Decompositions of Graphs into Fans and Single Edges

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Abstract

Given two graphs $G$ and $H$, an \textit{$H$-decomposition} of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a graph isomorphic to $H$. Let $\phi(n,H)$ be the smallest number $\phi$ such that any graph $G$ of order $n$ admits an $H$-decomposition with at most $\phi$ parts. Pikhurko and Sousa conjectured that $\phi(n,H) = \text{ex}(n,H)$ for $\chi(H) \geq 3$ and all sufficiently large $n$, where $\text{ex}(n,H)$ denotes the maximum number of edges in a graph on $n$ vertices not containing $H$ as a subgraph. Their conjecture has been verified by Özkahya and Person for all edge-critical graphs $H$. In this article, the conjecture is verified for the $k$-fan graph. The $k$-\textit{fan graph}, denoted by $F_k$, is the graph on $2k + 1$ vertices consisting of $k$ triangles which intersect in exactly one common vertex called the centre of the $k$-fan.

1 Introduction

Given two graphs $G$ and $H$, an \textit{$H$-decomposition} of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a graph isomorphic to $H$.

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Let \( \phi(G, H) \) be the smallest possible number of parts in an \( H \)-decomposition of \( G \). It is easy to see that, for non-empty \( H \), we have \( \phi(G, H) = e(G) - p_H(G)(e(H) - 1) \), where \( p_H(G) \) is the maximum number of pairwise edge-disjoint copies of \( H \) that can be packed into \( G \) and \( e(G) \) denotes the number of edges in \( G \). In this paper, we study the function

\[
\phi(n, H) = \max \{ \phi(G, H) \mid v(G) = n \},
\]

which is the smallest number \( \phi \) such that any graph \( G \) of order \( n \) admits an \( H \)-decomposition with at most \( \phi \) parts.

This function was first studied, in 1966, by Erdős, Goodman, and Pósa [6], who were motivated by the problem of representing graphs by set intersections. They proved that \( \phi(n, K_3) = \text{ex}(n, K_3) \), where \( K_s \) denotes the complete graph of order \( s \) and \( \text{ex}(n, H) \) denotes the maximum number of edges in a graph on \( n \) vertices not containing \( H \) as a subgraph. A decade later, Bollobás [2] proved that \( \phi(n, K_r) = \text{ex}(n, K_r) \), for all \( n \geq r \geq 3 \).

General graphs \( H \) were only considered recently by Pikhurko and Sousa [8]. They proved the following result.

**Theorem 1.1** (See Theorem 1.1 from [8]). Let \( H \) be any fixed graph of chromatic number \( r \geq 3 \). Then,

\[
\phi(n, H) = \text{ex}(n, H) + o(n^2).
\]

Pikhurko and Sousa also made the following conjecture.

**Conjecture 1.2.** [8] For any graph \( H \) of chromatic number \( r \geq 3 \), there exists \( n_0 = n_0(H) \) such that \( \phi(n, H) = \text{ex}(n, H) \) for all \( n \geq n_0 \).

A graph \( H \) is edge-critical if there exists an edge \( e \in E(H) \) such that \( \chi(H) > \chi(H - e) \), where \( \chi(H) \) denotes the chromatic number of \( H \). For \( r \geq 4 \), a clique-extension of order \( r \) is a connected graph that consists of a \( K_{r-1} \) plus another vertex, say \( v \), adjacent to at most \( r - 2 \) vertices of \( K_{r-1} \). Conjecture 1.2 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order \( r \geq 4 \) \( (n \geq r) \) [12] and the cycles of length \( 5 \) \( (n \geq 6) \) and \( 7 \) \( (n \geq 10) \) [11, 10]. Later, Özkahya and Person [7] verified the conjecture for all edge-critical graphs with chromatic number \( r \geq 3 \). Recall that the Turán graph \( T_{r-1}(n) \) is the complete balanced \( (r - 1) \)-partite graph on \( n \) vertices and does not contain \( K_r \) as a subgraph. Their result is the following.

**Theorem 1.3** (See Theorem 3 from [7]). For any edge-critical graph \( H \) with chromatic number \( r \geq 3 \), there exists \( n_0 = n_0(H) \) such that \( \phi(n, H) = \text{ex}(n, H) \), for all \( n \geq n_0 \). Moreover, the only graph attaining \( \text{ex}(n, H) \) is the Turán graph \( T_{r-1}(n) \).

Recently, as an extension of Özkahya and Person’s work, Allen, Böttcher, and Person [1] improved the error term obtained by Pikhurko and Sousa in Theorem 1.1. In fact, they proved that the error term \( o(n^2) \) can be replaced by \( O(n^{2-\alpha}) \) for some \( \alpha > 0 \). Furthermore, they also showed that this error term has the correct order of
magnitude. Their result is indeed an extension of Theorem 1.3 since the error term $O(n^{2-\alpha})$ that they obtained vanishes for every edge-critical graph $H$.

