Alternative formulae for the Transverse Mercator projection

Introduction

These notes describe an attempt to derive formulae for the Transverse Mercator projection that should be (a) somewhat simpler, for the same order of accuracy, than the Redfearn formulae [1, 2] used by the Ordnance Survey of Great Britain, and (b) capable of routine extension (perhaps with the help of computerized algebra) to higher orders of accuracy.

The formulae below, when applied to Great Britain, are similar in accuracy to the OS formulae if carried to the 6th order. If carried to the 12th order they agree with Karney's GeographicLib [3] to a fraction of nanometre when applied to Great Britain. They seem, however, less accurate than Karney's method when applied to a wider area.

Only formulae for coordinate conversion are given at present. Formulae for scale and convergence may be added later.

The method used here is not entirely original, as shown by the appendix to Deakin *et al.* [4]. The idea is to carry out the projection from the ellipsoid to the plane in two stages: (i) from the ellipsoid to an intermediate sphere, (ii) from the sphere to the plane. Part (ii) and its inverse can be done by simple closed formulae containing hyperbolic functions. The hard work lies in part (i).

Conformal mappings

Let S and Σ be surfaces (e.g. plane, surface of sphere). A differentiable mapping from S to Σ is called *conformal* iff it preserves the angle at which curves intersect, or (equivalently) iff the scale at any point is the same in all directions. Let (x, y) be rectangular coordinates in S such that at every point the scales in x and y are equal, i.e. if σ denotes distance in S then $\partial\sigma/\partial x = \partial\sigma/\partial y$. Let (ξ, η) be similar coordinates in Σ. Then the mapping is conformal iff at every point it satisfies the Cauchy–Riemann conditions:

$$\partial\xi/\partial x = \partial\eta/\partial y, \qquad \partial\xi/\partial y = -\partial\eta/\partial x.$$

Assuming that the mapping can be put into the form

$$\xi = f_0(y) + x f_1(y) + (x^2/2!) f_2(y) + (x^3/3!) f_3(y) + \ldots$$
$$\eta = g_0(y) + x g_1(y) + (x^2/2!) g_2(y) + (x^3/3!) g_3(y) + \ldots$$

the Cauchy–Riemann conditions are equivalent to

$$\begin{aligned} f_1 &= g_0^{(1)}, & g_1 &= -f_0^{(1)} \\ f_2 &= -f_0^{(2)}, & g_2 &= -g_0^{(2)} \\ f_3 &= -g_0^{(3)}, & g_3 &= f_0^{(3)} \\ f_4 &= f_0^{(4)}, & g_4 &= g_0^{(4)} \\ &\ldots \end{aligned}$$

So writing f, g for f_0, g_0 we see that the mapping is conformal iff it is of the form

$$\begin{aligned} \xi &= f(y) + x g^{(1)}(y) - (x^2/2!) f^{(2)}(y) - (x^3/3!) g^{(3)}(y) + (x^4/4!) f^{(4)}(y) + \ldots \\ \eta &= g(y) - x f^{(1)}(y) - (x^2/2!) g^{(2)}(y) + (x^3/3!) f^{(3)}(y) + (x^4/4!) g^{(4)}(y) - \ldots \end{aligned} \qquad \textbf{(1)}$$

Transverse Mercator projection

This is considered a suitable projection for a region such as Britain or New Zealand that extends more N–S than E–W. The Earth is taken to be an ellipsoid generated by rotating an ellipse about its smaller axis. A central meridian of longitude is chosen for the region to be mapped (e.g. for Britain the Ordnance Survey has chosen 2° W). The transverse Mercator projection from the ellipsoid to the (x, y) plane is such that

- It is conformal.
- The central meridian is mapped to the y-axis.
- The scale is the same at all points on the central meridian.

If μ_1 and μ_2 are two such mappings then $\mu_1^{-1}\mu_2$ is a conformal mapping of the plane taking $(0, y)$ to $(0, my + n)$, where m and n are constants. In the notation of the preceding section, we must then have $f(y) = 0$ and $g(y) = my + n$, so that (x, y) is sent to $(mx, my + n)$. Hence the transverse Mercator projection defined above is unique up to a change of scale and a shift parallel to the y-axis.

