Monochromatic $K_r$-Decompositions of Graphs 
(Extended Abstract)

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Abstract

Given graphs $G$ and $H$, and a colouring of the edges of $G$ with $k$ colours, a monochromatic $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a monochromatic graph isomorphic to $H$. Let $\phi_k(n, H)$ be the smallest number $\phi$ such that any $k$-edge-coloured graph $G$ of order $n$, admits a monochromatic $H$-decomposition with at most $\phi$ parts. Here we study the function $\phi_k(n, K_r)$ for $k \geq 2$ and $r \geq 3$.

1 Introduction

Given two graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of its edge set, such that, each part is either a single edge or forms an $H$-subgraph, i.e., a graph isomorphic to $H$. Let $\phi(G, H)$ be the smallest possible number of parts in an $H$-decomposition of $G$. It is easy to see that, for non-empty $H$, $\phi(G, H) = e(G) - p_H(G)(e(H) - 1)$,

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where \( p_H(G) \) is the maximum number of pairwise edge-disjoint \( H \) in \( G \). Consider the function
\[
\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\},
\]
which is the smallest number such that any graph \( G \) of order \( n \) admits an \( H \)-decomposition with at most \( \phi(n, H) \) parts. This function was first studied, in 1966, by Erdős, Goodman and Pósa [2], who were motivated by the problem of representing graphs by set intersections. They proved that \( \phi(n, K_3) = t_2(n) \), where \( t_{r-1}(n) \) denotes the number of edges in the Turán graph of order \( n \), \( T_{r-1}(n) \), which is the unique complete \((r-1)\)-partite graph on \( n \) vertices that has the maximum number of edges and contains no complete subgraph of order \( r \). Later, Bollobás [1] proved that \( \phi(n, K_r) = t_{r-1}(n) \), for all \( n \geq r \geq 3 \).

General graphs \( H \) were only considered recently by Pikhurko and Sousa [8] who proved the following result.

**Theorem 1.1.** [8] Let \( H \) be any fixed graph of chromatic number \( r \geq 3 \). Then,
\[
\phi(n, H) = t_{r-1}(n) + o(n^2).
\]

However, the exact value of the function \( \phi(n, H) \) is far from being known. Sousa determined it for a few special edge-critical graphs, namely for clique-extensions of order \( r \geq 4 \) \((n \geq r) \) [10] and the cycles of length 5 \((n \geq 6) \) and 7 \((n \geq 10) \) [9, 11]. Later, Özkahya and Person [7] determined it for all edge-critical graphs with chromatic number \( r \geq 3 \) and \( n \) sufficiently large. Let \( \text{ex}(n, H) \) denote the maximum number of edges in a graph of order \( n \), that does not contain \( H \) as a subgraph. Recall that \( \text{ex}(n, K_r) = t_{r-1}(n) \). They proved the following result.

**Theorem 1.2.** [7] Let \( H \) be any edge-critical graph with chromatic number \( r \geq 3 \). Then, there exists \( n_0 \) such that \( \phi(n, H) = \text{ex}(n, H) \), for all \( n \geq n_0 \). Moreover, the only graph attaining \( \phi(n, H) \) is the Turán graph \( T_{r-1}(n) \).

We consider a coloured version of the \( H \)-decomposition problem. We define the problem more precisely.

A \( k \)-edge-colouring of a graph \( G \) is a function \( c : E(G) \rightarrow \{1, \ldots, k\} \). Given a fixed graph \( H \), a graph \( G \) of order \( n \) and a \( k \)-edge-colouring of the edges of \( G \), a monochromatic \( H \)-decomposition of \( G \) is a partition of the edge set of \( G \) such that each part is either a single edge or a monochromatic copy of \( H \). Let \( \phi_k(G, H) \) be the smallest number such that, for any \( k \)-edge-colouring of \( G \), there exists a monochromatic \( H \)-decomposition of \( G \) with at most \( \phi_k(G, H) \) elements. The goal is to study the function
\[
\phi_k(n, H) = \max\{\phi_k(G, H) \mid v(G) = n\},
\]
which is the smallest number such that, any \( k \)-edge-coloured graph of order \( n \) admits a monochromatic \( H \)-decomposition with at most \( \phi_k(n, H) \) elements.