Here we will verify Conjecture 1.2 for the $k$-fan graph. The $k$-fan graph, denoted by $F_k$, is the graph on $2k+1$ vertices consisting of $k$ triangles which intersect in exactly one common vertex, called the centre of $F_k$. Observe that $\chi(F_k) = 3$ and for $k \geq 2$ the graph $F_k$ is not edge-critical.

In 1995, Erdős, Füredi, Gould, and Gunderson [5] have determined the value of the function $\text{ex}(n,F_k)$ as well as the $F_k$-extremal graphs for every fixed $k$ and whenever $n$ is large. They have proved the following result.

**Theorem 1.4.** [5] Let $F_{n,k}$ be the following family of graphs.

- If $k$ is odd and $n \geq 4k - 1$, then a member of $F_{n,k}$ is a Turán graph $T_2(n)$ with two vertex-disjoint copies of $K_k$ added into one class.

- If $k$ is even and $n \geq 4k - 3$, then a member of $F_{n,k}$ is a $T_2(n)$ with a graph having $2k - 1$ vertices, $k^2 - \frac{3}{2}k$ edges and maximum degree $k - 1$ added into one class.

For $k \geq 1$ and $n \geq 50k^2$, we have

$$\text{ex}(n,F_k) = \left\lceil \frac{n^2}{4} \right\rceil + g(k) = \begin{cases} \left\lceil \frac{n^2}{4} \right\rceil + k^2 - k & \text{if } k \text{ is odd}, \\ \left\lceil \frac{n^2}{4} \right\rceil + k^2 - \frac{3}{2}k & \text{if } k \text{ is even}. \end{cases}$$

Moreover, the only $F_k$-free graphs with $\text{ex}(n,F_k)$ edges are the members of $F_{n,k}$.

Here we will prove the following result.

**Theorem 1.5.** For $k \geq 1$, there exists $n_0 = n_0(k)$ such that $\phi(n,F_k) = \text{ex}(n,F_k)$ for all $n \geq n_0$. Moreover, the only graphs attaining $\text{ex}(n,F_k)$ are the members of $F_{n,k}$.

The lower bound $\phi(n,F_k) \geq \text{ex}(n,F_k)$ follows immediately by considering any member of $F_{n,k}$. The upper bound will be proved in Section 2.

Our notations throughout the paper are fairly standard. Let $G = (V,E)$ be a graph, $U \subset V$ and $v$ a vertex of $G$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree of $G$, respectively. The subgraph of $G$ induced by $U$ is denoted by $G[U]$ and $e_G(U) = e(G[U])$. We write $\deg_G(v)$ for the degree of $v$ in $G$ and $\deg_G(v,U)$ for the number of neighbours that $v$ has in $U$. If it is clear which graph is being considered we simply write $e(U)$, $\deg(v)$ and $\deg(v,U)$. Finally, for two disjoint subsets $U,W \subset V$, $e(U,W)$ denotes the number of edges of $G$ with one endpoint in $U$ and the other in $W$. 

3
2 Proof of Theorem 1.5

In this section we will prove the upper bound in Theorem 1.5. In outline, the proof is the following. Suppose we have a graph $G$ on $n$ vertices such that $\phi(G, F_k) \geq \text{ex}(n, F_k)$. We first apply a stability type result (Lemma 2.1) to deduce that $G$ must be a dense and near-balanced bipartite graph with $m = o(n^2)$ edges inside the classes. Then, we find too many edge-disjoint copies of $F_k$ in $G$ which would imply that $\phi(G, F_k) \leq \text{ex}(n, F_k)$, a contradiction to our initial assumption on $\phi(G, F_k)$. Such approach (stability method) has been widely used to study various problems in extremal graph theory. Our proof generally follows that of Özkahya and Person [7] except for the case when $m = O(k^2)$, when some further detailed analysis will be required.

Before presenting the proof we need to introduce the tools. Firstly, recall the following stability type result about graphs $G$ on $n$ vertices with $\phi(G, H) \geq \text{ex}(n, H) - o(n^2)$ due to Özkahya and Person [7]. Their result follows from a result of Pikhurko and Sousa ([8], Theorem 1.1) and an application of a stability result of Erdőös [4] and Simonovits [9].

**Lemma 2.1** (See Lemma 4 in [7]). Let $H$ be a graph with $\chi(H) = r \geq 3$ and $H \neq K_r$. Then, for every $\gamma > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every graph $G$ on $n \geq n_0$ vertices the following is true. If $\phi(G, H) \geq \text{ex}(n, H) - \varepsilon n^2$ then there exists a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

Secondly, let $f(\nu, \Delta) = \max \{e(G) \mid \nu(G) \leq \nu \text{ and } \Delta(G) \leq \Delta\}$, where $\nu(G)$ is the size of a maximum matching in $G$. We will need the following result of Chvátal and Hanson [3].

**Theorem 2.2.** [3] For $\nu, \Delta \geq 1$, we have

$$f(\nu, \Delta) = \nu \Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\Delta/2} \right\rfloor \leq \nu \Delta + \nu.$$

We are now able to complete the proof of Theorem 1.5.