Notation

Where lower-case and upper-case of the same letter are given, lower-case refers to the ellipsoid and upper-case refers to the intermediate sphere.

a = semi-major axis of ellipsoid
b = semi-minor axis of ellipsoid
e = eccentricity of ellipsoid; $e^2 = (a^2 - b^2) / a^2$
$\varepsilon = e^2 / (1 - e^2) = (a^2 - b^2) / b^2$
λ = longitude east of Greenwich meridian
$\lambda_0 = \lambda$ for central meridian of projection (e.g. OS uses $\lambda_0 = -2°$)
ω, Ω = geographical longitude, relative to central meridian
φ, Φ = geographical latitude
ψ, Ψ = isometric latitude
φ_m = fixed geographical latitude chosen by user, typically near mid-latitude of region of interest
Φ_m, ψ_m, Ψ_m = values of Φ, ψ, Ψ corresponding to φ_m
$c = \cos\varphi_m$
$s = \sin\varphi_m$
R = radius of intermediate sphere
ρ = radius of curvature in meridian of ellipsoid $= a(1 - e^2)(1 - e^2\sin^2\varphi)^{-3/2} = a(1 + \varepsilon)^{1/2}(1 + \varepsilon\cos^2\varphi)^{-3/2}$
F_0 = constant scale on central meridian
E_{off}, N_{off} = offsets added to easting and northing in order to give values measured from a conventional "false origin"

Geographical latitude φ at a point P on the ellipsoid is defined to be the angle between the normal at P and the plane of the equator. Isometric latitude ψ is defined in terms of φ by

$$\psi = \tanh^{-1}(\sin\varphi) - e\tanh^{-1}(e\sin\varphi). \qquad \textbf{(2)}$$

It is straightforward to verify that if σ denotes distance on the ellipsoid then $\partial\sigma/\partial\omega = \partial\sigma/\partial\psi$ at each point (ω, ψ). Hence the formulae for conformal mappings (first section) can be applied to the (ω, ψ) coordinate system.

Intermediate sphere

In the method of this web page, the transverse Mercator projection is split into two stages:

- (i) a conformal mapping from the ellipsoid to an intermediate sphere, taking the central meridian to the central meridian with constant scale;
- (ii) a transverse Mercator projection from the sphere to the (x, y) plane, taking the central meridian to the y-axis with unit scale.

From (2) with $e = 0$ we get Ψ as the inverse Gudermann function of Φ, which can be written in three equivalent ways:

$$\tanh\Psi = \sin\Phi, \qquad \operatorname{sech}\Psi = \cos\Phi, \qquad \sinh\Psi = \tan\Phi.$$

Stage (ii) above (transverse Mercator from sphere to plane) is therefore given by simple closed formulae

$$\begin{aligned} \tanh(x/R) &= \operatorname{sech}\Psi\sin\Omega = \cos\Phi\sin\Omega, \\ \tan(y/R) &= \sinh\Psi\sec\Omega = \tan\Phi\sec\Omega, \end{aligned} \qquad \textbf{(3)}$$

whose equally simple inverse is given by (14) below.

We now need to work out stage (i), the projection from ellipsoid to sphere.

Outline of method

We are looking for a conformal mapping (1) with a change of notation: (Ω, Ψ) for (ξ, η), and (ω, ψ) for (x, y). Since the meridian $\omega = 0$ is projected to $\Omega = 0$, we have $f = 0$, so that (1) becomes

$$\begin{aligned} \Omega &= \omega g^{(1)}(\psi) - (\omega^3/3!)g^{(3)}(\psi) + (\omega^5/5!)g^{(5)}(\psi) - \ldots \\ \Psi &= g(\psi) - (\omega^2/2!)g^{(2)}(\psi) + (\omega^4/4!)g^{(4)}(\psi) - \ldots \end{aligned} \qquad \textbf{(4)}$$

The problem is to find g, and for this it suffices to consider the mapping of the central meridian, on which $\Psi = g(\psi)$. Fix some geographical latitude φ_m near the centre of the region to be mapped (e.g. $\varphi_m = 55.6°$ is good for Britain) and let ψ_m, Φ_m, Ψ_m be the corresponding values of ψ, Φ, Ψ. Define $\delta = \psi - \psi_m$.

To simplify the working and the result, we assume that on the central meridian ($\omega = 0$) Ψ is linear to third order in ψ around ψ_m, and write

$$\Psi|_{\omega=0} = g(\psi) = g(\psi_m + \delta) = \Psi_m + \Delta,$$

where

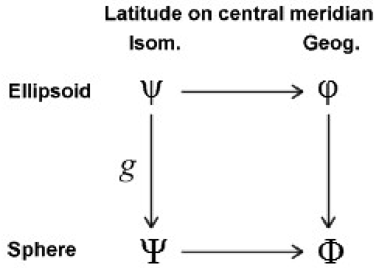
$$\Delta = a_1\delta + a_4\delta^4 + a_5\delta^5 + a_6\delta^6 + \ldots. \tag{5}$$

