Converting isometric latitude to geographic latitude

If the Earth is modelled as an oblate spheroid with eccentricity e, then the isometric latitude ψ is defined in terms of the geographic latitude φ by

$$\psi = \tanh^{-1}(\sin\varphi) - e\tanh^{-1}(e\sin\varphi). \qquad \textbf{(1)}$$

The problem is to invert (1) and find φ when ψ is given. We do this here by finding $\sin\varphi$. For other approaches, see e.g. Wolfram MathWorld [1].

If we define $s = \sin\varphi$ then (1) becomes

$$\psi = \tanh^{-1}(s) - e\tanh^{-1}(es), \qquad \textbf{(2)}$$

and we require s given ψ. Three methods are briefly described below.

Method 1: Iteration

This is straightforward. Starting with $s_0 = \tanh\psi$, or better $s_0 = \tanh\psi / (1 - e^2)$, we iterate

$$s_{n+1} = \tanh(\psi + e\tanh^{-1}(es_n))$$

until the desired accuracy has been achieved.

Method 2: Power series obtained by iteration

Write $t = \tanh\psi$. From (2)

$$\psi = \tanh^{-1}(s) - e^2 s - e^4 s^3/3 - e^6 s^5/5 - \ldots$$

whence

$$s = \tanh(\psi + e^2 s + e^4 s^3/3 + e^6 s^5/5 + \ldots).$$

Substituting the first approximation $s = t$ in the RHS gives

$$s = \tanh(\psi + e^2 t) + O(e^4) = (1 + e^2)t - e^2 t^3 + O(e^4).$$

Substituting the second approximation $s = (1 + e^2)t - e^2 t^3$ gives a third approximation with error $O(e^6)$, and so on. Taking the approximation up to terms in e^{12}, with error $O(e^{14})$, we find

$$\sin\varphi = F_1(e)t - F_3(e)t^3 + F_5(e)t^5 - F_7(e)t^7 + F_9(e)t^9 - F_{11}(e)t^{11} + F_{13}(e)t^{13} - \ldots$$

where

$$F_1(e) = 1 + e^2 + e^4 + e^6 + e^8 + e^{10} + e^{12} + \ldots$$
$$F_3(e) = e^2 + (8/3)e^4 + 5e^6 + 8e^8 + (35/3)e^{10} + 16e^{12} + \ldots$$
$$F_5(e) = (5/3)e^4 + (36/5)e^6 + (293/15)e^8 + (127/3)e^{10} + 80e^{12} + \ldots$$
$$F_7(e) = (16/5)e^6 + (6\,017/315)e^8 + (21\,319/315)e^{10} + (58\,111/315)e^{12} + \ldots$$
$$F_9(e) = (2\,069/315)e^8 + (1\,751/35)e^{10} + (619\,831/2\,835)e^{12} + \ldots$$
$$F_{11}(e) = (883/63)e^{10} + (2\,892\,031/22\,275)e^{12} + \ldots$$
$$F_{13}(e) = (1\,594\,444/51\,975)e^{12} + \ldots$$
$$\ldots.$$

It may be more convenient to rearrange this as a power series in e^2. Method 2 thus becomes:

$$\sin\varphi = t + e^2\,t\,(1 - t^2)[1 + G_1(t)e^2 + G_2(t)e^4 + G_3(t)e^6 + \ldots]$$

where

$$G_1(t) = 1 - (5/3)t^2,$$
$$G_2(t) = 1 - 4t^2 + (16/5)t^4,$$
$$G_3(t) = 1 - 7t^2 + (188/15)t^4 - (2069/315)t^6,$$
$$G_4(t) = 1 - (32/3)t^2 + (95/3)t^4 - (11\,344/315)t^6 + (883/63)t^8,$$
$$G_5(t) = 1 - 15t^2 + 65t^4 - (37\,636/315)t^6 + (281\,107/2\,835)t^8 - (1\,594\,444/51\,975)t^{10},$$
$$\ldots.$$

2016-10-22 Corrected the following errors.
The coefficient 16/5 in F_7 was mistyped as16/15.
The coefficient 188/15 in G_3 appeared as 1316/105 (correct value, but not in its lowest terms).