**Proof of the upper bound in Theorem 1.5.** The case $k = 1$ is the result of Erdős, Goodman, and Pósa [6], so assume $k \geq 2$. We choose $\gamma = \frac{1}{(288k)^2}$ and let $n_0(k) = \max(n_0, 505k^4) + \binom{n_0}{2}$ where $n_0$ is given by Lemma 2.1. Furthermore, suppose that there exists a graph $G$ on $n \geq n_0(k)$ vertices such that $\phi(G, F_k) \geq \text{ex}(n, F_k)$ and $G \notin \mathcal{F}_{n,k}$. We will derive a contradiction by finding sufficiently many edge-disjoint copies of $F_k$ in $G$, which will give

$$\phi(G, F_k) = e(G) - p_{F_k}(G)(e(F_k) - 1) < \text{ex}(n, F_k).$$

We first prove the following claim. Although the proof is similar to that of Claim 7 in [7], we include it for the sake of completeness.
Lemma 2.1, we have

Without loss of generality, let 

\[ G \]

Proof. If \( \delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor \), then set \( i = 0 \). Otherwise, there exists \( v \in V(G) \) with \( \deg_G(v) < \left\lfloor \frac{n}{2} \right\rfloor \). Then we delete \( v \) from \( G \) obtaining \( G_1 := G - v \) with

\[
\phi(G_1, F_k) \geq \phi(G, F_k) - \deg_G(v) \geq \text{ex}(n, F_k) + m - \left\lfloor \frac{n}{2} \right\rfloor + 1
\]

\[ = \text{ex}(n - 1, F_k) + m + 1, \]

since \( \text{ex}(n, F_k) - \text{ex}(n-1, F_k) = \left\lfloor \frac{n}{2} \right\rfloor \) by Theorem 1.4. If \( \delta(G_1) < \left\lfloor \frac{n-1}{2} \right\rfloor \), then we iterate this procedure until we arrive at a graph \( G' \) that has \( n - i \) vertices, \( \delta(G') \geq \left\lfloor \frac{n-i}{2} \right\rfloor \) and \( \phi(G', F_k) \geq \text{ex}(n - i, F_k) + m + i, \) or we stop when \( G' \) has \( n_0 \) vertices. But the latter case cannot occur since \( \phi(G', F_k) > (n_0)^2 \), which is a contradiction. \[ \square \]

By Claim 2.3, we may assume that \( \delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor \). Otherwise, we can consider the graph \( G' \) instead of \( G \). Note that if \( G' \neq G \), then \( \phi(G', F_k) > \text{ex}(n', F_k) \) so that \( G' \notin \mathcal{F}_{n',k} \).

Let \( V_0 \cup V_1 \) be a partition of \( V(G) \) such that \( e(V_0, V_1) \) is maximised and let \( m = e(V_0) + e(V_1) \). Observe that

\[
m = e(G) - e(V_0, V_1) \geq \text{ex}(n, F_k) - \left\lfloor \frac{n^2}{4} \right\rfloor = g(k),
\]

and that

\[
e(G) = m + e(V_0, V_1) \leq m + \left\lfloor \frac{n^2}{4} \right\rfloor = \text{ex}(n, F_k) + m - g(k). \quad (2.1)
\]

By Lemma 2.1, we also have \( m < \gamma n^2 \).

The following claim says that the partition \( V(G) = V_0 \cup V_1 \) is very close to being balanced.

Claim 2.4. For \( i = 0,1 \), we have

\[
\frac{n}{2} - \sqrt{\gamma}n \leq |V_i| \leq \frac{n}{2} + \sqrt{\gamma}n. \quad (2.2)
\]

Proof. Without loss of generality, let \( |V_0| \leq |V_1| \) and \( |V_0| = \frac{n}{2} - a \), where \( a \geq 0 \). By Lemma 2.1, we have

\[
e(G) \leq |V_0||V_1| + \gamma n^2 = \frac{n^2}{4} - a^2 + \gamma n^2.
\]

Also, by Theorem 1.4, we have

\[
e(G) \geq \text{ex}(n, F_k) = \left\lfloor \frac{n^2}{4} \right\rfloor + g(k) \geq \frac{n^2}{4}.
\]

Therefore, \( \frac{n^2}{4} - a^2 + \gamma n^2 \geq \frac{n^2}{4} \), which implies that \( a \leq \sqrt{\gamma}n \) and (2.2) holds. \[ \square \]
In order to obtain the required contradiction it suffices to show that we can find \( \lfloor \frac{m-g(k)}{3k-1} \rfloor + 1 \) edge-disjoint copies of \( F_k \) in \( G \). In fact, together with (2.1) we obtain
\[
\phi(G, F_k) = e(G) - (e(F_k) - 1)p_Fk(G)
\leq ex(n, F_k) + m - g(k) - (3k - 1) \left( \lfloor \frac{m-g(k)}{3k-1} \rfloor + 1 \right)
< ex(n, F_k).
\]

For this purpose we will describe a procedure that will find the required number of edge-disjoint copies of \( F_k \) in \( G \). We continue the proof by considering two different cases.

