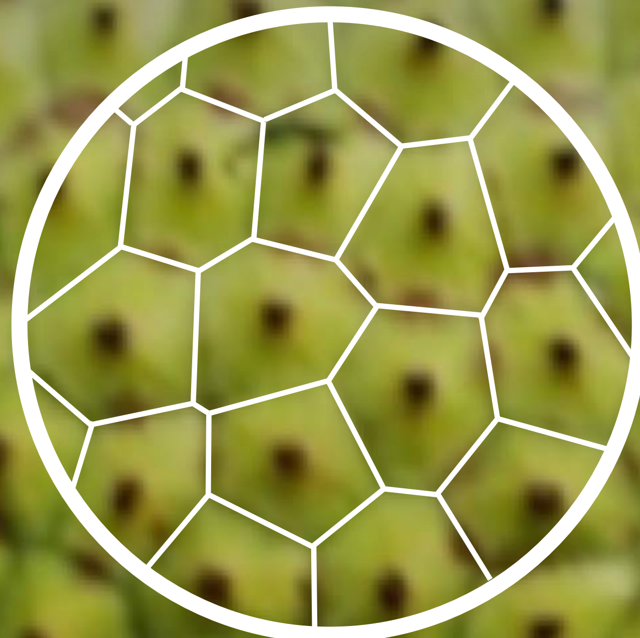
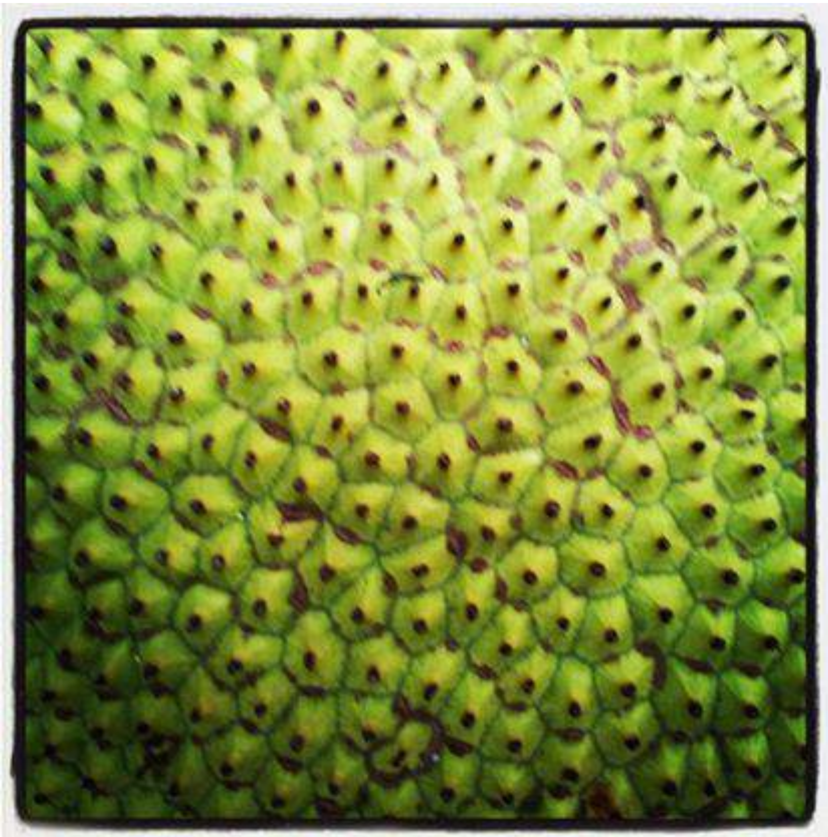
Guangzhou Discrete Mathematics Seminar



GEOMETRICAL PROPERTIES OF **SPHERICAL LAGUERRE VORONOI DIAGRAM** WITH APPLICATIONS

Supanut Chaidee
Department of Mathematics, Faculty of Science
Chiang Mai University, Thailand

Motivation



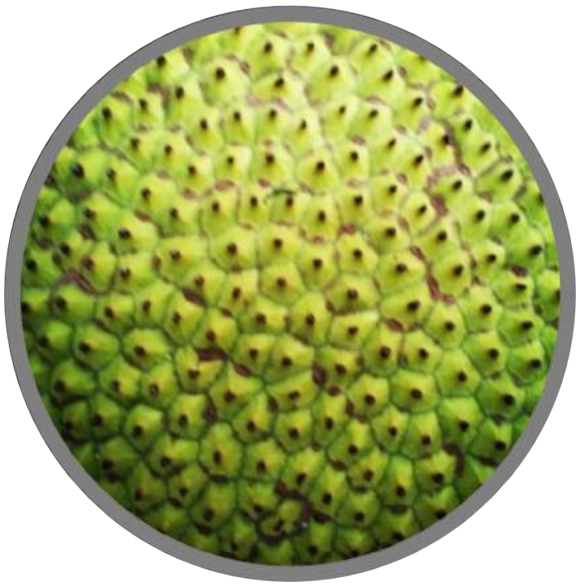
There are many phenomena related to polygonal net problems.

Patterns on the fruit skins

- Patterns on (approximated) spherical surface

Can we use mathematical concepts to model or understand the pattern formation?

Tessellation Patterns

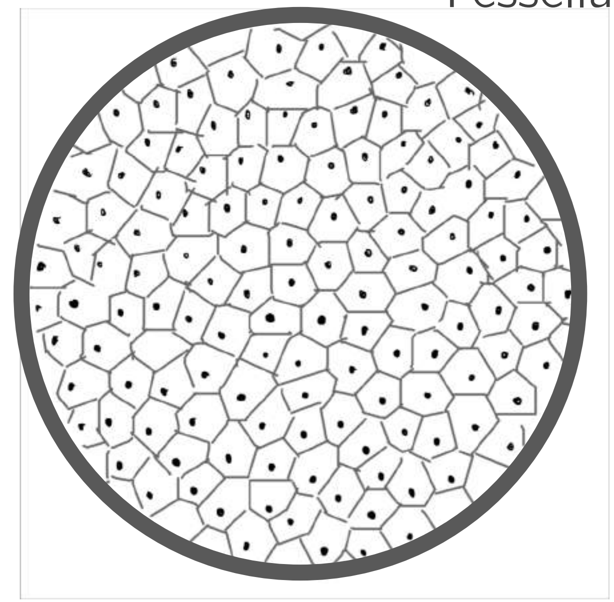


A skin pattern of "Jackfruit"
Artocarpus Heterophyllus

Problem Formulation →

Geometrical Viewpoint

← Understanding phenomena



Computational Geometry

the study of algorithms which can be stated in terms of geometry.

