

The implication (c) $\implies$ (b) goes as follows. Let $p = (0, 0, 1)$ denote the Mukai vector of a skyscraper sheaf. Given an isomorphism $\psi\colon H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ of the required type, put $v = \psi(p)$. Then v is algebraic and integral, and hence defines a primitive class in $\mathcal{N}(X)$. With a bit of jiggery-pokery involving known auto-equivalences of $D(X)$ we can even assume that $v = (r, D, s)$ is such that $r > 0$ and there exists a polarization ω with $\gcd(r, D\cdot\omega) = 1$. (If we use Gieseker stability instead of slope stability we don't need this last condition). Let $Z = \mathcal{M}_{X,\omega}(v)$ be the resulting fine moduli space, and $\Phi\colon D(Z) \to D(X)$ the corresponding equivalence. Let $\phi\colon H^*(Z, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ be the induced Hodge isometry; by definition it takes p to v. Now $\psi^{-1}\circ\phi$ is a Hodge isometry $H^*(Z, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ which preserves the class p. But since $p^\perp/\mathbb{Z}\cdot p = H^2(X, \mathbb{Z})$ this then induces a Hodge isometry $H^2(Z, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$. The usual Torelli theorem then implies $Y \cong Z$ and we are done. □

4.4. **Problems.**

4.4.1. *Moduli of bundles on an elliptic curve.* Let $\mathcal{M}_X(r, d)$ denote the moduli space of indecomposable vector bundles on an elliptic curve X of rank r and degree d. Prove that $\mathcal{M}_X(r, d) \cong X$. (If you are being careful about moduli spaces you might need Exercise 3.4.7).

4.4.2. *Auto-equivalences of an abelian surface.* Let X be an abelian surface. This is a smooth projective surface with $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = \mathbb{C}^2$.

(a) Prove that any nonzero object $E \in \mathrm{Coh}(X)$ satisfies $\dim_{\mathbb{C}} \mathrm{Ext}^1_X(E, E) \geqslant 2$. (Hint: When E is a vector bundle use the argument from the proof of Lemma 4.4; the general case can be reduced to this one by using the Fourier-Mukai trandform and Exercise 3.4.3).
(b) Use the spectral sequence (10) to show that if $\Phi\colon D(Y) \to D(X)$ is an equivalence then it takes skyscraper sheaves to shifts of sheaves.
(c) Show that any auto-equivalence which acts trivially on $\mathcal{N}(X)$ is standard and hence determine the kernel of the map $\mathrm{Aut}\, D(X) \to \mathrm{Aut}\, \mathcal{N}(X)$.

4.4.3. *Reflection functor.* Let X be a K3 surface and let $\mathcal{I}_\Delta \in \mathrm{Coh}(X \times X)$ denote the ideal sheaf of the diagonal $\Delta \subset X \times X$.

(a) Prove that $\Phi^{\mathcal{I}_\Delta}_{X\to X}$ defines an auto-equivalence $\Phi \in \mathrm{Aut}\, D(X)$.
(b) Prove that for any $E \in D(X)$ there is a distinguished triangle

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_{D(X)}(\mathcal{O}_X, E) \otimes_{\mathbb{C}} \mathcal{O}_X[-i] \longrightarrow E \longrightarrow \Phi(E).$$

(c) Let $\phi \in \operatorname{Aut} K_0(X)$ be the effect of Φ on the Grothendieck group, and set $q = [\mathcal{O}_X]$. Show that
$$\phi(v) = v - \chi(q, v)q.$$
Conclude that $\Phi^2 \in \operatorname{Aut} D(X)$ is a non-standard auto-equivalence which acts trivially on $K_0(X)$.

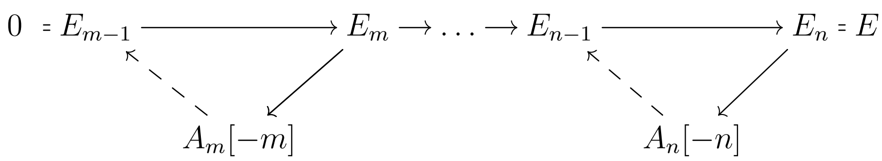
5. T-structures and tilting

We have seen interesting examples of equivalences
$$D^b(\mathcal{A}) \longrightarrow D^b(\mathcal{B}).$$
Recall that $\mathcal{A} \subset D^b(\mathcal{A})$ is a full subcategory. A slightly different way to think of this is to fix a triangulated category D and look for abelian categories $\mathcal{A} \subset D$. In fact we want more: namely the existence of cohomology and truncation functors. This leads to the definition of a t-structure.