**Case 1:** \( m \geq \frac{5}{2} k^2 - \frac{1}{2} k + 1 \).

For \( i = 0, 1 \) and a vertex \( v \in V_i \), we call \( v \) a **bad** vertex if \( \deg(v, V_i) > \frac{n}{72k} \). Otherwise, \( v \) is a **good** vertex. Observe that the total number of bad vertices in \( G \) is at most
\[
\frac{2\gamma n^2}{n/72k} = 144k\gamma n. \tag{2.3}
\]

For each bad vertex \( v \in V_i, i = 0, 1 \), we may choose \( k \left( \frac{1}{2k} \deg(v, V_i) \right) \) edges of \( G \) which connect \( v \) to good vertices in \( V_i \). This is possible since the number of bad vertices is at most \( 144k\gamma n \). We keep these edges and delete the remaining edges in \( G[V_i] \) incident with \( v \). We repeat this procedure for each bad vertex in \( G \). Let \( G_0 \) be the resulting graph. Writing \( U_i \subset V_i \) for the set of good vertices in \( V_i \), we have
\[
e_{G_0}(V_i) = e_{G_0}(U_i) + \sum_{v \text{ bad } \in V_i} \deg_{G_0}(v, V_i)
\leq e_G(U_i) + \sum_{v \text{ bad } \in V_i} k \left( \frac{1}{2k} \deg_G(v, V_i) \right)
\geq \frac{1}{2} e_G(U_i) + \frac{1}{2} \sum_{v \text{ bad } \in V_i} \deg_G(v, V_i) \geq \frac{1}{2} e_G(V_i),
\]
so that \( e_{G_0}(V_0) + e_{G_0}(V_1) \geq \frac{m}{2} \).

We now find sufficiently many edge-disjoint copies of \( F_k \) in \( G_0 \), with each copy having exactly \( k \) edges in either \( V_0 \) or \( V_1 \). Each time we find a copy of \( F_k \) we delete its edges. Let \( G_s \subset G_0 \) be the graph obtained after we have removed \( s \) copies.

We define a threshold
\[
t = \frac{n}{2} - \frac{n}{36k} - \sqrt{\gamma n}. \tag{2.4}
\]

The purpose for the introduction of the threshold \( t \) is that in our procedure, for any good vertex \( v \in V_i, i = 0, 1 \), we will ensure that \( \deg_{G_s}(v, V_{1-i}) \) will not be very much less than \( t \), for every subgraph \( G_s \subset G \).

For a good vertex \( v \in V_i, i = 0, 1 \), we say that \( v \) is **active** (in \( G_s \)) if \( \deg_{G_s}(v, V_{1-i}) \geq t \). Otherwise, \( v \) is **inactive**. Note that initially, all good vertices \( v \in V_i \) are active in
Then $\Delta(G) \geq \lfloor \frac{n}{2} \rfloor$ and $\deg_{G^*}(v, V_i) \leq \frac{n}{72k}$, so that
\[
\deg_{G_0}(v, V_{i-1}) \geq \frac{n}{2} - \frac{n}{72k} \geq t + \frac{n}{72k}.
\] (2.5)

We perform the following procedure to find edge-disjoint copies of $F_k$. If we cannot perform Step 1 then we proceed to Step 2.

**Step 1.** Let $s \geq 0$ and let $G_s \subset G_0$ be a subgraph at some point of the iteration. Suppose there exists a vertex $u \in V_i$ with $\deg_{G_s}(u, V_i) \geq k$, for some $i \in \{0, 1\}$. We take $k$ neighbours $v_1, \ldots, v_k \in V_i$ of $u$ in $G_s$. We then find good and active vertices $w_1, \ldots, w_k \in V_{i-1}$, where $w_j$ is a common neighbour of $u$ and $v_j$ in $G_s$ for $1 \leq j \leq k$. This gives a copy of $F_k$. We remove the copy of $F_k$ and update the status of the good vertices (whether they are active or inactive) and let $G_{s+1} \subset G_s$ be the new subgraph.

We perform Step 1 successively by considering first all bad vertices, followed by good vertices. Suppose that we have considered bad vertices for $a$ iterations and that Step 1 stops after $b \geq a$ iterates. That is, Step 1 is performed $a$ times for bad vertices and $b - a$ times for good vertices.