Voronoi Diagram

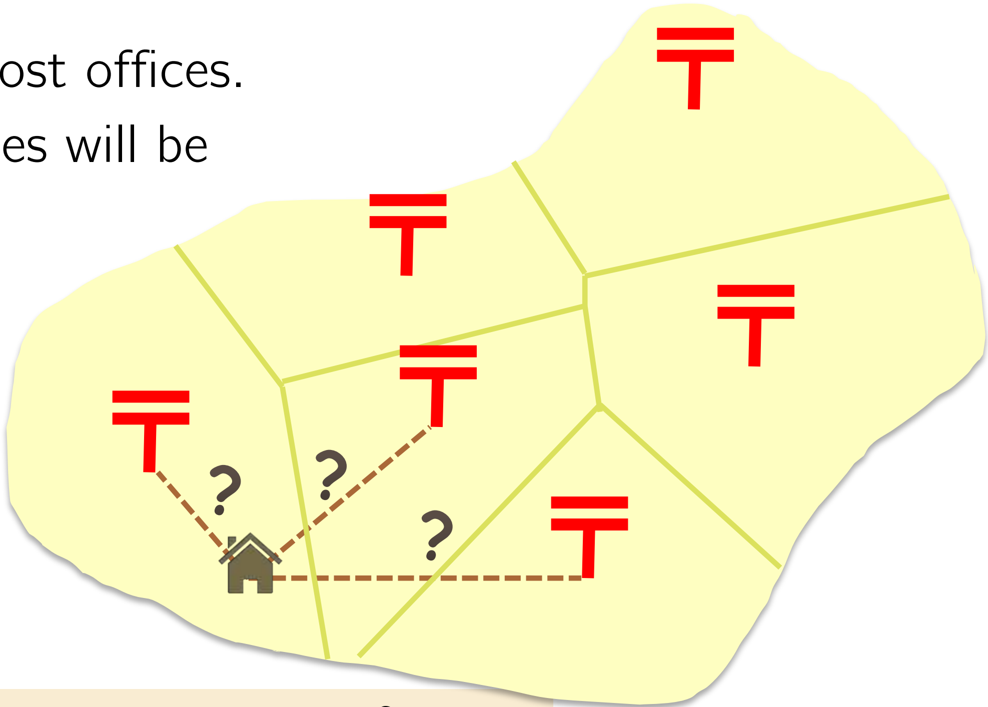
Some problems are related to the space partitioning.

Post Office Problem

Suppose that a city has a set of post offices.

We need to determine which houses will be operated by which office.

A resident needs to send a letter at a post office near his home!



A subdivision of a plane into these regions is called Voronoi diagram.

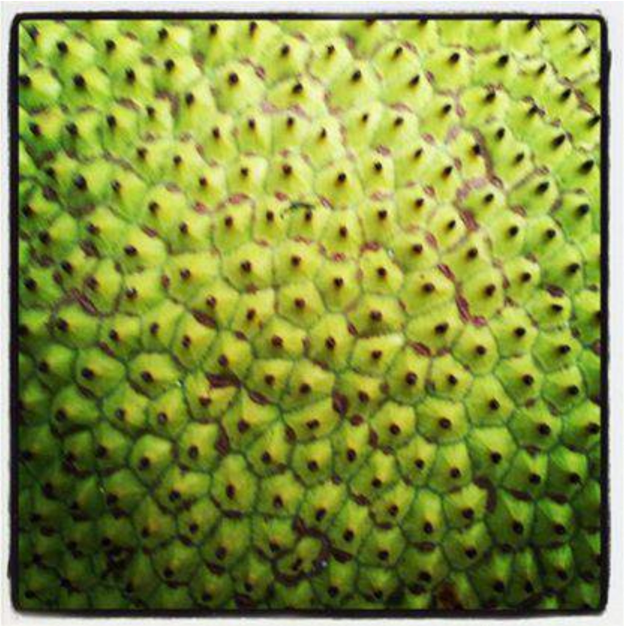
Let $P = \{p_1, p_2, ..., p_n\}$ be a set of n sites over $\mathbb{R}^2$.
The **Voronoi region** $V(p_i)$ of the site $p_i \in S$ is defined by

$$V(p_i) = \{x \in \mathbb{R}^2 | d(x, p_i) \le d(x, p_j) \text{ for } i \ne j\}$$

where $d(x, y)$ denotes the Euclidean distance between points x and y in the plane.

Considering on the Real-world Problem..

Is ordinary Voronoi diagram enough for modeling the pattern?



