SOME RECENT APPLICATIONS OF SZEMERÉDI'S REGULARITY LEMMA

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Outline

1. Szemerédi's Regularity Lemma
2. Locating vertices on Hamiltonian cycles
3. Sketch of the proof
4. Further works

Outline

Regular pair

- **Density**: Let G be a graph, for any two disjoint vertex sets X and Y of G. The density of the pair (X, Y) is the ratio $d(X, Y) := \frac{e(X,Y)}{|X||Y|}$.
- **ϵ-regularity**: Let $\epsilon > 0$, the pair (X, Y) is called *ϵ-regular* if for every $A \subseteq X$ and $B \subseteq Y$ such that $|A| > \epsilon|X|$ and $|B| > \epsilon|Y|$ we have $|d(A, B) - d(X, Y)| < \epsilon$.
- **Super-regularity**: Let $\delta > 0$, the pair (X, Y) is called *(ϵ, δ)-super-regular* if it is ϵ-regular, $deg_Y(x) > \delta|Y|$ for all $x \in X$ and $deg_X(y) > \delta|X|$ for all $y \in Y$.

PROPERTIES OF REGULAR PAIRS

LEMMA

Let (A, B) be an ϵ-regular pair of density d and $Y \subseteq B$ such that $|Y| > \epsilon|B|$. Then all but at most $\epsilon|A|$ vertices in A have more than $(d - \epsilon)|Y|$ neighbors in Y.

LEMMA (SLICING LEMMA)

Let $\alpha > \epsilon > 0$ and $\epsilon' := \max\{\frac{\epsilon}{\alpha}, 2\epsilon\}$. Let (A, B) be an ϵ-regular pair with density d. Suppose $A' \subseteq A$ such that $|A'| \geq \alpha|A|$, and $B' \subseteq B$ such that $|B'| \geq \alpha|B|$. Then (A', B') is an ϵ'-regular pair with density d' such that $|d' - d| < \epsilon$.

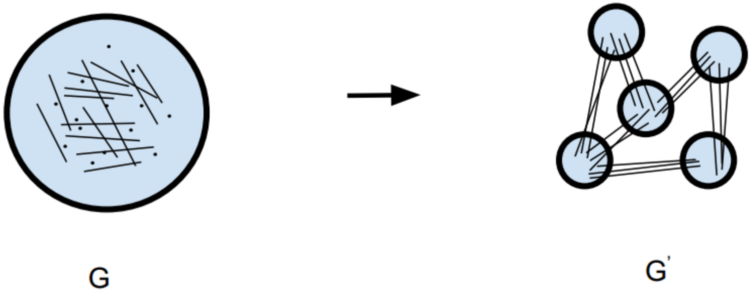
Regularity lemma

Lemma (Regularity Lemma-Degree Form)

For every $\epsilon > 0$ and every integer m_0 there is an $M_0 = M_0(\epsilon, m_0)$ such that if $G = (V, E)$ is any graph on at least M_0 vertices and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $l + 1$ clusters $V_0, V_1, ..., V_l$, and there is a subgraph $G^{'} = (V, E^{'})$ with the following properties:

- $m_0 \leq l \leq M_0$;
- $|V_0| \leq \epsilon|V|$, and V_i $(1 \leq i \leq l)$ are of the same size L;
- $deg_{G'}(v) > deg_G(v) - (d + \epsilon)|V|$ for all $v \in V$;
- $G^{'}[V_i] = \emptyset$ (i.e. V_i is an independant set in $G^{'}$) for all i;
- each pair (V_i, V_j), $1 \leq i < j \leq l$, is ϵ-regular, each with a density 0 or exceeding d.

Regularity lemma



Blow-up Lemma

Lemma (Blow-up Lemma-Bipartite Version)

For every $\delta, \Delta > 0$, there exists an $\epsilon = \epsilon(\delta, \Delta) > 0$ such that the following holds. Let (X, Y) be an (ϵ, δ)-super-regular pair with $|X| = |Y| = N$. If a bipartite graph H with $\Delta(H) \leq \Delta$ can be embedded in $K_{N,N}$ by a function ϕ, then H can be embedded in (X, Y).

Lemma

For every $\delta > 0$ there are $\epsilon_{BL} = \epsilon_{BL}(\delta)$, $n_{BL} = n_{BL}(\delta) > 0$ such that if $\epsilon \leq \epsilon_{BL}$ and $n \geq n_{BL}$, $G = (A, B)$ is an (ϵ, δ)-super-regular pair with $|A| = |B| = n$ and $x \in A$, $y \in B$, then there is a Hamiltonian path in G starting with x and ending with y.