**Step 2.** When Step 1 is completed, we obtain the subgraph $G_b \subset G_0$ such that for $i = 0, 1$, we have $\deg_{G_b}(u, V_i) = 0$ for all bad vertices $u \in V_i$, and $\Delta(G_b[V_i]) < k$. Suppose that $G_s \subset G_b$ ($s \geq b$) is a subgraph at a further point of the iteration, and there exists a matching in $G_s[V_i]$ of size $k$ for some $i \in \{0, 1\}$. Let $v_1w_1, \ldots, v_kw_k$ be the matching. We find a good and active vertex $u \in V_{i-1}$ which is a common neighbour of $v_1, w_1, \ldots, v_k, w_k$ in $G_s$. As before, remove the resulting copy of $F_k$, let $G_{s+1} \subset G_s$ be the new subgraph and update the status of the good vertices. Step 2 is repeated until we have exhausted all such matchings. Let $G^*$ be the subgraph obtained after Step 2 is terminated.

**Claim 2.5.** Steps 1 and 2 can be successfully iterated.

For $i = 0, 1$, we have $\deg_{G^*}(u, V_i) = 0$ for all bad vertices $u \in V_i$, since $\deg_{G_0}(u, V_i)$ is a multiple of $k$. Also, $\Delta(G^*[V_i]) < k$ and $G^*[V_i]$ does not contain a matching of size $k$. Thus, by Theorem 2.2, $e_{G^*}(V_i) \leq f(k-1, k-1) \leq (k-1)^2 + (k-1)$ implying that $e_{G^*}(V_0) + e_{G^*}(V_1) \leq 2k(k-1)$. Therefore, since $e_{G_0}(V_0) + e_{G_0}(V_1) \geq \frac{mn}{2}$, for $m \geq 14k^2$, we have found and removed at least
\[
\frac{1}{k} \left( \frac{m}{2} - 2k(k-1) \right) \geq \left[ \frac{m-k^2 + \frac{3}{2}k}{3k-1} \right] + 1 \geq \left[ \frac{m-g(k)}{3k-1} \right] + 1
\] edge-disjoint copies of $F_k$ from $G$. Suppose now that $\frac{3}{2}k^2 - \frac{1}{2}k + 1 \leq m \leq 14k^2$ holds. Then $\Delta(G[V_i]) \leq 14k^2 \leq \frac{m}{72k}$ for $i = 0, 1$, so that $G$ has no bad vertices and $G_0 = G$.

In this case we have found and removed at least
\[
\frac{1}{k} \left( m - 2k(k-1) \right) \geq \left[ \frac{m-k^2 + \frac{3}{2}k}{3k-1} \right] + 1 \geq \left[ \frac{m-g(k)}{3k-1} \right] + 1
\] edge-disjoint copies of $F_k$ from $G$, as required. Therefore, to complete the proof of Case 1 it remains to prove Claim 2.5.
Proof of Claim 2.5. The maximality of $e(V_0, V_1)$ implies that for every $v \in V_i, i = 0, 1$ we have

$$\deg_{G_i}(v, V_{1-i}) \geq \max \left( \deg_{G_i}(v, V_i), \left\lfloor \frac{n}{4} \right\rfloor \right). \quad (2.6)$$

For Step 1, let $G_s \subseteq G_0$ be a subgraph at some point of the iteration, where $0 \leq s < b$. Suppose firstly that $0 \leq s < a$, so that we have a bad vertex $u \in V_i$ with neighbours $v_1, \ldots, v_k \in V_i$ in $G_s$, for some $i \in \{0, 1\}$. Note that $v_1, \ldots, v_k$ are good vertices. Then, $u$ was involved in at most $\frac{1}{k} \deg_{G_0}(u, V_i)$ previous iterates, and for each iterate the number of edges that $u$ sends to $V_{1-i}$ was reduced by $k$. Therefore, by (2.6),

$$\deg_{G_s}(u, V_{1-i}) \geq \deg_{G_0}(u, V_{1-i}) - k \cdot \frac{1}{k} \deg_{G_0}(u, V_i) \geq \deg_G(u, V_{1-i}) - \frac{1}{2} \deg_G(u, V_i) - k \geq \frac{1}{2} \deg_G(u, V_{1-i}) - k \geq \frac{1}{2} \left\lfloor \frac{n}{4} \right\rfloor - k. \quad (2.7)$$

Also, for every $1 \leq j \leq k$, we have

$$\deg_{G_s}(v_j, V_{1-i}) \geq t - \Delta(F_k) - \deg_{G_0}(v_j, V_i) \geq t - 2k - \frac{n}{72k}, \quad (2.8)$$

since after $v_j$ becomes inactive, the number of edges that $v_j$ sends to $V_{1-i}$ decreases by at most $\deg_{G_0}(v_j, V_i)$. Finally, note that there are $s \leq \frac{m}{k} < \frac{n^2}{k}$ previous iterates, and for each iterate, the number of edges that a good and active vertex of $V_{1-i}$ sends to $V_i$ is reduced by at most $\Delta(F_k) = 2k$. By (2.5), the number of inactive good vertices of $G_s$ in $V_{1-i}$ is at most

$$\frac{2k - \gamma n^2}{k} = 144k\gamma n. \quad (2.9)$$

Let $L(u, v_j) \subseteq V_{1-i}$ be the set of good and active common neighbours of $u$ and $v_j$ in $G_s$. Then using (2.3) and (2.9), we have