This is permissible because, as noted above, the projection has two degrees of freedom (scale and *y*-shift), so that we can choose it to make the terms in δ^2 and δ^3 vanish. If for convenience we define $a_0 = \Phi_m$, $a_2 = a_3 = 0$, then (4) and (5) can be developed as (cf. Deakin *et al.* [4], page 19)

$$\begin{aligned}\Psi + i\Omega &= g(\psi) + (i\omega)g^{(1)}(\psi) + \frac{(i\omega)^2}{2!}g^{(2)}(\psi) + \frac{(i\omega)^3}{3!}g^{(3)}(\psi) + \ldots \\ &= g(\psi + i\omega) = g(\psi_m + \delta + i\omega) \\ &= \sum_{n=0}^{\infty} a_n(\delta + i\omega)^n = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} \delta^{n-m}(i\omega)^m \\ &= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} (i\omega)^m \binom{n}{m} a_n\delta^{n-m} = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} (i\omega)^m \binom{m+r}{m} a_{m+r}\delta^r,\end{aligned} \tag{6}$$

from which Ψ and Ω can be found by equating real and imaginary parts.

To find the coefficients a_n we will calculate Φ as a power series in δ in two ways, going via φ and Ψ respectively (see diagram). Equating coefficients of δ^n will then give the result.



Details of method

φ from ψ. Recalling that $\delta = \psi - \psi_m$, we express φ as a Taylor series in δ. We require the first few $d^n\varphi/d\psi^n$. Differentiating (2) w.r.t. φ gives

$$d\psi/d\varphi = (1 - e^2)/(1 - e^2\sin^2\varphi)\cos\varphi = 1/(\cos\varphi + \varepsilon\cos^3\varphi)$$

whence

$$\begin{aligned}d\varphi/d\psi &= \cos\varphi + \varepsilon\cos^3\varphi, \\ d^2\varphi/d\psi^2 &= -\sin\varphi[\cos\varphi + 4\varepsilon\cos^3\varphi + 3\varepsilon^2\cos^5\varphi], \\ d^3\varphi/d\psi^3 &= \cos\varphi + (-2 + 13\varepsilon)\cos^3\varphi + (-18\varepsilon + 27\varepsilon^2)\cos^5\varphi + (-34\varepsilon^2 + 15\varepsilon^3)\cos^7\varphi - 18\varepsilon^3\cos^9\varphi, \\ d^4\varphi/d\psi^4 &= -\sin\varphi[\cos\varphi + (-6 + 40\varepsilon)\cos^3\varphi + (-96\varepsilon + 174\varepsilon^2)\cos^5\varphi + (-328\varepsilon^2 + 240\varepsilon^3)\cos^7\varphi + (-400\varepsilon^3 + 105\varepsilon^4)\cos^9\varphi \\ &\quad - 162\varepsilon^4\cos^{11}\varphi], \\ &\ldots\end{aligned}$$

This and similar lists are to be extended, preferably with the aid of computerized algebra, up to the order of accuracy desired.

Φ from φ. If σ denotes distance along the central meridian then $d\sigma/d\varphi = \rho = \text{const} \times (1 + \varepsilon\cos^2\varphi)^{-3/2}$. By hypothesis, the scale along the central meridian is constant, so that

$$\Phi = p\int (1 + \varepsilon\cos^2\varphi)^{-3/2}d\varphi + q, \tag{7}$$

where *p* and *q* are constants. The first few deriatives of Φ w.r.t. φ are therefore

$$\begin{aligned}d\Phi/d\varphi &= p(1 + \varepsilon\cos^2\varphi)^{-3/2}, \\ d^2\Phi/d\varphi^2 &= 3p\varepsilon\sin\varphi\cos\varphi(1 + \varepsilon\cos^2\varphi)^{-5/2},\end{aligned}$$

$$d^3\Phi/d\varphi^3 = 3p\varepsilon(1 + \varepsilon\cos^2\varphi)^{-7/2}[-1 + (2 + 4\varepsilon)\cos^2\varphi - 3\varepsilon\cos^4\varphi],$$
$$d^4\Phi/d\varphi^4 = 3p\varepsilon(1 + \varepsilon\cos^2\varphi)^{-9/2}\sin\varphi\cos\varphi[-(4 + 15\varepsilon) + (22\varepsilon + 20\varepsilon^2)\cos^2\varphi - 9\varepsilon^2\cos^4\varphi]$$
$$d^5\Phi/d\varphi^5 = 3p\varepsilon(1 + \varepsilon\cos^2\varphi)^{-11/2}[(4 + 15\varepsilon) - (8 + 128\varepsilon + 180\varepsilon^2)\cos^2\varphi + (116\varepsilon + 362\varepsilon^2 + 120\varepsilon^3)\cos^4\varphi - (164\varepsilon^2 + 136\varepsilon^3)\cos^6\varphi + 27\varepsilon^3\cos^8\varphi],$$
$$\ldots$$