Method 3: Another power series method

If we define $\varepsilon = e^2 / (1 - e^2)$, then up to $O(e^{14})$ the first two F's in the preceding method are given by

$$F_1(e) = 1 + \varepsilon, \quad F_3(e) = (1 + \varepsilon)(3 + 2\varepsilon)\varepsilon/3.$$

In fact these expressions are exact, and all the F's are finite polynomials in ε. To see this, note that (2) gives

$$\frac{d\,s}{d\,t} = \frac{(1 - s^2)(1 - e^2 s^2)}{(1 - e^2)(1 - t^2)} = \frac{(1 - s^2)(1 + \varepsilon - \varepsilon\, s^2)}{(1 - t^2)}.$$

Consequently, if we set

$$\frac{d^n s}{d\, t^n} = \frac{(1 - s^2)(1 + \varepsilon - \varepsilon\, s^2)}{(1 - t^2)^n} \sum c(n; i, j) s^i t^j$$

then after some routine calculation we find the recurrence

$$c(n + 1; i, j) = \varepsilon(i + 1)c(n; i - 3, j) - (i + 1)(1 + 2\varepsilon)c(n; i - 1, j) + (i + 1)(1 + \varepsilon)c(n; i + 1, j) + (2n - j + 1)c(n; i, j - 1) + (j + 1)c(n; i, j + 1),$$

from which the $d^n s/d\, t^n$ can be calculated for $n = 1, 2, 3, \ldots$. Hence we obtain s as a power series in t,

$$\sin\varphi = (1 + \varepsilon)\left[\, t - \varepsilon P_3(\varepsilon) t^3 + \varepsilon^2 P_5(\varepsilon) t^5 - \varepsilon^3 P_7(\varepsilon) t^7 + \varepsilon^4 P_9(\varepsilon) t^9 - \ldots \right], \qquad \textbf{(3)}$$

where

$$3P_3(\varepsilon) = 3 + 2\varepsilon,$$
$$15P_5(\varepsilon) = 25 + 33\varepsilon + 11\varepsilon^2,$$
$$315P_7(\varepsilon) = 1\,008 + 1\,985\varepsilon + 1\,314\varepsilon^2 + 292\varepsilon^3,$$
$$2\,835P_9(\varepsilon) = 18\,621 + 48\,726\varepsilon + 48\,160\varepsilon^2 + 21\,288\varepsilon^3 + 3\,548\varepsilon^4,$$
$$155\,925P_{11}(\varepsilon) = 2\,185\,425 + 7\,131\,667\varepsilon + 9\,368\,049\varepsilon^2 + 6\,187\,111\varepsilon^3 + 2\,053\,245\varepsilon^4 + 273\,766\varepsilon^5,$$
$$6\,081\,075P_{13}(\varepsilon) = 186\,549\,948 + 729\,283\,824\varepsilon + 1\,194\,598\,888\varepsilon^2 + 1\,048\,861\,221\varepsilon^3 + 520\,345\,408\varepsilon^4 + 138\,241\,602\varepsilon^5 + 15\,360\,178\varepsilon^6,$$
$$638\,512\,875P_{15}(\varepsilon) = 43\,645\,743\,540 + 198\,807\,682\,245\varepsilon + 390\,064\,483\,656\varepsilon^2 + 427\,111\,938\,764\varepsilon^3 + 281\,769\,302\,772\varepsilon^4 + 111\,954\,199\,526\varepsilon^5 + 24\,798\,632\,628\varepsilon^6 + 2\,361\,774\,536\varepsilon^7,$$
$$10\,854\,718\,875P_{17}(\varepsilon) = 1\,675\,357\,363\,125 + 8\,712\,734\,726\,988\varepsilon + 19\,914\,368\,526\,088\varepsilon^2 + 26\,118\,610\,367\,256\varepsilon^3 + 21\,491\,782\,658\,226\varepsilon^4 + 11\,358\,058\,960\,776\varepsilon^5 + 3\,763\,853\,754\,228\varepsilon^6 + 714\,894\,255\,024\varepsilon^7 + 59\,574\,521\,252\varepsilon^8,$$
$$1\,856\,156\,927\,625P_{19}(\varepsilon) = 653\,523\,328\,781\,091 + 3\,820\,411\,433\,944\,980\varepsilon + 9\,967\,712\,402\,926\,242\varepsilon^2 + 15\,228\,851\,331\,982\,346\varepsilon^3 + 15\,010\,468\,930\,152\,492\varepsilon^4 + 9\,895\,994\,198\,275\,836\varepsilon^5 + 4\,362\,794\,587\,093\,944\varepsilon^6 + 1\,240\,022\,346\,019\,032\varepsilon^7 + 206\,147\,974\,037\,364\varepsilon^8 + 15\,270\,220\,299\,064\varepsilon^9,$$
$$\ldots.$$