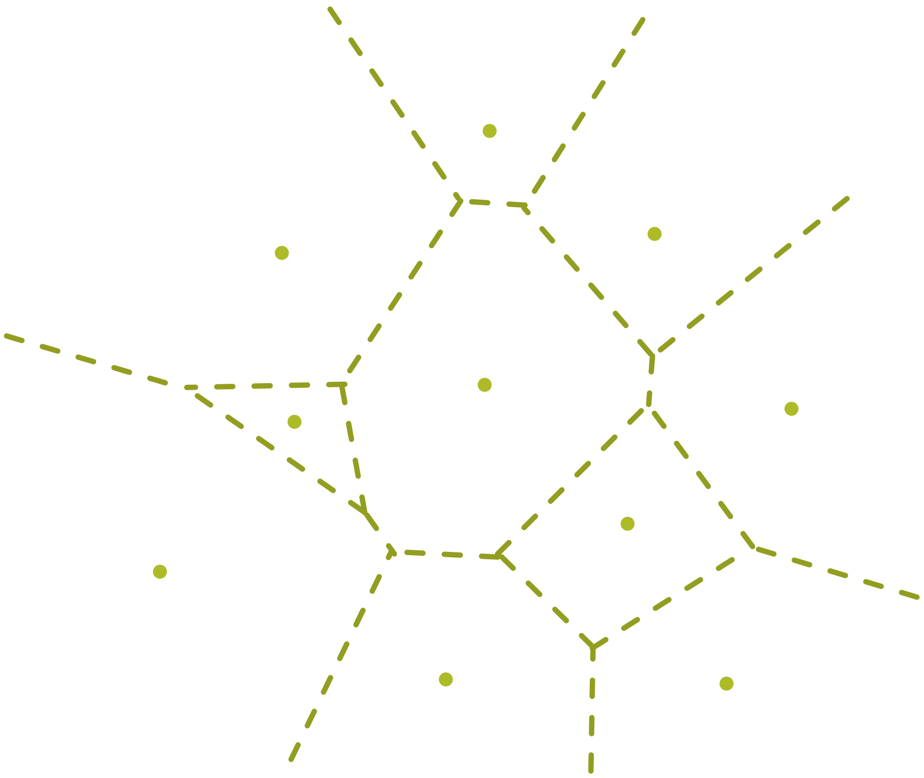
Jackfruit skin pattern

Lychee skin pattern

To model this kind of tessellation, **weights** of each generator is important due to real-world phenomena.

Considering on the Real-world Problem..

Voronoi Diagram

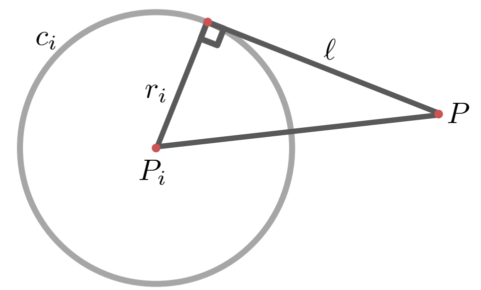


Voronoi Diagram

$$\mathrm{V(space/generator/distance)}$$

K. Sugihara, Journal for Geometry and Graphics (2002)
Laguerre Voronoi Diagram on the Sphere

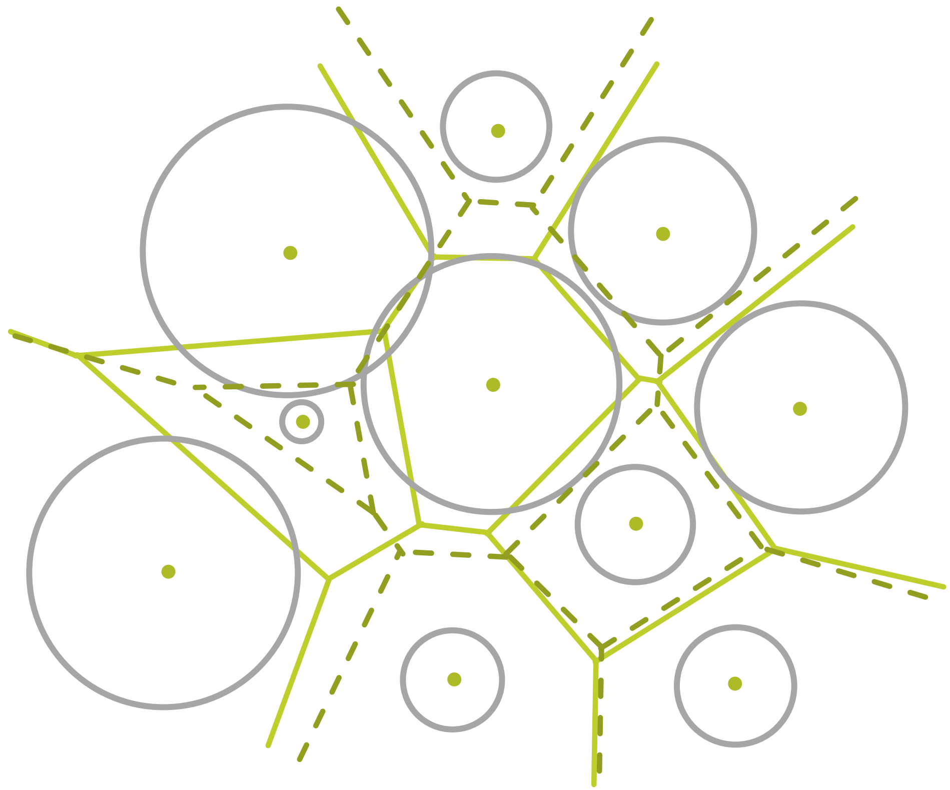
Voronoi Diagram



Each generator comes with its circle.

$$d_{\mathrm{L}}(P, c_i) = d(P, P_i)^2 - r_i^2$$

Laguerre Voronoi Diagram

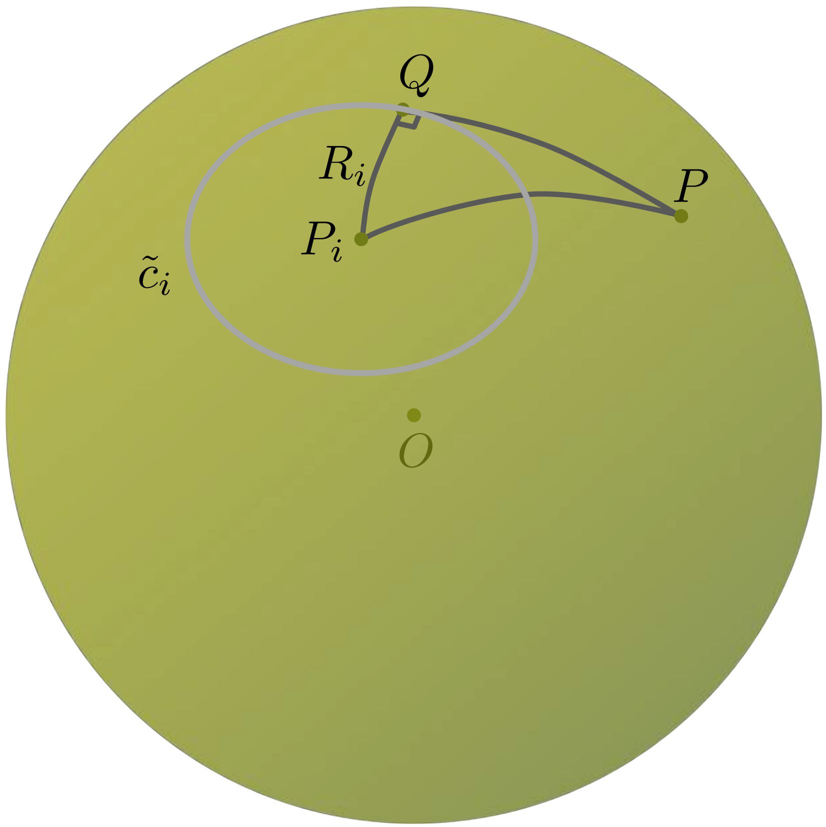


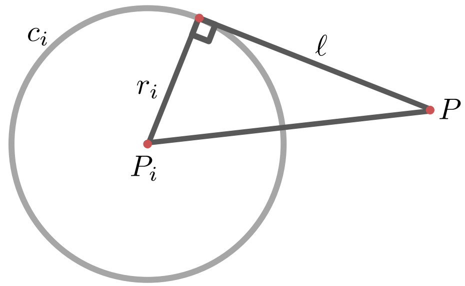
Voronoi Diagram

V(space/generator/distance)