Φ *from* ψ *via* φ. Using the above, express Φ as a Taylor series in $\varphi - \varphi_m$, and substitute for $\varphi - \varphi_m$ a Taylor series in δ (= $\psi - \psi_m$). Setting $c = \cos\varphi_m$, $s = \sin\varphi_m$, we find after a routine calculation

$$\Phi = \Phi_m + rc(1 + \varepsilon c^2)^{-1/2}\{\delta - (\delta^2/2!)s + (\delta^3/3!)(1 - 2c^2 - \varepsilon c^4) - (\delta^4/4!)s(1 - 6c^2 - 9\varepsilon c^4 - 4\varepsilon^2c^6) + (\delta^5/5!)[1 - 20c^2 + (24 - 58\varepsilon)c^4 + (72\varepsilon - 64\varepsilon^2)c^6 + (77\varepsilon^2 - 24\varepsilon^3)c^8 + 28\varepsilon^3c^{10}] + \ldots\}. \quad \textbf{(8)}$$

Φ *from* ψ *via* Ψ. The first few derivatives of Φ w.r.t. Ψ, written in terms of Φ, are

$$d\Phi/d\Psi = \cos\Phi$$
$$d^2\Phi/d\Psi^2 = -\sin\Phi\cos\Phi,$$
$$d^3\Phi/d\Psi^3 = \cos\Phi - 2\cos^3\Phi,$$
$$d^4\Phi/d\Psi^4 = -\sin\Phi(\cos\Phi - 6\cos^3\Phi),$$
$$d^5\Phi/d\Psi^5 = \cos\Phi - 20\cos^3\Phi + 24\cos^5\Phi.$$

So from (5) we get

$$\Phi = \Phi_m + \delta a_1\cos\Phi_m - (\delta^2/2!)a_1^2\sin\Phi_m\cos\Phi_m + (\delta^3/3!)a_1^3(\cos\Phi_m - 2\cos^3\Phi_m) + \delta^4[a_4\cos\Phi_m - (1/4!)a_1^4\sin\Phi_m(\cos\Phi_m - 6\cos^3\Phi_m)] + \delta^5[a_5\cos\Phi_m - a_1a_4\sin\Phi_m\cos\Phi_m + (1/5!)a_1^5(\cos\Phi_m - 20\cos^3\Phi_m + 24\cos^5\Phi_m) + \ldots. \quad \textbf{(9)}$$

On to the result. Equating coefficients of δ, δ^2, and δ^3 in (8) and (9) gives respectively

$$a_1\cos\Phi_m = rc(1 + \varepsilon c^2)^{-1/2}$$
$$a_1^2\sin\Phi_m\cos\Phi_m = rsc(1 + \varepsilon c^2)^{-1/2}$$
$$a_1^3(\cos\Phi_m - 2\cos^3\Phi_m) = rc(1 - 2c^2 - \varepsilon c^4)(1 + \varepsilon c^2)^{-1/2}$$

For a given φ_m these three equations are to be solved for Φ_m, p, and a_1. A straightforward calculation gives

$$\tan\Phi_m = \tan\varphi_m(1 + \varepsilon\cos^2\varphi_m)^{-1/2} \quad \textbf{(10)}$$
$$p = 1 + \varepsilon\cos^2\varphi_m \quad \textbf{(11)}$$
$$a_1 = (1 + \varepsilon\cos^4\varphi_m)^{1/2}.$$

To find a_4, note that

$$a_1\cos\Phi_m = rc(1 + \varepsilon c^2)^{-1/2} = c(1 + \varepsilon c^2)^{1/2}, \qquad a_1\sin\Phi_m = s,$$

so that equating coefficients of δ^4 in (5) and (9) and dividing by $a_1\cos\Phi_m$ gives

$$-(1/4!)s(1 - 6c^2 - 9\varepsilon c^4) = a_4/a_1 - (1/4!)s[1 + \varepsilon c^4 - 6c^2(1 + \varepsilon c^2)]$$

whence $a_4 = (1/6)a_1s\varepsilon c^4(1 + \varepsilon c^2)$.

Similarly one calculates a_5, a_6, … as far as desired. This is probably not the best method for finding the a_i, as it is rather lengthy and does not explain why the factor $(1 + \varepsilon c^2)$ occurs in each a_i. However, with the help of computerized algebra it yields the following up to 12th order, where $k = a_1\varepsilon c^4(1 + \varepsilon c^2)$. Note that the factor $s = \sin\varphi_m$ appears in a_i for even but not odd i.