$$|L(u, v_j)| \geq \deg_{G_s}(u, V_{1-i}) + \deg_{G_s}(v_j, V_{1-i}) - |V_{1-i}| - 144k\gamma n - 144k\gamma n,$n

$$\geq \frac{1}{2} \left\lfloor \frac{n}{4} \right\rfloor - k + t - 2k - \frac{n}{72k} - \frac{n}{2} - \sqrt{\gamma} n - 288k\gamma n \geq \frac{n}{12}, \quad (2.10)$$

where the second inequality follows from (2.2), (2.7) and (2.8), and the last one follows from (2.4). Hence, there exist $w_1, \ldots, w_k \in V_{1-i}$ such that for all $1 \leq j \leq k$, $w_j$ is a good and active vertex of $G_s$, and is a common neighbour of $u$ and $v_j$ in $G_s$.

Next, suppose that $a \leq s < b$, so that $\deg_{G_i}(u, V_i) = 0$ for all bad vertices $u \in V_i, i = 0, 1$. We have a good vertex $u \in V_i$ with $\deg_{G_i}(u, V_i) \geq k$ for some $i \in \{0, 1\}$. Let $v_1, \ldots, v_k \in V_i$ be (good) neighbours of $u$ in $G_s$, and $L(u, v_j) \subseteq V_{1-i}$ be the good and active common neighbours of $u$ and $v_j$ in $G_s (1 \leq j \leq k)$. Then, we have

$$\deg_{G_i}(u, V_{1-i}) \geq t - \Delta(F_k) - \deg_{G_0}(u, V_i) \geq t - 2k - \frac{n}{72k}, \quad (2.11)$$
since after \( u \) becomes inactive, in each subsequent iterate, the number of edges that \( u \) sends to \( V_i \) and \( V_{1-i} \) are both reduced by either 1 or \( k \). Similarly, for \( 1 \leq j \leq k \) we have

\[
\deg_{G_s}(v_j, V_i) \geq t - 2k - \frac{n}{72k}.
\]

Therefore,

\[
|L(u, v_j)| \geq 2\left( t - 2k - \frac{n}{72k} \right) - \frac{n}{2} - \sqrt{\gamma}n - 288k\gamma n \geq \frac{n}{3}.
\]

As before, we conclude that the required vertices \( w_1, \ldots, w_k \in V_{1-i} \) do exist.

Now, consider an iterate of Step 2 and let \( G_s \) be the graph at some point of the iteration. Suppose that we have a matching \( M \) in \( G_s[V_i] \) of size \( k \), for some \( i \in \{0, 1\} \). Let \( v_1w_1, \ldots, v_kw_k \) be the edges of \( M \). Then, as in (2.11), we have

\[
\deg_{G_s}(v_j, V_{1-i}) \geq t - 2k - \frac{n}{72k},
\]

\[
\deg_{G_s}(w_j, V_{1-i}) \geq t - 2k - \frac{n}{72k},
\]

for \( 1 \leq j \leq k \).

Let \( L(M) \subset V_{1-i} \) be the set of good and active vertices which are adjacent to \( v_1, w_1, \ldots, v_k, w_k \) in \( G_s \). Then, similar to (2.10), we have

\[
|L(M)| \geq 2k\left( t - 2k - \frac{n}{72k} \right) - (2k - 1)\left( \frac{n}{2} + \sqrt{\gamma}n \right) - 288k\gamma n \geq \frac{n}{3}.
\]

Hence, there exists a good and active vertex \( u \in V_{1-i} \) which is adjacent to \( v_1, w_1, \ldots, v_k, w_k \) in \( G_s \). This completes the proof of Claim 2.5 and Case 1 follows.

\[ \text{Case 2: } g(k) \leq m \leq \frac{5}{2}k^2 - \frac{1}{2}k. \]

We may assume that \( |V_0| \leq |V_1| \). For \( i = 0, 1 \), let \( A_i \subset V_i \) be the set of vertices incident to at least one edge of \( G[V_i] \) and \( B_i = V_i \setminus A_i \) the isolated vertices of \( G[V_i] \). Then, \( |A_i| \leq 2m \leq 5k^2 \) and since \( |V_i| \geq \frac{n}{2} - \sqrt{\gamma}n \) by Claim 2.4, we must have \( B_i \neq \emptyset \).

Since \( \delta(G) \geq \lceil \frac{n}{2} \rceil \), by considering any vertex of \( B_1 \), it follows that \( G \) has the following structure. The partition \( V(G) = V_0 \cup V_1 \) must be exactly balanced, so that \( |V_0| = \lceil \frac{n}{2} \rceil \) and \( |V_1| = \lceil \frac{n}{2} \rceil \). Moreover, if \( n \) is even, then for \( v \in B_1, v \) is adjacent to all vertices in \( V_{1-i} \). If \( n \) is odd, then for \( v \in B_1, v \) is adjacent to all vertices of \( V_0 \), and for \( v \in B_0 \), all but at most one vertex of \( V_1 \) are neighbours of \( v \).