K. Sugihara, Journal for Geometry and Graphics (2002)
Laguerre Voronoi Diagram on the Sphere

Voronoi Diagram





Each generator comes with its circle.

$$d_{\mathrm{L}}(P, c_i) = d(P, P_i)^2 - r_i^2$$

Laguerre Voronoi Diagram

A *spherical circle* on U corresponding to P_i is

$$\tilde{c}_i = \{Q \in U | \tilde{d}(P_i, Q) = R_i\}$$

where $0 \leq R_i/R < \pi/2$.

The *Laguerre Proximity*

$$\tilde{d}_L(P, \tilde{c}_i) = \frac{\cos\left(\tilde{d}(P, P_i)/R\right)}{\cos(R_i/R)}$$

$\mathrm{V(sphere/points/Laguerre)}$

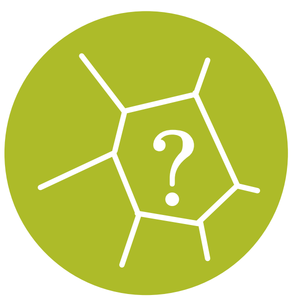
Spherical Laguerre Voronoi Diagram (SLVD)

K. Sugihara, Journal for Geometry and Graphics (2002)
Laguerre Voronoi Diagram on the Sphere

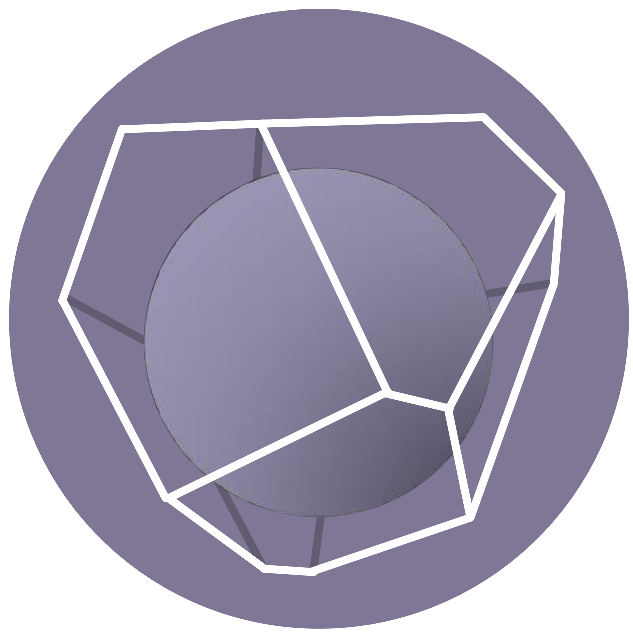
Research Scopes

to construct mathematical models for understanding the polygonal tessellation on the fruit skins

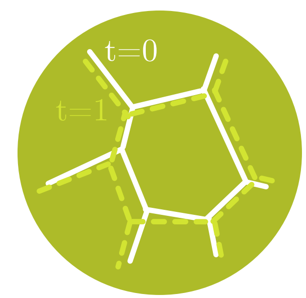
We use the spherical Laguerre Voronoi diagram as a main tool for solving the problem.