$$a_4 = ks/6,$$
$$a_5 = -(k/30)[(5 - 6c^2) + \varepsilon c^2(6 - 7c^2)],$$
$$a_6 = (ks/180)[(23 - 39c^2) + \varepsilon c^2(66 - 104c^2) + \varepsilon^2c^4(48 - 70c^2)],$$
$$a_7 = -(k/1\,260)[(97 - 366c^2 + 285c^4) + \varepsilon c^2(534 - 1\,834c^2 + 1\,354c^4) + \varepsilon^2c^4(912 - 2\,826c^2 + 1\,974c^4) + \varepsilon^3c^6(480 - 1\,368c^2 + 910c^4)],$$
$$a_8 = (ks/10\,080)[(399 - 2\,259c^2 + 2\,340c^4) + \varepsilon c^2(3\,786 - 18\,201c^2 + 17\,091c^4) + \varepsilon^2c^4(11\,568 - 47\,664c^2 + 41\,176c^4)$$

$$
\begin{aligned}
&\quad + \varepsilon^3 c^6\,(13\,920 - 50\,832c^2 + 40\,964c^4) \\
&\quad + \varepsilon^4 c^8\,(5\,760 - 19\,152c^2 + 14\,560c^4)],
\end{aligned}
$$

$$
\begin{aligned}
a_9 = -(k/90\,720)[\,&(1\,617 - 15\,822c^2 + 35\,145c^4 - 21\,420c^6) \\
&+ \varepsilon c^2\,(25\,110 - 197\,721c^2 + 390\,216c^4 - 220\,761c^6) \\
&+ \varepsilon^2 c^4\,(122\,832 - 810\,108c^2 + 1\,456\,888c^4 - 777\,368c^6) \\
&+ \varepsilon^3 c^6\,(254\,880 - 1\,476\,432c^2 + 2\,465\,576c^4 - 1\,253\,156c^6) \\
&+ \varepsilon^4 c^8\,(236\,160 - 1\,240\,992c^2 + 1\,951\,360c^4 - 951\,748c^6) \\
&+ \varepsilon^5 c^{10}\,(80\,640 - 392\,832c^2 + 587\,664c^4 - 276\,640c^6)],
\end{aligned}
$$

$$
\begin{aligned}
a_{10} = (ks/907\,200)[\,&(6\,511 - 96\,003c^2 + 284\,580c^4 - 216\,720c^6) \\
&+ \varepsilon c^2\,(160\,362 - 1\,741\,062c^2 + 4\,357\,044c^4 - 2\,979\,576c^6) \\
&+ \varepsilon^2 c^4\,(1\,183\,536 - 10\,218\,210c^2 + 22\,614\,831c^4 - 14\,293\,933c^6) \\
&+ \varepsilon^3 c^6\,(3\,777\,120 - 27\,694\,080c^2 + 55\,715\,568c^4 - 33\,050\,400c^6) \\
&+ \varepsilon^4 c^8\,(5\,892\,480 - 38\,260\,512c^2 + 71\,319\,600c^4 - 40\,132\,128c^6) \\
&+ \varepsilon^5 c^{10}\,(4\,435\,200 - 26\,210\,304c^2 + 45\,899\,424c^4 - 24\,698\,800c^6) \\
&+ \varepsilon^6 c^{12}\,(1\,290\,240 - 7\,070\,976c^2 + 11\,753\,280c^4 - 6\,086\,080c^6)],
\end{aligned}
$$

$$
\begin{aligned}
a_{11} = -(k/9\,979\,200)[\,&(26\,129 - 612\,942c^2 + 2\,973\,465c^4 - 4\,769\,100c^6 + 2\,404\,080c^8) \\
&+ \varepsilon c^2\,(1\,001\,238 - 16\,079\,440c^2 + 64\,079\,286c^4 - 91\,147\,284c^6 + 42\,371\,064c^8) \\
&+ \varepsilon^2 c^4\,(10\,751\,184 - 133\,032\,942c^2 + 462\,342\,495c^4 - 603\,358\,772c^6 + 264\,215\,043c^8) \\
&+ \varepsilon^3 c^6\,(49\,606\,560 - 513\,290\,016c^2 + 1\,608\,671\,808c^4 - 1\,961\,068\,704c^6 + 818\,045\,920c^8) \\
&+ \varepsilon^4 c^8\,(116\,035\,200 - 1\,054\,446\,048c^2 + 3\,047\,708\,544c^4 - 3\,514\,163\,808c^6 + 1\,407\,298\,464c^8) \\
&+ \varepsilon^5 c^{10}\,(144\,587\,520 - 1\,190\,221\,056c^2 + 3\,223\,093\,824c^4 - 3\,547\,726\,976c^6 + 1\,372\,021\,728c^8) \\
&+ \varepsilon^6 c^{12}\,(91\,607\,040 - 697\,432\,320c^2 + 1\,789\,882\,560c^4 - 1\,894\,042\,304c^6 + 710\,673\,040c^8) \\
&+ \varepsilon^7 c^{14}\,(23\,224\,320 - 165\,934\,080c^2 + 407\,062\,656c^4 - 416\,391\,360c^6 + 152\,152\,000c^8)],
\end{aligned}
$$