It may be more convenient to rearrange this as a power series in ε. Method 3 thus becomes:

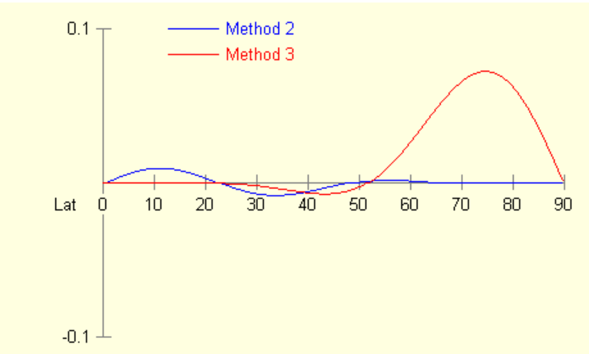
$$\sin\varphi = t + \varepsilon\, t\,(1 - t^2)[1 + Q_1(t)\varepsilon + Q_2(t)\varepsilon^2 + Q_3(t)\varepsilon^3 + \ldots]$$

where

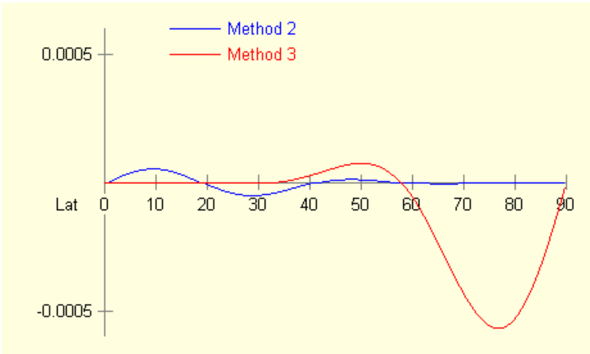
$$Q_1(t) = -(5/3)\, t^2,$$
$$Q_2(t) = -(2/15)\, t^2\,(5 - 24t^2),$$
$$Q_3(t) = (1/315)\, t^4\,(924 - 2\,069t^2),$$
$$Q_4(t) = (1/315)\, t^4\,(231 - 3\,068t^2 + 4\,415t^4),$$
$$Q_5(t) = -(2/155\,925)\, t^6\,(397\,485 - 2\,266\,880t^2 + 2\,391\,666t^4),$$
$$\ldots.$$

Discussion

The diagrams show the error on the ground when one includes terms up to and including e^8 and e^{10} respectively. Method 3 has a lower error than Method 2 up to about 30° latitude, but is much worse around 70°–80°.



Error on the ground in mm, using powers up to e^8



Error on the ground in mm, using powers up to e^10

For a method that suits all latitudes, it seems best therefore to use either iteration or Method 2. The maximum error on the ground with Method 2 is as follows:

Highest power of e included	6	8	10	12
Maximum error on ground in mm	1.9	0.010	5.9×10^{-5}	3.4×10^{-7}

Reference

[1] http://mathworld.wolfram.com/IsometricLatitude.html