Inverse Voronoi Diagram Problem

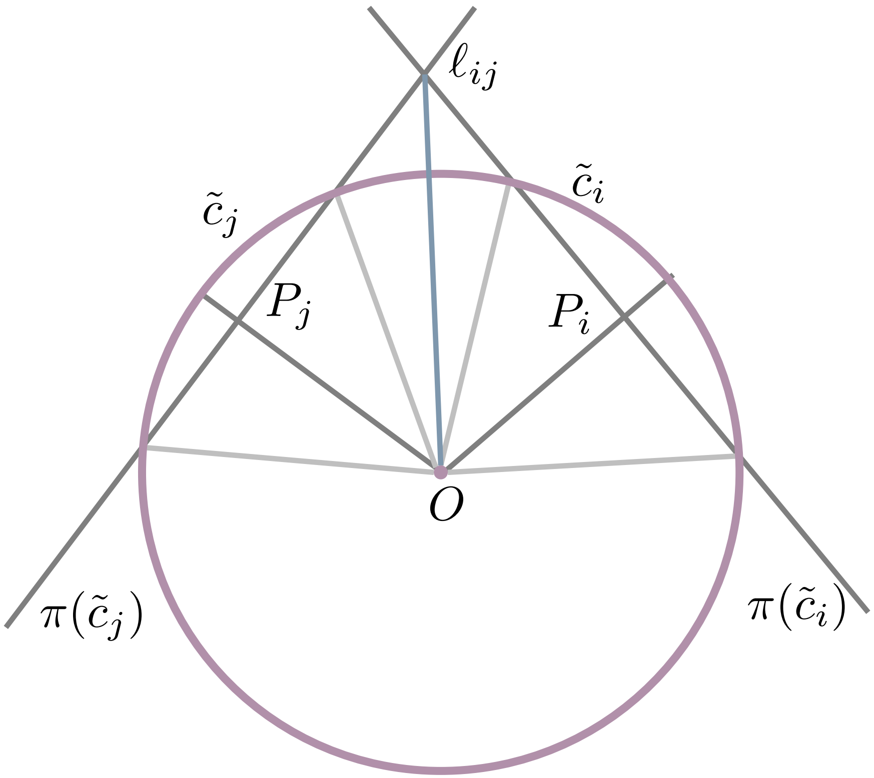


Properties of the **spherical Laguerre Voronoi diagram**

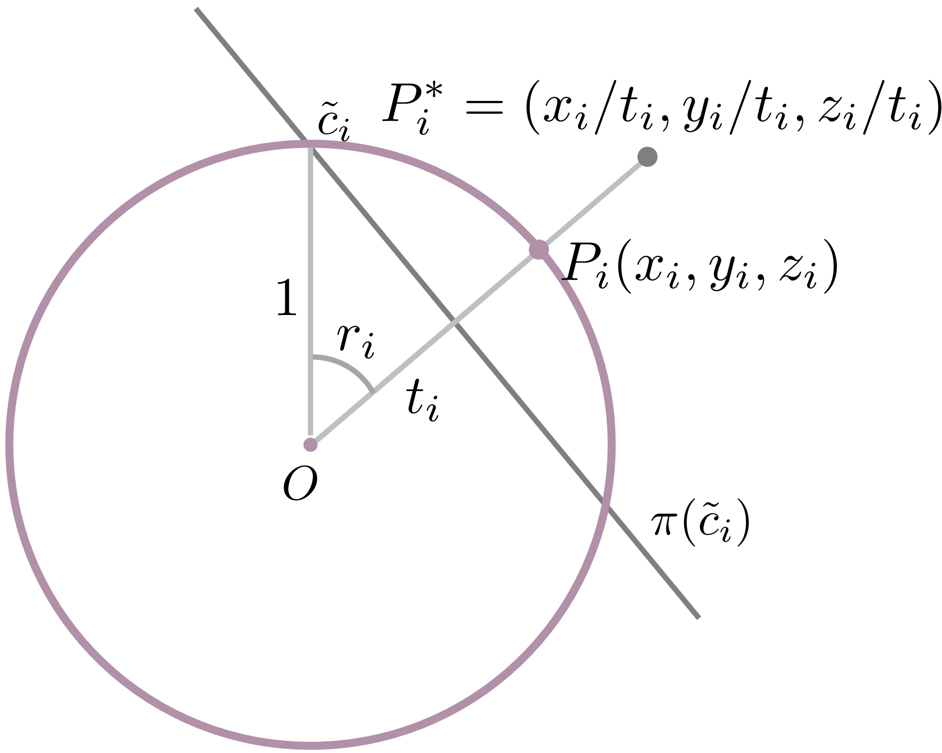


Voronoi-based Modeling

Construction of SLVD

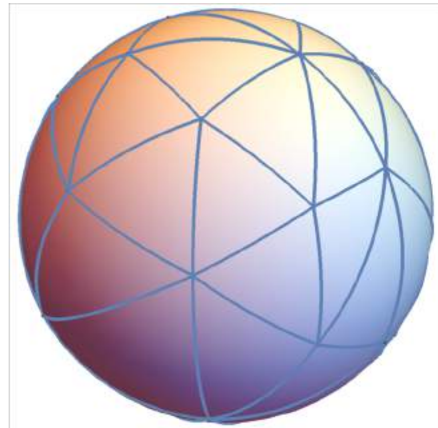


Spherical Laguerre
Voronoi diagram

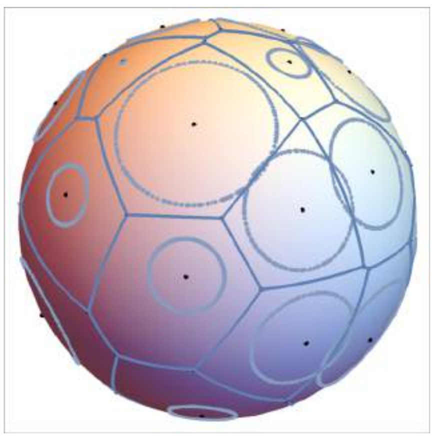


Spherical Laguerre
Delaunay diagram

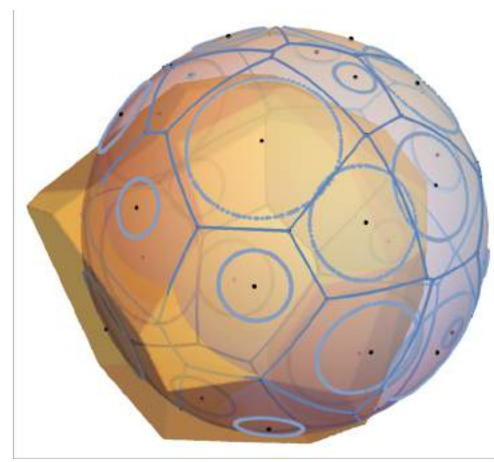
Corresponding Structures



Spherical Laguerre Delaunay Diagram



Spherical Laguerre Voronoi diagram

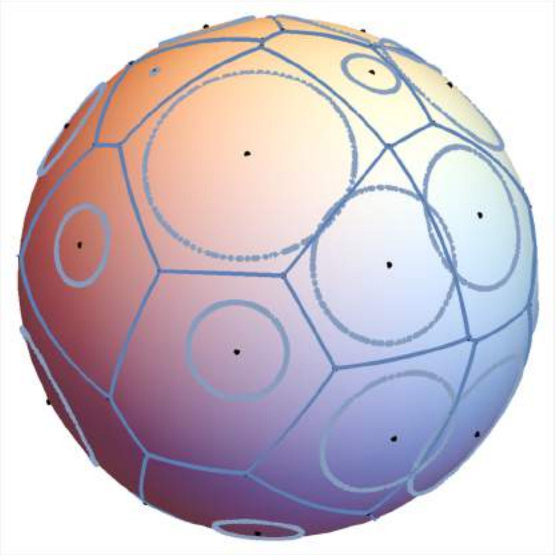


Convex polyhedron

[1] K. Sugihara. Laguerre Voronoi Diagram on the Sphere, Journal for Geometry and Graphics, 6:1, 69–81 (2002).