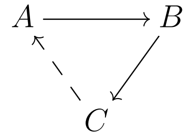
5.1. **Hearts.** Let D be a triangulated category. A *heart* $\mathcal{A} \subset D$ (or heart of a bounded t-structure) is a full additive subcategory such that:

(a) for every $k < 0$
$$\operatorname{Hom}_D(A, B[k]) = 0 \text{ for all } A, B \in \mathcal{A}.$$

(b) for every nonzero object $E \in D$ there are integers $m < n$ and objects $A_j \in \mathcal{A}$ fitting into triangles



It follows from the axioms that any heart $\mathcal{A} \subset D$ is an abelian category (see Exercise 5.4.1). The short exact sequences of $\mathcal{A}$ are the triangles



with $A, B, C \in \mathcal{A}$. We write $H^m_{\mathcal{A}}(E) = A_m \in \mathcal{A}$. This defines functors
$$H^i_{\mathcal{A}} \colon D \to \mathcal{A}.$$

The basic example of a heart is $\mathcal{A} \subset D^b(\mathcal{A})$. But not all hearts are equivalent to a heart of this form: it is not true that if $\mathcal{A} \subset D$ is a heart then $D \cong D^b(\mathcal{A})$.

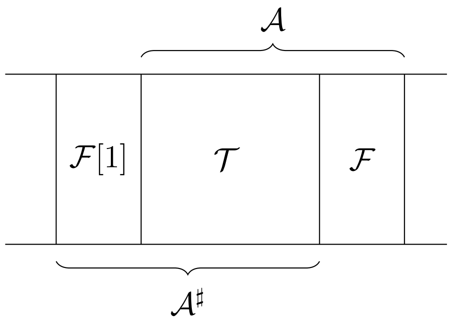


Figure 1. Tilting a heart by a torsion pair.

5.2. **Tilting.** There is a very general and important method for constructing hearts known as tilting. To explain this operation we first need

Definition 5.1. A *torsion pair* in an abelian category $\mathcal{A}$ is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ such that

(a) $\operatorname{Hom}_{\mathcal{A}}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$,
(b) for every object $E \in \mathcal{A}$ there is a short exact sequence
$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$$
for some pair of objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The objects of $\mathcal{T}$ and $\mathcal{F}$ are called *torsion* and *torsion-free* respectively. This terminology is explained by

Example 5.2. Let $\mathcal{A} = \operatorname{Coh}(X)$ for some variety X. Then there is a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ for which $\mathcal{T}$ consists of torsion sheaves (in the usual sense), and $\mathcal{F}$ consists of torsion-free sheaves.

The tilting operation is defined by the following easy result.

Lemma 5.3. *Suppose that $\mathcal{A} \subset D$ is a heart and $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$. Then the full subcategory*
$$\mathcal{A}^\sharp = \left\{ E \in D : \begin{array}{l} H^i(E) = 0 \textit{ for } i \notin \{-1, 0\}, \\ H^{-1}(E) \in \mathcal{F},\ H^0(E) \in \mathcal{T}. \end{array} \right\} \subset D$$
is a heart.

Proof. This is more-or-less obvious from the 'filmstrip' picture: Figure 1. □

We call $\mathcal{A}^\sharp$ (or sometimes $\mathcal{A}^\sharp[-1]$) the tilt of $\mathcal{A}$ with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$. Note that $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{A}^\sharp$ and tilting again gives the back the heart $\mathcal{A}[1] \subset D$.

Example 5.4. Take $D = D(X)$ with X a smooth projective variety, and $\mathcal{A} = \mathrm{Coh}(X) \subset D(X)$ the standard heart. Fix an arbitrary subset $S \subset X$ and define $\mathcal{T} \subset \mathcal{A}$ to be the full subcategory consisting of those sheaves which are supported on a zero-dimensional subset of S. Then $\mathcal{F} \subset \mathrm{Coh}(X)$ consists of those sheaves for which

$$\mathrm{Hom}_X(\mathcal{O}_x, E) = 0$$

for all $x \in S$. Tilting with respect to this torsion pair gives a different t-structure on D for each subset $S \subset X$. So the set of t-structures is very big!

