Length of meridian arc on an ellipsoid

Suppose the Earth is modelled as an ellipsoid with semi-major axis a, semi-minor axis b, and eccentricity e. A standard problem is to find the length of a meridian arc between two geographical latitudes, say φ_1 and φ_2. Three methods are suggested below. For more on this topic see e.g. the lecture notes by Deakin and Hunter [1].

The radius of curvature in meridian at latitude φ is given by

$$\rho = a(1-e^2)(1-e^2\sin^2\varphi)^{-3/2} = a(1+\varepsilon)^{1/2}(1+\varepsilon\cos^2\varphi)^{-3/2}$$

where $\varepsilon = e^2/(1-e^2) = (a^2-b^2)/b^2$.

We work here in terms of ε, which is assumed to be small ($\varepsilon \approx 1/150$ for the Earth). The arc length is given by

$$a(1+\varepsilon)^{1/2}\int_{\varphi_1}^{\varphi_2}(1+\varepsilon\cos^2\varphi)^{-3/2}\,d\varphi.$$

Method 1: Numerical integration

In theory this method is subject to rounding errors, due to the addition of many terms. In practice it seems to work well enough, provided one uses an accurate integration formula such as Weddle's rule (rather than the better known Simpson's and three-eighths rules). In Weddle's rule the number of intervals is a multiple of 6, and the integral over a block of 6 intervals is estimated as

$$\int_0^{6h} y(x)\,dx = \frac{3h}{10}[\,y(0)+5y(h)+y(2h)+6y(3h)+y(4h)+5y(5h)+y(6h)].$$

The number of intervals can be repeatedly doubled until the value stays constant within the desired order of accuracy. This method is slow, but is useful as a check on other methods.

Method 2: integration of the power series

Here we expand the integrand $(1+\varepsilon\cos^2\varphi)^{-3/2}$ by the binomial theorem, taking as many terms as are required to achieve the desired accuracy. The powers $\cos^{2m}\varphi$ are integrated by rewriting them in terms of multiple angles:

$$\cos^{2m}\varphi = 2^{1-2m}\left[\frac{1}{2}\binom{2m}{m}+\binom{2m}{m-1}\cos2\varphi+\binom{2m}{m-2}\cos4\varphi+\binom{2m}{m-3}\cos6\varphi+\ldots+\cos2m\varphi\right]$$

We may as well multiply the result by the binomial expansion of $(1+\varepsilon)^{1/2}$, since this does not make it any more complicated. If this is done, the length of the meridian arc is

$$a\,[c_0(\varphi_2-\varphi_1)+c_1(\sin2\varphi_2-\sin2\varphi_1)+c_2(\sin4\varphi_2-\sin4\varphi_1)+c_3(\sin6\varphi_2-\sin6\varphi_1)+\ldots],$$

where the c_m are power series in ε. To reduce the effect of rounding errors when $\varphi_2-\varphi_1$ is small, it may be better to rewrite $\sin2m\varphi_2-\sin2m\varphi_1$ as $2\sin m(\varphi_2-\varphi_1)\cos m(\varphi_2+\varphi_1)$. The coefficients c_m are given below up to ε^8 (equivalent to e^{16}), which should be more than enough for most purposes.

$$c_0 = 1-\frac{1}{4}\varepsilon+\frac{13}{64}\varepsilon^2-\frac{45}{256}\varepsilon^3+\frac{2\,577}{16\,384}\varepsilon^4-\frac{9\,417}{65\,536}\varepsilon^5+\frac{139\,613}{1\,048\,576}\varepsilon^6-\frac{522\,821}{4\,194\,304}\varepsilon^7+\frac{126\,287\,705}{1\,073\,741\,824}\varepsilon^8-\ldots$$

$$c_1 = -\frac{3}{8}\varepsilon+\frac{9}{32}\varepsilon^2-\frac{237}{1\,024}\varepsilon^3+\frac{819}{4\,096}\varepsilon^4-\frac{23\,325}{131\,072}\varepsilon^5+\frac{84\,711}{524\,288}\varepsilon^6-\frac{4\,993\,233}{33\,554\,432}\varepsilon^7+\frac{18\,593\,103}{134\,217\,728}\varepsilon^8-\ldots$$