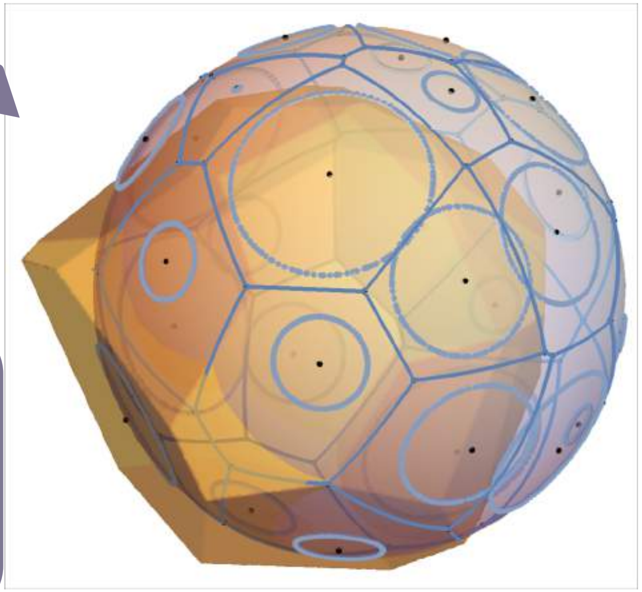
[2] S. Chaidee, K. Sugihara. Recognition of Spherical Laguerre Voronoi Diagram, submitted

Correspondence between SLVD and Polyhedra



Spherical Laguerre Voronoi diagram

By definition and construction algorithms

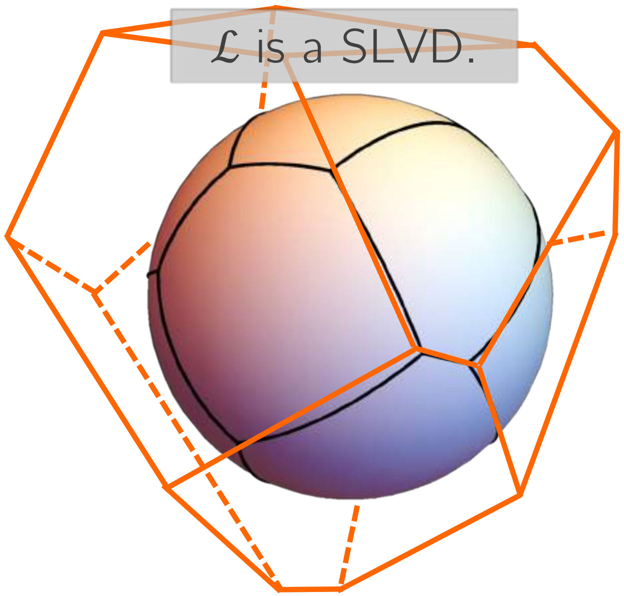


Convex polyhedron

Proposition

$\mathcal{L}$ is a SLVD if and only if there is a convex polyhedron $\mathcal{P}$ containing the center of the sphere whose central projection coincides with $\mathcal{L}$.

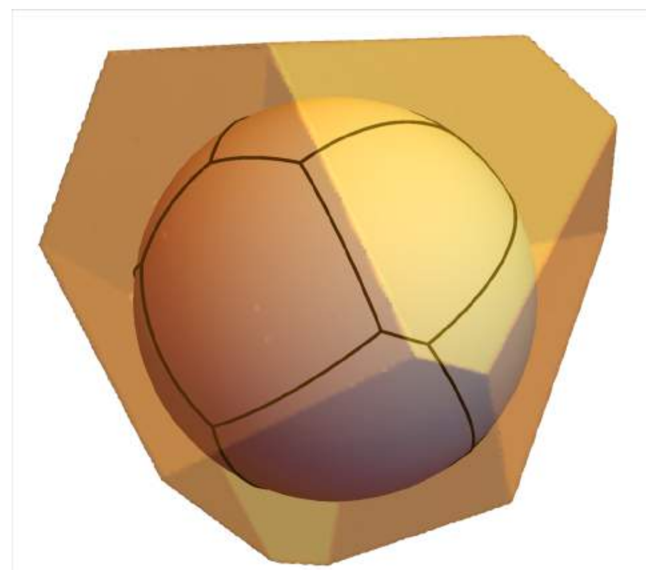
Correspondence between SLVD and Polyhedra



Polyhedron transformation

For a point $\mathbf{v}_a = (t_a, x_a, y_a, z_a) \in P^3(\mathbb{R})$ in the homogeneous coordinate system, define a map $f : P^3(\mathbb{R}) \to P^3(\mathbb{R})$ such that

$$f(\mathbf{v}_a) = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix} \mathbf{v}_a$$



Theorem

There exists a transformation satisfying the projection preservation properties.

We proposed algorithms for constructing a polyhedron with respect to SLVD.

Tessellation Analysis

inverse

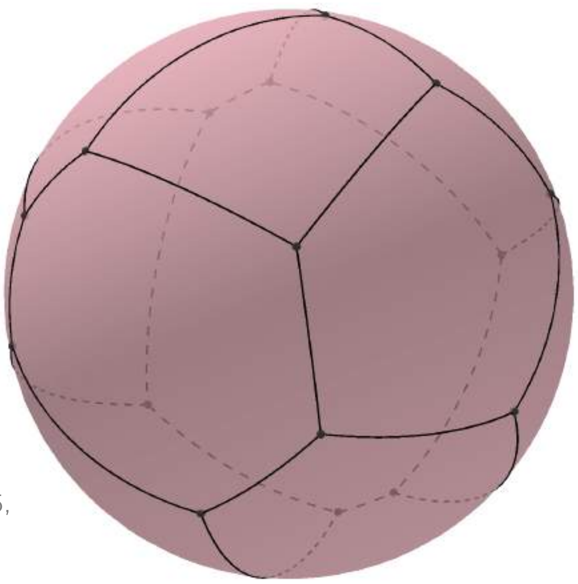
SLVD

Recognition Problem

find the SLVD which best fits to the given tessellation

Approximation Problem

S. Chaidee and K. Sugihara (2018), *Spherical Laguerre Voronoi Diagram Approximation of Tessellation without Generators*, Graphical Models 95, pp. 1 – 13



recover the generators and their weights.