In this note we study the function \( \phi_k(n, K_r) \) for all \( k \geq 2 \) and \( r \geq 3 \).
2 Monochromatic $K_r$-decompositions

In this work we will determine the asymptotic value of the function $\phi_k(n, K_3)$ for all $k \geq 2$ (see Theorem 2.1) and the exact value of the function $\phi_k(n, K_r)$ for all $k \geq 2$ and $r \geq 4$ (see Theorem 2.2). Our results involve the Ramsey numbers and the Turán numbers. Recall that for $r \geq 3$ and $k \geq 2$, the Ramsey number for $K_r$, denoted by $R_k(r)$, is the smallest value of $s$ for which every $k$-edge-colouring of $K_s$ contains a monochromatic $K_r$. The Ramsey numbers are notoriously difficult to calculate, even though, it is known that their values are finite for all $r \geq 3$ and $k \geq 2$. In fact, for the Ramsey numbers $R_k(r)$, only three of them are currently known. In 1955, Greenwood and Gleason [3] were the first to determine $R_2(3) = 6$, $R_3(3) = 17$ and $R_2(4) = 18$.

Theorem 2.1. For all $k \geq 2$, we have

$$\phi_k(n, K_3) = t_{R_k(3)-1}(n) + o(n^2).$$

For larger cliques we are able to find the exact value of the function $\phi_k(n, K_r)$ for all $k \geq 2$ and $r \geq 4$.

Theorem 2.2. Let $r \geq 4$, $k \geq 2$. There is an $n_0 = n_0(r, k)$ such that, for all $n \geq n_0$, we have

$$\phi_k(n, K_r) = t_{R_k(r)-1}(n).$$

Moreover, the only graph attaining $\phi_k(n, K_r)$ is the Turán graph $T_{R_k(r)-1}(n)$.

Before presenting the proofs, we need to introduce the tools and some auxiliary results.

A $K_r$-cover in a graph is a set of edges meeting all $K_r$s, that is, the removal of a $K_r$-cover results in a $K_r$-free graph. A $K_r$-packing in a graph is a set of pairwise edge-disjoint $K_r$s. The $K_r$-covering number of a graph $G$, denoted by $\tau_r(G)$, is the minimum size of a $K_r$-cover of $G$ and the $K_r$-packing number of $G$, denoted by $\nu_r(G)$, is the maximum size of a $K_r$-packing of $G$.

A long-standing conjecture of Tuza, states the following.

Conjecture 2.3. [12] For every graph $G$, we have $\tau_3(G) \leq 2\nu_3(G)$.

Conjecture 2.3 remains open, and many partial results have been proved. By combining results of Krivelevich [6], and Haxell and Rödl [5], Yuster [13] observed that, asymptotically, Tuza’s conjecture holds. We have the following result which is crucial to the proofs of Theorems 2.1 and 2.2.

Theorem 2.4. [13] Let $G$ be a graph on $n$ vertices. Then,

(i) $\tau_3(G) \leq 2\nu_3(G) + o(n^2)$;

(ii) $\tau_r(G) \leq \left\lceil \frac{r^2}{4} \right\rceil \nu_r(G) + o(n^2)$, for $r \geq 4$. 

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implies that \( \sum \) implies contradiction, and the upper bound of Theorem 2.1 follows.

Let \( k \leq n \leq R \) be sufficiently large. Let \( G \) be a monochromatic copy of \( K \), and let \( n \) be the subgraph of \( G \). We are now able to prove Theorem 2.1 and Theorem 2.2. We will start by proving the lower bound for both theorems.

Proof of the lower bound in Theorem 2.1. Let \( k \geq 2 \) be fixed, let \( \varepsilon > 0 \) be arbitrary and let \( n_0 \) be sufficiently large. Let \( G \) be a \( k \)-edge-coloured graph on \( n \geq n_0 \) vertices. For the sake of simplicity, let \( R = R_k(3) \). We will show that \( G \) admits a monochromatic \( K_3 \)-decomposition with at most \( t_{R-1}(n) + \varepsilon n^2 \) parts.

Let \( e(G) = t_{R-1}(n) + \varepsilon n^2 + m \), where \( m \) is an integer. If \( m \leq 0 \) then \( G \) can be decomposed into single edges and we are done.

Suppose that \( m > 0 \). Observe that it suffices to show that we can find at least \( \frac{m}{2} \) edge-disjoint monochromatic copies of \( K_3 \), since then \( G \) admits a monochromatic \( K_3 \)-decomposition with at most \( e(G) - 2 \cdot \frac{m}{2} = t_{R-1}(n) + \varepsilon n^2 \) parts, as required. Therefore, and in order to get a contradiction, assume that the maximum number of edge-disjoint monochromatic copies of \( K_3 \) in our graph \( G \) is at most \( \frac{m}{2} \). For \( 1 \leq i \leq k \), let \( G_i \) be the subgraph of \( G \) on \( n \) vertices, containing all the edges in colour \( i \). Our assumption implies that \( \sum_{i=1}^{k} \nu_3(G_i) \leq \frac{m}{2} \). By Theorem 2.4, we have \( \tau_3(G_i) \leq 2 \nu_3(G_i) + \frac{\varepsilon}{2k} n^2 \) for every \( 1 \leq i \leq k \). Therefore, we have

\[
\sum_{i=1}^{k} \tau_3(G_i) \leq \sum_{i=1}^{k} \left( 2 \nu_3(G_i) + \frac{\varepsilon}{2k} n^2 \right) \leq m + \frac{\varepsilon}{2} n^2.
\]