Outline

Locating vertices on Hamiltonian cycles

Theorem (Kaneko and Yoshimoto, 2001)

Let G be a graph of order n with $\delta(G) \geq \frac{n}{2}$, and let d be a positive integer such that $d \leq \frac{n}{4}$. Then, for any vertex subset S with $|S| \leq \frac{n}{2d}$, there is a Hamiltonian cycle C such that $dist_C(u, v) \geq d$ for any $u, v \in S$.

- The result is sharp ($|S|$ can not be larger) as can be seen from the graph $2K_{\frac{n}{2}-1} + K_2$. When all the vertices of S are placed in one of the copies of $K_{\frac{n}{2}-1}$, then the bound becomes clear.

Locating vertices on Hamiltonian cycles

Theorem (Sárközy and Selkow, 2008)

There are $\omega, n_0 > 0$ such that if G is a graph with $\delta(G) \geq \frac{n}{2}$ on $n \geq n_0$ vertices, d is an arbitrary integer with $3 \leq d \leq \frac{\omega n}{2}$ and S is an arbitrary subset of $V(G)$ with $2 \leq |S| = k \leq \frac{\omega n}{2}$, then for every sequence of integers with $3 \leq d_i \leq d$, and $1 \leq i \leq k-1$, there is a Hamiltonian cycle C of G and an ordering of the vertices of S, $a_1, a_2, ..., a_k$, such that the vertices of S are encountered in this order on C and we have $|dist_C(a_i, a_{i+1}) - d_i| \leq 1$, for all but one $1 \leq i \leq k-1$.

- Almost all of the distances between successive pairs of S can be specified almost exactly.

LOCATING VERTICES ON HAMILTONIAN CYCLES

The two discrepancies by 1 can not be eliminated:

- $|dist_C(a_i, a_{i+1}) - d_i| \leq 1$: parity reason, e.g. $G = K_{\frac{n}{2},\frac{n}{2}}$, S in one side and d_i is odd.
- for all but one $1 \leq i \leq k-1$: Take two complete graphs on U and V with $|U| = |V| = \frac{n}{2}$. Let $S = S^{'} \cup S^{''}$ with $S^{'} \subset U$, $S^{''} \subset V$ and $|S^{'}| = |S^{''}| = \frac{|S|}{2}$, and add the complete bipartite graphs between $S^{'}$ and V, and between $S^{''}$ and U.

Locating vertices on Hamiltonian cycles

Theorem (Faudree and Gould, 2013)

Let $n_1, \ldots, n_{k-1}$ be a set of $k-1$ integers each at least 2 and $\{x_1, \ldots, x_k\}$ be a fixed set of k ordered vertices in a graph G of order n. If $\delta(G) \geq \frac{n+2k-2}{2}$, then there is $N = N(k, n_1, \ldots, n_{k-1})$ such that if $n \geq N$, there is a Hamiltonian cycle C of G such that $dist_C(x_i, x_{i+1}) = n_i$ for all $1 \leq i \leq k-1$.

- Degree condition is sharp: $G = \bar{K}_{\frac{n-2k+3}{2}} + (\frac{n+2k-3}{2(2k-2)} K_{2k-2})$, if k vertices are all selected from one of the copies of K_{2k-2}.

LOCATING VERTICES ON HAMILTONIAN CYCLES

THEOREM (GOULD, MAGNANT AND NOWBANDEGANI, 2017)

Given an integer $k \geq 3$, let G be a graph of sufficiently large order n. Then there exists $n_0 = n_0(k, n)$ such that if $n_1, n_2, ..., n_k$ are a set of k positive integers with $n_i \geq n_0$ for all i ,$\sum n_i = n$, and $\delta(G) \geq \frac{n+k}{2}$, then for any k distinct vertices $x_1, x_2, ..., x_k$ in G, there exists a Hamiltonian cycle such that the length of the path between x_i to x_{i+1} on the Hamiltonian cycle is n_i.

- Degree condition is sharp when k is even: Consider two complete graphs A and B each of order $\frac{n-(k+1)}{2}$. Let C be the remaining $k+1$ vertices. Let $G = (A + C) \cup (C + B)$ where the copies of vertices of C are identified. If all of the vertices $x_1, ..., x_k$ are chosen from A and each length n_i is chosen to be $\frac{n}{k}$.