Example 5.5. Take X to be a smooth projective variety, and fix a polarization $\omega \in H^2(X, \mathbb{Z})$. Let $\mathcal{A} = \mathrm{Coh}(X) \subset D$ be the standard t-structure on $D = D(X)$. For a fixed real number $\mu_0 \in \mathbf{R}$ define a torsion pair in $\mathcal{A}$ by

$$\mathcal{T} = \{E : E/\mathrm{Tor}(E) \text{ has HN factors of slope } \geqslant \mu_0\}.$$

$$\mathcal{F} = \{E : E \text{ is torsion-free with HN factors of slope } \leqslant \mu_0\}.$$

Here $\mathrm{Tor}(E)$ denotes the torsion part of the sheaf E in the usual sense. Thus all torsion sheaves are contained in $\mathcal{T}$. The HN means Harder-Narasimhan: the subcategories $\mathcal{T}$ and $\mathcal{F}$ are defined in terms of the μ-semistable sheaves appearing as factors in the unique filtration (6).

For many varieties X, for example for elliptic curves, these torsion pairs are different for every $\mu_0 \in \mathbf{R}$. The resulting tilts are important for constructing stability conditions.

Example 5.6. Consider X a variety and $\mathcal{A} = \mathrm{Coh}(X) \subset D = D(X)$. Define a torsion pair in $\mathcal{A}$ by

$$\mathcal{T} = \{E \in \mathcal{A} : \dim_{\mathbb{C}} \mathrm{supp}(E) = 0\},$$

$$\mathcal{F} = \{E \in \mathcal{A} : \mathrm{Hom}_X(\mathcal{O}_x, E) = 0 \ \forall x \in X\}.$$

Consider the heart $\mathcal{B} = \mathcal{A}^\sharp[-1] \subset D$ be the tilt shifted to the right. Note that $\mathcal{O}_X \in \mathcal{B}$. Consider the 'Hilbert scheme of the category $\mathcal{B}$'. It parameterises short exact sequences

$$0 \longrightarrow J \xrightarrow{f} \mathcal{O}_X \xrightarrow{g} Q \longrightarrow 0$$

in $\mathcal{B}$. Taking cohomology with respect to the standard t-structure gives a long exact sequence of sheaves

$$0 \longrightarrow H^0(J) \longrightarrow \mathcal{O}_X \longrightarrow H^0(Q) \longrightarrow H^1(J) \longrightarrow 0.$$

Thus $Q \in \mathcal{A}$ is a sheaf having no zero-dimensional torsion, and $g\colon \mathcal{O}_X \to Q$ is a map of sheaves whose cokernel is supported in dimension 0. These are precisely the stable pairs considered by Pandharipande-Thomas. Conversely, given such a map g one can take the cone and obtain a short exact sequence in $\mathcal{B}$.

5.3. **Threefold flops.** Let $f\colon Y \to X$ be a proper, birational map such that

(a) Y is a smooth projective threefold,
(b) f contracts only finitely many rational curves,
(c) $K_Y \cdot C = 0$ for every curve contracted by f.

Define a torsion pair in $\mathrm{Coh}(Y)$ by

$$\mathcal{T} = \big\{T \in \mathrm{Coh}(Y) : \text{the natural map } f^*(f_*(T)) \to T \text{ is surjective}\big\}.$$

$$\mathcal{F} = \big\{F \in \mathrm{Coh}(Y) : f_*(F) = 0\big\}.$$

The resulting tilt is denoted $\mathrm{Per}(Y/X) \subset D(Y)$. Note that $\mathcal{O}_Y \in \mathcal{T} \subset \mathrm{Per}(Y/X)$. Define a *perverse point sheaf* to be an object $E \in \mathrm{Per}(Y/X)$ which is a quotient of $\mathcal{O}_Y$ and has same Chern character as $\mathcal{O}_y$.

Theorem 5.7. *There is a fine moduli scheme $\mathcal{M}$ for perverse point sheaves in* $\mathrm{Per}(Y/X)$*. This scheme has a natural map $g\colon \mathcal{M} \to X$ which is the flop of $f\colon Y \to X$. The universal perverse point sheaf $\mathcal{P}$ on $\mathcal{M} \times X$ gives rise to an equivalence $\Phi\colon D(\mathcal{M}) \to D(X)$.*

Since any birational map between Calabi-Yau threefolds can be decomposed into a sequence of flops it follows that birational Calabi-Yau threefolds have equivalent derived categories. It is expected that this result is also true in higher dimensions but this seems to be a very difficult problem.