$$c_2 = \frac{15}{256}\varepsilon^2-\frac{75}{1\,024}\varepsilon^3+\frac{1\,245}{16\,384}\varepsilon^4-\frac{4\,905}{65\,536}\varepsilon^5+\frac{607\,125}{8\,388\,608}\varepsilon^6-\frac{2\,332\,785}{33\,554\,432}\varepsilon^7+\frac{35\,785\,995}{536\,870\,912}\varepsilon^8-\ldots$$

$$c_3 = -\frac{35}{3\,072}\varepsilon^3+\frac{245}{12\,288}\varepsilon^4-\frac{19\,985}{786\,432}\varepsilon^5+\frac{90\,475}{3\,145\,728}\varepsilon^6-\frac{3\,093\,755}{100\,663\,296}\varepsilon^7+\frac{12\,812\,765}{402\,653\,184}\varepsilon^8-\ldots$$

$$c_4 = \frac{315}{131\,072}\varepsilon^4-\frac{2\,835}{524\,288}\varepsilon^5+\frac{68\,859}{8\,388\,608}\varepsilon^6-\frac{354\,627}{33\,554\,432}\varepsilon^7+\frac{26\,776\,197}{2\,147\,483\,648}\varepsilon^8-\ldots$$

$$c_5 = -\frac{693}{1\,310\,720}\varepsilon^5+\frac{7\,623}{5\,242\,880}\varepsilon^6-\frac{430\,353}{167\,772\,160}\varepsilon^7+\frac{498\,267}{134\,217\,728}\varepsilon^8-\ldots$$

$$c_6 = \frac{1\,001}{8\,388\,608}\varepsilon^6-\frac{13\,013}{33\,554\,432}\varepsilon^7+\frac{418\,847}{536\,870\,912}\varepsilon^8-\ldots$$

$$c_7 = -\frac{6\,435}{234\,881\,024}\varepsilon^7+\frac{96\,525}{939\,524\,096}\varepsilon^8-\ldots$$

$$c_8 = \frac{109\,395}{17\,179\,869\,184}\varepsilon^8 - \ldots$$

Method 3: Another power series method

The following method may be of interest in that it can be extended by computer to any desired degree of accuracy, without the need to program-in complicated coefficients like those in Method 2. It finds the arc length along the meridian from the equator to a given latitude φ. The arc length from φ_1 to φ_2 can be found by applying the method to φ_1 and φ_2, and taking the difference.

We expand $(1 + \varepsilon\cos^2\varphi)^{-3/2}$ by the binomial theorem, as in Method 2, but now integrate the $\cos^{2m}\varphi$ as follows. Setting

$$K_m = \int_0^\varphi \cos^{2m}\varphi\, d\varphi,$$

we find easily that $2mK_m = (2m-1)K_{m-1} + \sin\varphi\cos^{2m-1}\varphi$, whence

$$K_0 = \varphi, \qquad K_1 = \frac{1}{2}\varphi + \frac{1}{2}\sin\varphi\cos\varphi, \qquad K_2 = \frac{3}{8}\varphi + \frac{3}{8}\sin\varphi\cos\varphi + \frac{1}{4}\sin\varphi\cos^3\varphi, \qquad \text{etc.}$$

A routine calculation shows that the integral, which has to be multiplied by $a(1+\varepsilon)^{1/2}$ $(= a^2/b)$ to get the arc length, is