Locating pairs of vertices on Hamiltonian cycles

Conjecture (Enomoto)

If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G, there is a Hamiltonian cycle C of G such that $dist_C(x, y) = \lfloor \frac{n}{2} \rfloor$.

Conjecture (Faudree and Li, 2012)

If G is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G and any integer $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle C of G such that $dist_C(x, y) = k$.

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Sharpness of the minimum degree condition

- The degree condition is sharp.
  - Example 1: there is no Hamiltonian cycle such that x and y have distance $\frac{n}{2}$.

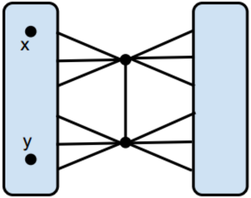
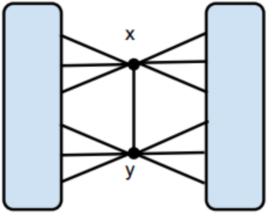


FIGURE: $2K_{\frac{n}{2}-1} + K_2$

Sharpness of the minimum degree condition

- The degree condition is sharp.
  - Example 2: x and y will be at distance $\frac{n}{2}$ in any Hamiltonian cycle of the graph.



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Locating pairs of vertices on Hamiltonian cycles

Theorem (Faudree and Li, 2012)

If p is a positive integer with $2 \leq p \leq \frac{n}{2}$ and G is a graph of order n with $\delta(G) \geq \frac{n+p}{2}$, then for any pair of vertices x and y in G, there is a Hamiltonian cycle C of G such that $dist_C(x, y) = k$ for any $2 \leq k \leq p$.

Corollary (Faudree and Li, 2012)

If G is a graph of order n with $\delta(G) \geq \lfloor \frac{3n}{4} \rfloor$, then for any pair of vertices x and y of G and any positive integer $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$, there is a Hamiltonian cycle C of G such that $dist_C(x, y) = k$.

Locating pairs of vertices on Hamiltonian cycles

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Our result

Theorem (He, Li and Sun, 2016)

There exists a positive integer n_0 such that for all $n \geq n_0$, if G is a graph of order n with $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices x, y in G, there is a Hamiltonian cycle C of G such that $dist_C(x, y) = \lfloor \frac{n}{2} \rfloor$.

Outline

Preparation of the proof

Theorem (He, Li and Sun, 2015)

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- Only need to consider the graphs with even order.
- Suppose $0 < \epsilon \ll d \ll \alpha \ll 1$, and n is sufficiently large.
- A *balanced partition* of $V(G)$ into V_1 and V_2 is a partition of $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = \frac{n}{2}$.
  - **Extremal Case 1**: There exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \geq 1 - \alpha$.
  - **Extremal Case 2**: There exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \leq \alpha$.

PREPARATION OF THE PROOF

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Let G be a graph not in either of the extremal cases. We apply the Regularity Lemma to G.

- **Reduced graph** R: the vertices of R are $r_1, r_2, ..., r_l$, and there is an edge between r_i and r_j if the pair (V_i, V_j) is ϵ-regular in G' with density exceeding d.
  - R inherits the minimum degree condition: $\delta(R) \geq (\frac{1}{2} - 2d)l$.
  - R is a Hamiltonian graph.

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  - R is a Hamiltonian graph.

STEP 2: CONSTRUCTING PATHS TO CONNECT CLUSTERS

- By the Hamiltonian cycle in R, we find a perfect matching in R. Denote the clusters by X_i, Y_i according to the matching. (X_i, Y_i) is called a pair of clusters.
- Construct paths P_i's and Q_i's to connect different pairs of clusters and to include x, y.

NON-EXTREMAL CASE

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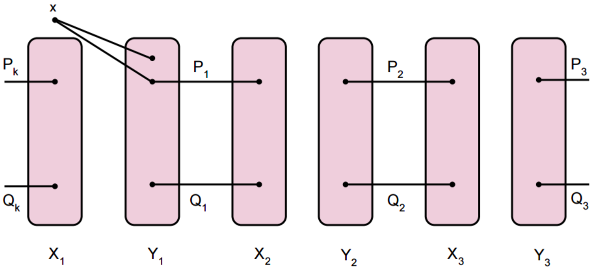


FIGURE: Construction of P_i's and Q_i's.

NON-EXTREMAL CASE

STEP 3: EXTENDING THE PATHS BY ALL THE VERTICES OF V_0

- Deal with the vertices of V_0 pair by pair.