S. Chaidee and K. Sugihara, *Recognition of the Spherical Laguerre Voronoi Diagram*, preprint.

Convex Spherical Tessellation

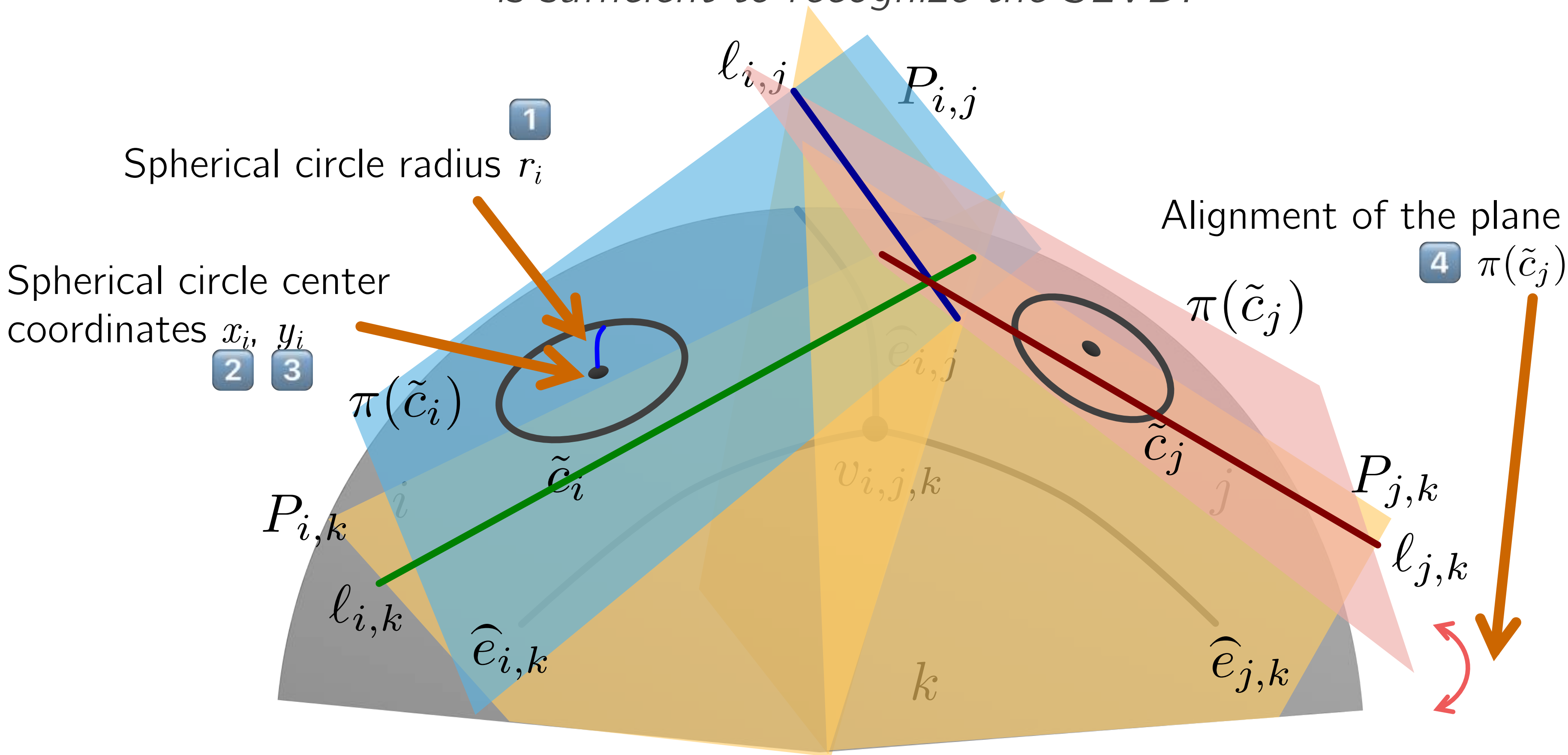
$\mathcal{T}=\{T_1, \ldots, T_n\}$

SLVD Recognition Problem

Theorem

There are exactly four degrees of freedom in the choice of a polyhedron $\mathcal{P}$ with respect to the given SLVD.

"Any choice of the initial pair of planes is sufficient to recognize the SLVD."



SLVD Approximation Problem

Voronoi Approximation of the Spike-containing Objects

S. Chaidee and K. Sugihara (2017), Pattern Analysis and Applications

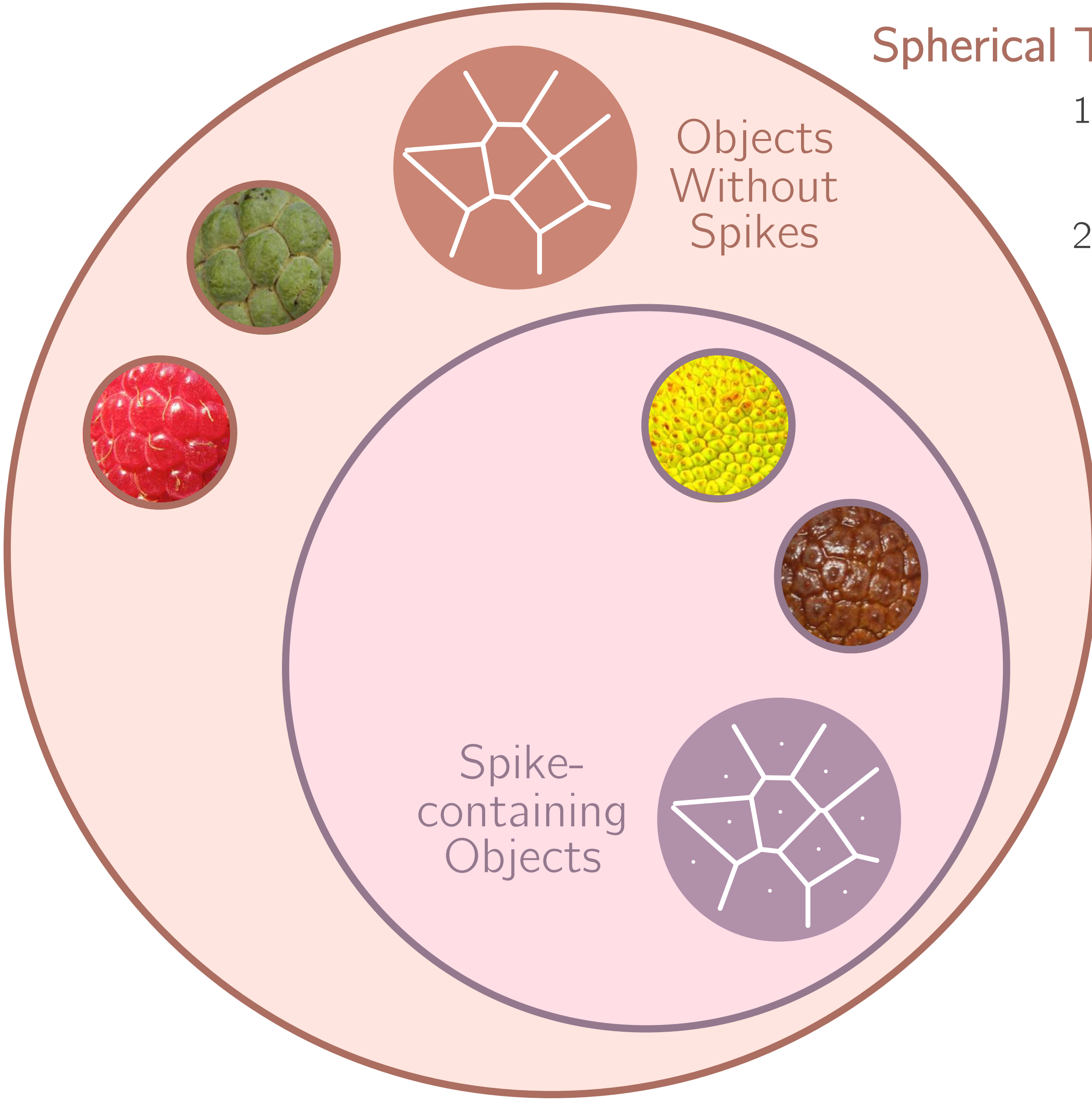
Approximation of Fruit Skin Patterns Using Spherical Voronoi Diagram