$$
\begin{aligned}
a_{12} = (ks/119\,750\,400)[\,&(104\,687 - 3\,695\,043c^2 + 23\,932\,260c^4 - 47\,995\,920c^6 + 29\,030\,400c^8) \\
&+ \varepsilon c^2\,(6\,164\,202 - 134\,900\,843c^2 + 679\,980\,393c^4 - 1\,168\,763\,544c^6 + 636\,663\,600c^8) \\
&+ \varepsilon^2 c^4\,(94\,019\,376 - 1\,503\,338\,308c^2 + 6\,400\,822\,965c^4 - 9\,876\,665\,101c^6 + 4\,988\,315\,196c^8) \\
&+ \varepsilon^3 c^6\,(603\,577\,440 - 7\,805\,457\,864c^2 + 29\,310\,906\,258c^4 - 41\,588\,254\,065c^6 + 19\,773\,581\,543c^8) \\
&+ \varepsilon^4 c^8\,(1\,987\,701\,120 - 22\,054\,179\,456c^2 + 75\,126\,837\,408c^4 - 99\,606\,227\,856c^6 + 45\,043\,964\,496c^8) \\
&+ \varepsilon^5 c^{10}\,(3\,648\,718\,080 - 36\,037\,592\,064c^2 + 113\,548\,973\,184c^4 - 142\,326\,334\,672c^6 + 61\,685\,211\,984c^8) \\
&+ \varepsilon^6 c^{12}\,(3\,779\,112\,960 - 34\,051\,258\,368c^2 + 100\,637\,983\,296c^4 - 120\,305\,983\,424c^6 + 50\,268\,203\,216c^8) \\
&+ \varepsilon^7 c^{14}\,(2\,066\,964\,480 - 17\,285\,068\,800c^2 + 48\,419\,217\,792c^4 - 55\,579\,694\,016c^6 + 22\,494\,311\,840c^8) \\
&+ \varepsilon^8 c^{16}\,(464\,486\,400 - 3\,650\,549\,760c^2 + 9\,769\,503\,744c^4 - 10\,826\,175\,360c^6 + 4\,260\,256\,000c^8)].
\end{aligned}
$$

Radius of intermediate sphere

We chose above that the projection from intermediate sphere to plane should have unit scale on the central meridian. We therefore need to choose R (the radius of the intermediate sphere) so that the projection from ellipsoid to intermediate sphere has the required constant scale F_0 on the central meridian. It suffices to do this at the base latitude φ_m. If σ denotes distance along the central meridian, then at φ_m we have

$$
\begin{aligned}
\mathrm{d}\varphi/\mathrm{d}\sigma &= \rho^{-1} = a^{-1}(1+\varepsilon)^{-1/2}(1+\varepsilon c^2)^{3/2} \\
\mathrm{d}\Phi/\mathrm{d}\varphi &= p(1+\varepsilon c^2)^{-3/2} = (1+\varepsilon c^2)^{-1/2} \quad \text{from (7) and (11)}
\end{aligned}
$$

whence the scale on the central meridian is $Ra^{-1}(1+\varepsilon)^{-1/2}(1+\varepsilon c^2)$. Equating this to F_0 gives

$$R = aF_0(1+\varepsilon)^{1/2}(1+\varepsilon c^2)^{-1}. \qquad \textbf{(12)}$$

Geographical to grid

The method for converting geographical coordinates (φ, ω) on the ellipsoid to grid coordinates (E, N) can now be given.

Find the isometric latitude ψ from (2), and set $\delta = \psi - \psi_m$. Calculate R from (12) and calculate as many of a_1, a_4, a_5, … as desired.

Longitude and isometric latitude on the sphere can now be found from from (6). If we equate real and imaginary parts, and evaluate the binomial coefficients, this gives the following series for Ψ and Ω. Terms containing a_i beyond those calculated should be ignored.