That is, by deleting at most \( m + \frac{\varepsilon}{2} n^2 \) edges from \( G \), we obtain a subgraph \( G' \) which does not contain a monochromatic copy of \( K_3 \). On the other hand, we have \( e(G') \geq t_{R-1}(n) + \frac{\varepsilon}{2} n^2 > t_{R-1}(n) \). Turán’s Theorem implies that \( G' \) must contain \( K_R \) as a subgraph and hence \( G' \) contains a monochromatic copy of \( K_3 \). We have a contradiction, and the upper bound of Theorem 2.1 follows.

Proof of the upper bound in Theorem 2.2. Let \( n_0 = n_0(r, k) \) be sufficiently large, let \( n \geq n_0 \) and let \( G \) be any \( k \)-edge-coloured graph on \( n \) vertices. For the sake of simplicity, let \( R = R_k(r) \). We will show that \( \phi_k(G, K_r) \leq t_{R-1}(n) \) with equality if and only if \( G = T_{R-1}(n) \).
Let \( e(G) = t_{R-1}(n) + m \), where \( m \) is an integer. If \( m < 0 \), we can decompose \( G \) into single edges and there is nothing to prove. If \( m = 0 \) and \( G \) contains a monochromatic copy of \( K_r \) then \( G \) admits a monochromatic \( K_r \)-decomposition with at most \( t_{R-1}(n) - \binom{r}{2} + 1 \) parts and we are done. If \( G \) does not contain a monochromatic \( K_r \), then the definition of the Ramsey number implies that \( G \) does not contain a copy of \( K_r \).

Therefore, \( G = T_{R-1}(n) \) by Turán’s Theorem. Now, let \( m > 0 \) and let \( \ell \) be the maximum number of edge-disjoint monochromatic \( K_r \)'s in \( G \). If \( \ell > \frac{m}{\binom{r}{2}-1} \), then

\[
\phi_k(G, K_r) \leq \ell + e(G) - \left( \frac{r}{2} \right) \ell < t_{R-1}(n).
\]

Therefore, it suffices to show that \( \ell > \frac{m}{\binom{r}{2}-1} \).

Consider first the case \( m = o(n^2) \). By Theorem 2.5 the graph \( G \) contains \( (1 - o(1))m \) edge-disjoint copies of \( K_r \). Since each \( K_r \) contains a monochromatic copy of \( K_r \), this implies that \( \ell > \frac{m}{\binom{r}{2}-1} \) and we are done.

Finally, assume that \( m \geq Cn^2 \), for some constant \( C > 0 \). In order to get a contradiction, suppose that \( \ell \leq \frac{m}{\binom{r}{2}-1} \). For \( 1 \leq i \leq k \), let \( G_i \) be the subgraph of \( G \) on \( n \) vertices that contains all edges coloured with colour \( i \). By Theorem 2.4, our assumption implies that

\[
\sum_{i=1}^{k} \tau_r(G_i) \leq \sum_{i=1}^{k} \left\lfloor \frac{r^2}{4} \nu_r(G_i) + o(n^2) \right\rfloor \leq \left\lfloor \frac{r^2}{4} \right\rfloor \ell + o(n^2)
\]

\[
\leq \frac{r^2}{4} \frac{m}{\binom{r}{2}-1} + o(n^2) \leq \frac{4}{5} m + o(n^2),
\]

since \( r \geq 4 \).

That is, by deleting at most \( \frac{4}{5} m + o(n^2) \) edges from \( G \), we obtain a subgraph \( G' \) that does not contain a monochromatic copy of \( K_r \). But

\[
e(G') \geq e(G) - \frac{4}{5} m - o(n^2) \geq t_{R-1}(n) + \frac{1}{5} m - o(n^2) > t_{R-1}(n).
\]

Therefore, Turán’s Theorem implies that \( G' \) must contain a copy of \( K_r \) which contains a monochromatic copy of \( K_r \). This is a contradiction and our proof is complete. \( \square \)

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References