Suppose that \( e(G) \geq \text{ex}(n, F_k) + s(3k - 1) \) for some integer \( s \geq 0 \). Then by (2.1), we have

\[
s(3k - 1) \leq e(G) - \text{ex}(n, F_k) \leq m - g(k) \leq \frac{5}{2}k^2 - \frac{1}{2}k - g(k)
\]

\[
= \begin{cases} 
\frac{3}{2}k^2 + \frac{1}{2}k & \text{if } k \text{ is odd}, \\
\frac{3}{2}k^2 + k & \text{if } k \text{ is even}, 
\end{cases}
\]

so that \( s \leq \frac{k}{2} + \frac{3}{8} \) if \( k \) is odd and \( s \leq \frac{k}{2} + \frac{3}{5} \) if \( k \) is even. Therefore, \( s \leq \lceil \frac{k}{2} \rceil \) since \( s \) is an integer. Our goal is to prove the following claim.
Claim 2.6. Let $G$ have the structure described above and let $\text{ex}(n, F_k) + s(3k - 1) \leq e(G) < \text{ex}(n, F_k) + (s + 1)(3k - 1)$ for some integer $0 \leq s \leq \lfloor \frac{k}{2} \rfloor$. Then, $G$ contains at least $s + 1$ edge-disjoint copies of $F_k$.

Claim 2.6 will then give us
\[
\phi(G, F_k) \leq e(G) - (s + 1)(e(F_k) - 1) < \text{ex}(n, F_k),
\]
and thus completing the proof of Case 2. Before we prove Claim 2.6, we need an auxiliary claim.

Claim 2.7. Let $G$ have the structure as described. Then for $i = 0, 1$ and any non-empty set $A \subset A_i$ with $|A| \leq 2k^2$, there exists a set $B \subset B_{1-i}$ with $|B| = 2k^2$, such that, $G$ contains a complete bipartite subgraph with classes $A$ and $B$.

Proof. Note that for $i = 0, 1$, we have $|B_i| \geq \lfloor \frac{n}{2} \rfloor - 5k^2 \geq 2k^2$. Hence, the only non-trivial case is when $n$ is odd and $i = 1$. For this case, assume that the claim does not hold. Then, the number of vertices of $B_0$ which are not adjacent to one vertex of $A$ is at least $\lfloor \frac{n}{2} \rfloor - 5k^2 - 2k^2 = \lfloor \frac{n}{2} \rfloor - 7k^2$. Hence, there exists a vertex $u \in A$ which is not adjacent to at least $\frac{1}{2k^2}(\lfloor \frac{n}{2} \rfloor - 7k^2)$ vertices of $B_0$. We have
\[
\deg_G(u) \leq \left\lfloor \frac{n}{2} \right\rfloor + \frac{5}{2}k^2 - \frac{1}{2}k - \frac{1}{2k^2}(\lfloor \frac{n}{2} \rfloor - 7k^2) < \left\lfloor \frac{n}{2} \right\rfloor,
\]
which contradicts $\delta(G) \geq \left\lfloor \frac{n}{2} \right\rfloor$.

Proof of Claim 2.6. The claim holds for $s = 0$ since we have $e(G) \geq \text{ex}(n, F_k)$ and $G \not\in \mathcal{F}_{n,k}$, so that by Theorem 1.4 $G$ contains a copy of $F_k$. Now, let $1 \leq s \leq \lfloor \frac{k}{2} \rfloor$. Note that since $e(G) \geq \text{ex}(n, F_k) + s(3k - 1)$ and $s < 3k$, we may apply Theorem 1.4 successively $s$ times to obtain and delete $s$ edge-disjoint copies of $F_k$ from $G$. Moreover, we will first greedily delete $q \leq s$ copies of $F_k$ that have a specific property, followed by a further $s - q$ copies. Let $G' \subset G$ be a subgraph on $V(G)$ obtained after deleting $s$ edge-disjoint copies of $F_k$ from $G$, so that $e(G') \geq \text{ex}(n, F_k) - s$. We will show that we can always obtain $G'$, so that $G'$ contains a further copy of $F_k$, which will imply Claim 2.6.

We consider several cases. In each case, we shall find the graph $G'$ and a vertex $u \in V(G')$ with $\deg_{G'}(u) \leq \left\lfloor \frac{n}{2} \right\rfloor - s$.