$$
\begin{aligned}
\Psi \;=\;& \Psi_m + a_1\delta + a_4\delta^4 + a_5\delta^5 + a_6\delta^6 + a_7\delta^7 + a_8\delta^8 + a_9\delta^9 + a_{10}\delta^{10} + a_{11}\delta^{11} + a_{12}\delta^{12} + \ldots \\
& - \omega^2(6a_4\delta^2 + 10a_5\delta^3 + 15a_6\delta^4 + 21a_7\delta^5 + 28a_8\delta^6 + 36a_9\delta^7 + 45a_{10}\delta^8 + 55a_{11}\delta^9 + 66a_{12}\delta^{10} + \ldots) \\
& + \omega^4(a_4 + 5a_5\delta + 15a_6\delta^2 + 35a_7\delta^3 + 70a_8\delta^4 + 126a_9\delta^5 + 210a_{10}\delta^6 + 330a_{11}\delta^7 + 495a_{12}\delta^8 + \ldots) \\
& - \omega^6(a_6 + 7a_7\delta + 28a_8\delta^2 + 84a_9\delta^3 + 210a_{10}\delta^4 + 462a_{11}\delta^5 + 924a_{12}\delta^6 + \ldots) \\
& + \omega^8(a_8 + 9a_9\delta + 45a_{10}\delta^2 + 165a_{11}\delta^3 + 495a_{12}\delta^4 + \ldots) \\
& - \omega^{10}(a_{10} + 11a_{11}\delta + 66a_{12}\delta^2 + \ldots) \\
& + \omega^{12}(a_{12} + \ldots) \\
& - \ldots \qquad\qquad \textbf{(13)} \\
\Omega \;=\;& \omega(a_1 + 4a_4\delta^3 + 5a_5\delta^4 + 6a_6\delta^5 + 7a_7\delta^6 + 8a_8\delta^7 + 9a_9\delta^8 + 10a_{10}\delta^9 + 11a_{11}\delta^{10} + 12a_{12}\delta^{11} + \ldots \\
& - \omega^3(4a_4\delta + 10a_5\delta^2 + 20a_6\delta^3 + 35a_7\delta^4 + 56a_8\delta^5 + 84a_9\delta^6 + 120a_{10}\delta^7 + 165a_{11}\delta^8 + 220a_{12}\delta^9 + \ldots) \\
& + \omega^5(a_5 + 6a_6\delta + 21a_7\delta^2 + 56a_8\delta^3 + 126a_9\delta^4 + 252a_{10}\delta^5 + 462a_{11}\delta^6 + 792a_{12}\delta^7 + \ldots) \\
& - \omega^7(a_7 + 8a_8\delta + 36a_9\delta^2 + 120a_{10}\delta^3 + 330a_{11}\delta^4 + 792a_{12}\delta^5 + \ldots) \\
& + \omega^9(a_9 + 10a_{10}\delta + 55a_{11}\delta^2 + 210a_{12}\delta^3 + \ldots) \\
& - \omega^{11}(a_{11} + 12a_{12}\delta + \ldots) \\
& + \ldots
\end{aligned}
$$

Having found Ψ and Ω, calculate the grid coordinates as

$$
\begin{aligned}
E &= R\tanh^{-1}(\sin\Omega/\cosh\Psi) + E_{off}, \\
N &= R\tan^{-1}(\sinh\Psi/\cos\Omega) + N_{off}.
\end{aligned}
$$

It remains to find the constant offsets E_{off} and N_{off}. There is no difficulty with E_{off}, which is simply the conventional easting of points on the central meridian (e.g. for Great Britain the OS uses E_{off} = 400 000 m).

N_{off} is implicitly defined by choosing a point on the central meridian, say with latitude φ_0, and specifying its conventional northing, say N_0. (E.g. for Great Britain the OS chooses φ_0 = 49° and N_0 = −100000 m.) One could estimate N_{off} by feeding $\varphi = \varphi_0$ into the above procedure, but since φ_0 may be outside the region of greatest accuracy the following method is preferable.

Let N_m be the northing of the point on the central meridian with latitude φ_m, before applying the conventional offset. Let D be the distance along the central meridian measured northwards from φ_0 to φ_m, perhaps calculated by one of the methods suggested on this website. Then we require $N_{off} = F_0D + N_0 - N_m$. The value of N_m can be found from (10), whence

$$
N_{off} = F_0D + N_0 - R\tan^{-1}((1 + \varepsilon c^2)^{-1/2}\tan\varphi_m).
$$

Grid to geographical

The problem here is: Given grid coordinates E and N, find geographical latitude φ and longitude λ. It is a question of reversing the above-described method of finding grid from geographical coordinates.

Having found the constants E_{off} and N_{off} as above, define

$$
x = E - E_{off}, \qquad y = N - N_{off}.
$$

The inverse of (3) above is given by

$$
\begin{aligned}
\tanh\Psi &= \operatorname{sech}(x/R)\sin(y/R), \\
\tan\Omega &= \sinh(x/R)\sec(y/R),
\end{aligned} \qquad \textbf{(14)}
$$

from which we get the isometric latitude Ψ and longitude Ω on the intermediate sphere.