Example 5.8. Consider the case when f contracts a single rational curve $C \cong \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. For any $x \in C$ there is a short exact sequences $\mathcal{O}_C(-1) \to \mathcal{O}_C \to \mathcal{O}_x$. By rotating the triangle this can be rewritten as a short exact sequence in $\mathrm{Per}(Y/X)$

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{f} \mathcal{O}_y \xrightarrow{g} \mathcal{O}_C(-1)[1] \longrightarrow 0. \tag{13}$$

The objects $\mathcal{O}_C$ and $\mathcal{O}_C(-1)[1]$ are perverse sheaves, and hence so is $\mathcal{O}_x$. But the unique map $\mathcal{O}_Y \to \mathcal{O}_y$ is not surjective in $\mathrm{Per}(Y/X)$ because its composite with g is zero; in fact its cokernel is $\mathcal{O}_C(-1)[1]$.

There is a different set of extensions in $\mathrm{Per}(Y/X)$ that are perverse point sheaves, namely

$$0 \longrightarrow \mathcal{O}_C(-1)[1] \xrightarrow{f} E \xrightarrow{g} \mathcal{O}_C \longrightarrow 0. \tag{14}$$

In fact we can show that the relevant extension group is 2-dimensional, so there is also a $\mathbb{P}^1$ parameterizing these extensions. This defines the rational curve in the flop $\mathcal{M}$ which is contracted by the natural map $\mathcal{M} \to X$.

To compute the extension group let $i\colon C \to Y$ be the closed embedding. Then by adjunction

$$\operatorname{Hom}^k_{D(Y)}(i_*(\mathcal{O}_C), i_*(\mathcal{O}_C(-1))) = \operatorname{Hom}^k_{D(C)}(\mathbf{L}i^*(i_*(\mathcal{O}_C)), \mathcal{O}_C(-1)).$$

Whenever we have an embedding of smooth varieties $C \subset Y$ there are identifications

$$H_q(\mathbf{L}i^*(i_*(\mathcal{O}_C)) \cong \bigwedge^q \mathcal{N}^*_{C\subset Y}$$

where $\mathcal{N}^*_{C\subset Y}$ is the conormal bundle. We now use the spectral sequence (10)

$$E_2^{p,q} = \operatorname{Hom}^p_{D(C)}(\bigwedge^q \mathcal{O}_C(1)^{\oplus 2}, \mathcal{O}_C(-1)) \implies \operatorname{Hom}^{p+q}_{D(Y)}(\mathcal{O}_C, \mathcal{O}_C(-1)).$$

Since C is a curve we have $E_2^{p,q} = 0$ unless $0 \leqslant p \leqslant 1$ so the spectral sequence degenerates at the E_2-term. We can rewrite

$$E_2^{p,q} = H^p\left(\mathbb{P}^1, \mathcal{O}(-1) \otimes \bigwedge^q \mathcal{O}(-1)^{\oplus 2}\right).$$

We thus conclude that

$$\operatorname{Hom}^k_{D(Y)}(\mathcal{O}_C, \mathcal{O}_C(-1)) = \begin{cases} \mathbb{C}^2 & \text{if } k = 2, 3, \\ 0 & \text{otherwise,} \end{cases}$$

which shows that the extension groups controlling both (13) and (14) are two-dimensional.

5.4. **Problems.**

5.4.1. Prove from the axioms that a heart is an abelian category.

5.4.2. Define a distance function on hearts by

$$d(\mathcal{A}, \mathcal{B}) = \inf_{n \geqslant 0} \left\{ \exists a \in \mathbb{Z} : \forall A \in \mathcal{A}, H^i_{\mathcal{B}}(A) = 0 \text{ for } i \notin [a, a+n] \right\}.$$

Prove that

(a)

$$d(\mathcal{A}, \mathcal{B}) = 0 \iff \mathcal{A} = \mathcal{B}[a] \text{ for some } a \in \mathbb{Z}.$$

(b) The function d is symmetric and satisfies the triangle inequality, and hence defines a metric on the set of t-structures up to shift.

(c) Given hearts $\mathcal{A}, \mathcal{B} \subset D$ show that there is a torsion pair in $\mathcal{A}$ whose tilt is some shift of $\mathcal{B}$ precisely if $d(\mathcal{A}, \mathcal{B}) \leqslant 1$.

5.4.3. What is the tilt of the standard heart in $D(\mathbb{P}^1)$ with respect to the torsion pair of Example 5.5 corresponding to some parameter $\mu_0 \in \mathbf{R}$?

5.4.4. Let X be an elliptic curve and $\mu_0 \in \mathbb{Q}$ a rational number. What is the tilt of the standard heart in $D(X)$ with respect to the torsion pair of Example 5.5? What happens when $\mu_0 \in \mathbf{R} \setminus \mathbb{Q}$ is irrational?