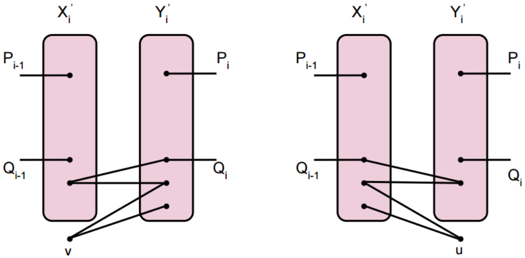
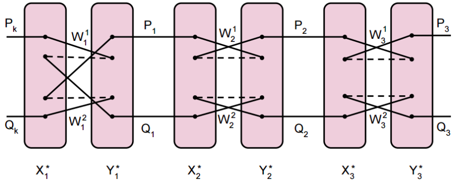


FIGURE: Insert $u, v \in V_0$ to Q_i's.

NON-EXTREMAL CASE

STEP 4: CONSTRUCTING THE DESIRED HAMILTONIAN CYCLE

- Construct paths W_i^1's and W_i^2's in each pair of clusters by Blow-up lemma and make sure x and y have distance $\frac{n}{2}$ on this cycle.



Extremal Case 1: There exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \geq 1 - \alpha$.

LEMMA

If G is in extremal case 1, then G contains a balanced spanning bipartite subgraph G^* with parts U_1, U_2 and G^* has the following properties:
(a) there is a vertex set W such that there exist vertex-disjoint 2-paths (paths of length two) in G^* with the vertices of W as the middle vertices (not the end vertices) in each 2-path and $|W| \leq \alpha_2 n$;
(b) $deg_{G^*}(v) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$ for all $v \notin W$.

Extremal case 1

The proof has some sub-cases discussions depending on the position of x,y and the parity of $\frac{n}{2}$. And the Blow-up lemma is the main tool.

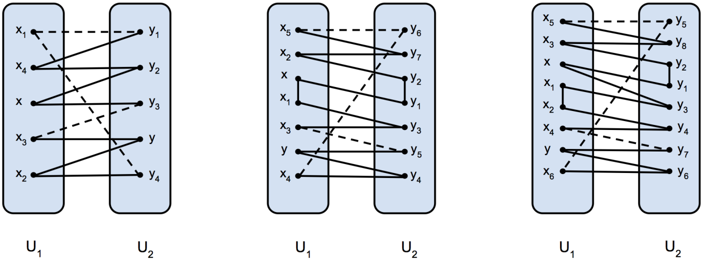


FIGURE: Extremal case 1.

Extremal case 2

Extremal Case 2: There exists a balanced partition of $V(G)$ into V_1 and V_2 such that the density $d(V_1, V_2) \leq \alpha$.

Lemma

If G is in extremal case 2, then $V(G)$ can be partitioned into two balanced parts U_1 and U_2 such that
(a) there is a set $W_1 \subseteq U_1$ (resp. $W_2 \subseteq U_2$) such that there exist vertex-disjoint 2-paths in $G[U_1]$ (resp. $G[U_2]$) with the vertices of W_1 (resp. W_2) as the middle vertices in each 2-path and $|W_1| \leq \alpha_2 \frac{n}{2}$ (resp. $|W_2| \leq \alpha_2 \frac{n}{2}$);
(b) $deg_{G[U_1]}(u) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$ for all $u \in U_1 - W_1$ and $deg_{G[U_2]}(v) \geq (1 - \alpha_1 - 2\alpha_2)\frac{n}{2}$ for all $v \in U_2 - W_2$.

Extremal case 2

The proof has some sub-cases discussions depending on the position of x and y.

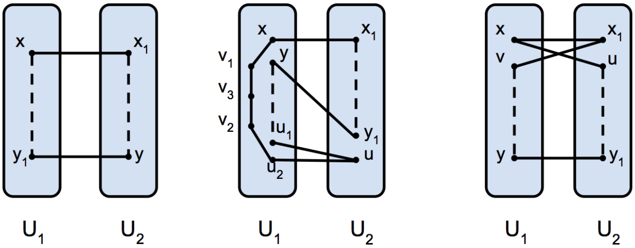


Figure: Extremal case 2.

Outline

Further works

- To avoid using Szemerédi's regularity lemma?
- To locate more vertices (≥ 3) on Hamiltonian cycles with precise distances?

FURTHER WORKS

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- To locate more vertices (≥ 3) on Hamiltonian cycles with precise distances?

Thank you!