The projection from sphere to ellipsoid is analogous to the projection from ellipsoid to sphere. We first need to invert the power series (5), so as to get δ as a power series in $\Delta = \Psi - \Psi_m$. The absence of terms in δ^2 and δ^3 simplifies the result, which can be written

$$
\delta = b_1\Delta - b_4\Delta^4 - b_5\Delta^5 - b_6\Delta^6 - \ldots,
$$

where up to 12th order

$$
\begin{aligned}
b_1 &= 1 / a_1, \\
b_4 &= a_4 / a_1^5, \\
b_5 &= a_5 / a_1^6, \\
b_6 &= a_6 / a_1^7, \\
b_7 &= (a_1a_7 - 4a_4^2) / a_1^9,
\end{aligned}
$$

$$b_8 = (a_1a_8 - 9a_4a_5) / a_1^{10},$$
$$b_9 = (a_1a_9 - 5a_5^2 - 10a_4a_6) / a_1^{11},$$
$$b_{10} = (a_1^2a_{10} - 11a_1a_5a_6 - 11a_1a_4a_7 + 22a_4^3) / a_1^{13},$$
$$b_{11} = (a_1^2a_{11} - 6a_1a_6^2 - 12a_1a_5a_7 - 12a_1a_4a_8 + 78a_4^2a_5) / a_1^{14},$$
$$b_{12} = (a_1^2a_{12} - 13a_1a_6a_7 - 13a_1a_5a_8 - 13a_1a_4a_9 + 91a_4a_5^2 + 91a_4^2a_6) / a_1^{15}.$$

The reasoning that led to (13) above leads to exactly analogous formulae for the ellipsoid coordinates ψ and ω:

$$\begin{aligned}\psi = {} & \psi_m + b_1\Delta + b_4\Delta^4 + b_5\Delta^5 + b_6\Delta^6 + b_7\Delta^7 + b_8\Delta^8 + b_9\Delta^9 + b_{10}\Delta^{10} + b_{11}\Delta^{11} + b_{12}\Delta^{12} + \ldots \\ & - \Omega^2(6b_4\Delta^2 + 10b_5\Delta^3 + 15b_6\Delta^4 + 21b_7\Delta^5 + 28b_8\Delta^6 + 36b_9\Delta^7 + 45b_{10}\Delta^8 + 55b_{11}\Delta^9 + 66b_{12}\Delta^{10} + \ldots) \\ & + \Omega^4(b_4 + 5b_5\Delta + 15b_6\Delta^2 + 35b_7\Delta^3 + 70b_8\Delta^4 + 126b_9\Delta^5 + 210b_{10}\Delta^6 + 330b_{11}\Delta^7 + 495b_{12}\Delta^8 + \ldots) \\ & - \Omega^6(b_6 + 7b_7\Delta + 28b_8\Delta^2 + 84b_9\Delta^3 + 210b_{10}\Delta^4 + 462b_{11}\Delta^5 + 924b_{12}\Delta^6 + \ldots) \\ & + \Omega^8(b_8 + 9b_9\Delta + 45b_{10}\Delta^2 + 165b_{11}\Delta^3 + 495b_{12}\Delta^4 + \ldots) \\ & - \Omega^{10}(b_{10} + 11b_{11}\Delta + 66b_{12}\Delta^2 + \ldots) \\ & + \Omega^{12}(b_{12} + \ldots) \\ & - \ldots\end{aligned}$$

$$\begin{aligned}\omega = {} & \Omega(b_1 + 4b_4\Delta^3 + 5b_5\Delta^4 + 6b_6\Delta^5 + 7b_7\Delta^6 + 8b_8\Delta^7 + 9b_9\Delta^8 + 10b_{10}\Delta^9 + 11b_{11}\Delta^{10} + 12b_{12}\Delta^{11} + \ldots \\ & - \Omega^3(4b_4\Delta + 10b_5\Delta^2 + 20b_6\Delta^3 + 35b_7\Delta^4 + 56b_8\Delta^5 + 84b_9\Delta^6 + 120b_{10}\Delta^7 + 165b_{11}\Delta^8 + 220b_{12}\Delta^9 + \ldots) \\ & + \Omega^5(b_5 + 6b_6\Delta + 21b_7\Delta^2 + 56b_8\Delta^3 + 126b_9\Delta^4 + 252b_{10}\Delta^5 + 462b_{11}\Delta^6 + 792b_{12}\Delta^7 + \ldots) \\ & - \Omega^7(b_7 + 8b_8\Delta + 36b_9\Delta^2 + 120b_{10}\Delta^3 + 330b_{11}\Delta^4 + 792b_{12}\Delta^5 + \ldots) \\ & + \Omega^9(b_9 + 10b_{10}\Delta + 55b_{11}\Delta^2 + 210b_{12}\Delta^3 + \ldots) \\ & - \Omega^{11}(b_{11} + 12b_{12}\Delta + \ldots) \\ & + \ldots\end{aligned}$$

The isometric latitude ψ on the ellipsoid needs to be converted to geographical latitude φ by inverting equation (2). This can be done by iteration or one of the other methods suggested on this website.

Finally the longitude λ with respect to Greenwich is given by $\lambda = \lambda_0 + \omega$.

Michael Behrend, October 2011

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