$$\begin{aligned}
&\varphi\,(1 - \tfrac{3}{4}\varepsilon + \tfrac{3}{4}\cdot\tfrac{15}{16}\varepsilon^2 - \tfrac{3}{4}\cdot\tfrac{15}{16}\cdot\tfrac{35}{36}\varepsilon^3 + \tfrac{3}{4}\cdot\tfrac{15}{16}\cdot\tfrac{35}{36}\cdot\tfrac{63}{64}\varepsilon^4 - \tfrac{3}{4}\cdot\tfrac{15}{16}\cdot\tfrac{35}{36}\cdot\tfrac{63}{64}\cdot\tfrac{99}{100}\varepsilon^5 + \ldots)\\
&+ \sin\varphi\cos\varphi\,(- \tfrac{3}{4}\varepsilon + \tfrac{3}{4}\cdot\tfrac{15}{16}\varepsilon^2 - \tfrac{3}{4}\cdot\tfrac{15}{16}\cdot\tfrac{35}{36}\varepsilon^3 + \tfrac{3}{4}\cdot\tfrac{15}{16}\cdot\tfrac{35}{36}\cdot\tfrac{63}{64}\varepsilon^4 - \tfrac{3}{4}\cdot\tfrac{15}{16}\cdot\tfrac{35}{36}\cdot\tfrac{63}{64}\cdot\tfrac{99}{100}\varepsilon^5 + \ldots)\\
&+ \tfrac{1}{2}\sin\varphi\cos^3\varphi\,(\tfrac{15}{16}\varepsilon^2 - \tfrac{15}{16}\cdot\tfrac{35}{36}\varepsilon^3 + \tfrac{15}{16}\cdot\tfrac{35}{36}\cdot\tfrac{63}{64}\varepsilon^4 - \tfrac{15}{16}\cdot\tfrac{35}{36}\cdot\tfrac{63}{64}\cdot\tfrac{99}{100}\varepsilon^5 + \ldots)\\
&+ \tfrac{1}{2}\cdot\tfrac{3}{4}\sin\varphi\cos^5\varphi\,(- \tfrac{35}{36}\varepsilon^3 + \tfrac{35}{36}\cdot\tfrac{63}{64}\varepsilon^4 - \tfrac{35}{36}\cdot\tfrac{63}{64}\cdot\tfrac{99}{100}\varepsilon^5 + \ldots)\\
&+ \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\sin\varphi\cos^7\varphi\,(\tfrac{63}{64}\varepsilon^4 - \tfrac{63}{64}\cdot\tfrac{99}{100}\varepsilon^5 + \ldots)\\
&+ \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{8}\sin\varphi\cos^9\varphi\,(- \tfrac{99}{100}\varepsilon^5 + \ldots)\\
&+ \ldots
\end{aligned}$$

It is clear how the pattern can be continued indefinitely. The following tested piece of code, written in Delphi Pascal, shows how the method can be applied to give any degree of precision up to the limit of the language used.

```
const
  MAX_POWER_EPSILON = 8;  // arbitrary: higher if machine precision warrants it
var
  coeff : array [0..MAX_POWER_EPSILON] of extended;
  // Delphi "extended" is floating-point with 64-bit precision (19 sig. fig.)

{-----------------------------------------------------------------------------------------------------
Compute the coefficients of phi, sin(phi)*cos(phi), sin(phi)*cos^3(phi), etc.
This procedure need only be called once for a given ellipsoid (a, b).
}
procedure ComputeCoefficients( a, b : extended);
var
  epsilon : extended;
  j, k : integer;
  denom, mult : extended;
begin
  epsilon := (a - b)*(a + b) / (b*b);

  // Compute sums shown inside large () in Web text
  j := MAX_POWER_EPSILON;
  denom := 4.0*j*j;
  coeff[j] := (denom - 1.0)/denom;
  for j := MAX_POWER_EPSILON - 1 downto 0 do begin
    for k := MAX_POWER_EPSILON downto j + 1 do coeff[k] := -epsilon * coeff[k];
    if (j > 0) then begin
      denom := 4.0*j*j;
      coeff[j] := (coeff[j+1] + 1.0) * (denom - 1.0)/denom;
      end
    else coeff[0] := coeff[1] + 1.0;
  end;
```

```
  // Apply factors 1/2, 1.3/2.4, 1.3.5/2.4.6, etc. and dimensions of ellipsoid.
  mult := a*a/b;
  coeff[0] := mult*coeff[0];
  coeff[1] := mult*coeff[1];
  for j := 2 to MAX_POWER_EPSILON do begin
    mult := mult*(2*j - 3)/(2*j - 2);
    coeff[j] := mult*coeff[j];
  end;
end;

{----------------------------------------------------------------------------------------------------
Apply the above coefficients to compute arc length on meridian
  from equator to passed-in latitude phi.
}
function ComputeArcLengthFromEquator( phi_degrees : extended) : extended;
var
  phi, c, c2, total : extended;
  j : integer;
begin
  phi := (PI/180.0) * phi_degrees;  // degrees to radians
  c := Cos( phi);
  c2 := c*c;
  total := coeff[MAX_POWER_EPSILON];
  for j := MAX_POWER_EPSILON - 1 downto 1 do total := total*c2 + coeff[j];
  result := phi*coeff[0] + Sin(phi)*c*total;
end;
```

Reference

[1] R.E. Deakin and M.N. Hunter, *Geometric Geodesy A*. Downloaded from
http://user.gs.rmit.edu.au/rod/files/publications/Geometric%20Geodesy%20A(2010).pdf