(i) Suppose that $G$ contains a copy of $F_{qk}$ with centre $u \in A_0 \cup A_1$ and $q \geq 2$. We choose $u$ so that $q$ is maximum. We have $q$ copies of $F_k$ and we are done if $q \geq s + 1$. If $q \leq s$, then $\deg_G(u) < \left\lfloor \frac{n}{2} \right\rfloor + (q + 1)k$, otherwise Claim 2.7 implies that there exists a copy of $F_{(q+1)k}$ with centre $u$, contradicting the choice of $q$. Obtain $G'$ by deleting the copy of $F_{qk}$, followed by a further $s - q$ copies of $F_k$. We have $\deg_{G'}(u) \leq \left\lfloor \frac{n}{2} \right\rfloor + (q + 1)k - 1 - 2qk \leq \left\lfloor \frac{n}{2} \right\rfloor - (q - 1)k < \left\lfloor \frac{n}{2} \right\rfloor - s$.

(ii) Suppose that there are $s$ vertices $u_1, \ldots, u_s \in A_0 \cup A_1$ such that $\deg_G(u_j, A_i) \geq \left\lfloor \frac{3}{2}k \right\rfloor - 1$, if $u_j \in A_i$. Without loss of generality we may assume that for $q \geq \left\lfloor \frac{s}{2} \right\rfloor$ we have $u_1, \ldots, u_q \in A_i$, for some $i \in \{0, 1\}$. However, in the special case when $q = \left\lfloor \frac{s}{2} \right\rfloor$
and $s$ is even, we will consider that $u_1, \ldots, u_q \in A_0$. For each $u_j \in A_i$, $1 \leq j \leq q$, we can choose $k$ neighbours of $u_j$, say $v_{j,1}, \ldots, v_{j,k} \in A_i \setminus \{u_1, \ldots, u_q\}$. This is possible since there are at least $\left\lfloor \frac{s}{2} k \right\rfloor - 1 - (q - 1) \geq k$ such neighbours. We may further assume that $v_{j,1} \neq v_{j,1}$ for $1 \leq j < \ell \leq q$. By Claim 2.7, we can find and delete $q$ copies of $F_k$ with centres $u_1, \ldots, u_q$ as follows. For $1 \leq j \leq q$, the copy with centre $u_j$ has triangles $u_jv_{1,j}u, u_jv_{2,j}w_{1,j}, \ldots, u_jv_{k,j}w_{k,j}$, for some $u, w_{j,2}, \ldots, w_{j,k} \in B_{1-i}$, with the vertices $w_{j,p}$ distinct for $1 \leq j \leq q$ and $2 \leq p \leq k$. Obtain the subgraph $G'$ by deleting a further $s - q$ copies of $F_k$. Then, $\deg_{G'}(u) \leq \deg_{G}(u) - 2q$. In the special case when $s$ is even and $q = \left\lceil \frac{s}{2} \right\rceil$ we must have $u \in B_1$, therefore $\deg_{G'}(u) \leq \left\lceil \frac{n}{2} \right\rceil - 2q \leq \left\lceil \frac{n}{2} \right\rceil - s$. In all other cases we have $\deg_{G'}(u) \leq \left\lceil \frac{n}{2} \right\rceil - 2q = \left\lceil \frac{n}{2} \right\rceil - s$, as required.

(iii) Suppose that (i) and (ii) do not hold. We obtain $G'$ by deleting any $s$ copies of $F_k$. If some copy has centre $u \in B_0 \cup B_1$, then $\deg_{G'}(u) \leq \left\lceil \frac{n}{2} \right\rceil - 2k < \left\lceil \frac{n}{2} \right\rceil - s$. Otherwise, all the centres lie in $A_0 \cup A_1$, and must be distinct. Moreover, at most $s - 1$ vertices $v \in A_0 \cup A_1$ satisfy $\deg_G(v, A_i) \geq \left\lfloor \frac{s}{2} k \right\rfloor - 1$ if $v \in A_i$. Hence, there exists a centre $u \in A_i$ with $\deg_G(u, A_i) \leq \left\lfloor \frac{s}{2} k \right\rfloor - 2$ for some $i \in \{0, 1\}$. We have $\deg_{G'}(u) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{s}{2} k \right\rfloor - 2 - 2k < \left\lceil \frac{n}{2} \right\rceil - s$.

Now, $\text{ex}(n, F_k) - \text{ex}(n - 1, F_k) = \left\lceil \frac{n}{2} \right\rceil$ by Theorem 1.4. Therefore in every case,

$$e(G' - u) = e(G') - \deg_{G'}(u) \geq \text{ex}(n, F_k) - s - \left\lfloor \frac{n}{2} \right\rceil + s = \text{ex}(n - 1, F_k). \quad (2.12)$$

Observe that equality in (2.12) can only happen in (ii). However, in this case, we have $\deg_{G'-u}(w_{1,2}) \leq \left\lceil \frac{n}{2} \right\rceil - 2 < \left\lceil \frac{n+1}{2} \right\rceil$. Thus, $G' - u \not\in \mathcal{F}_{n-1,k}$, since otherwise one must have $\delta(G' - u) \geq \left\lceil \frac{n+1}{2} \right\rceil$. Therefore, by Theorem 1.4, $G' - u$ contains a copy of $F_k$. This completes the proof of Claim 2.6 and Case 2 follows.

The proof of Theorem 1.5 is now complete. \hfill \Box

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