Voronoi Approximation of the Objects without Spikes

S. Chaidee, K. Sugihara (2016), Discrete and Computational Geometry and Graphs (LNCS 9943)

Fitting Spherical Laguerre Voronoi Diagrams to Real World Tessellations Using Planar Photographic Images

Object Classification



Spherical Tessellation Object

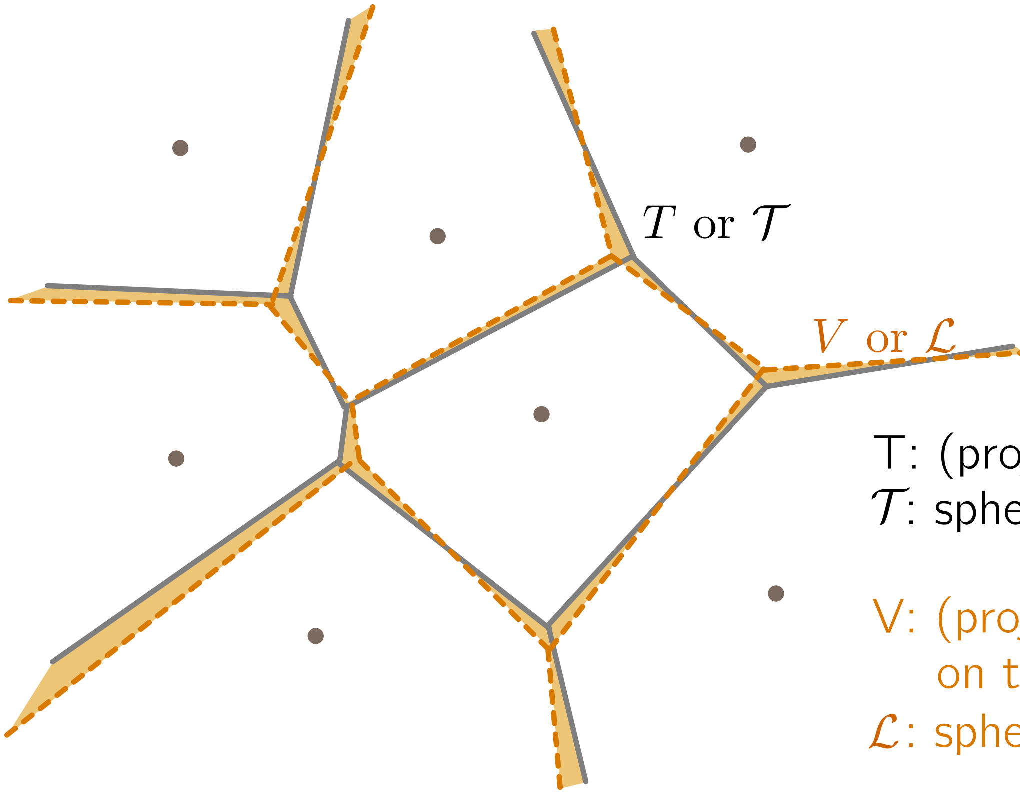
1. The object is a convex surface which can be approximated by a sphere.
2. There exists a polygonal net on the surface.

Spike-containing Object

1. The object is spherical tessellation object.
2. Each unit of the polygonal net contains exactly one spike.
3. The heights of spikes are approximately uniform.

Voronoi Approximation Problem

Find the spherical (Laguerre) Voronoi diagram which best fits to the given tessellation.



'Discrepancy' is defined as the ratio of sum of different areas to sum of total areas.

T: (projected) tessellation on the plane
$\mathcal{T}$: spherical tessellation on the unit sphere

V: (projected) spherical Voronoi diagram on the plane
$\mathcal{L}$: spherical Laguerre Voronoi diagram

minimized 'Discrepancy' ≡ the **best fitted Voronoi diagram**

Voronoi Approximation of
the Spike-containing Objects

Main Framework

Tessellation Fitting using ordinary spherical Voronoi diagram

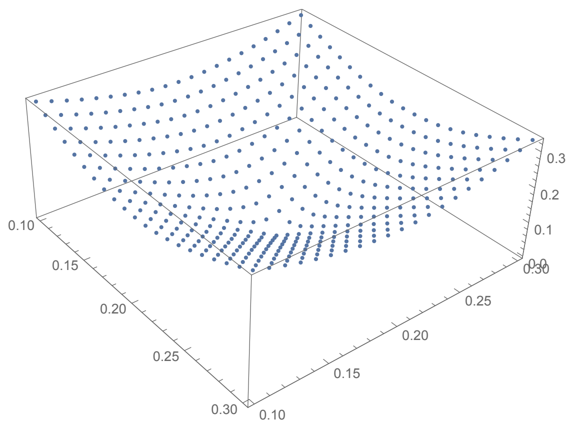
The discrepancy depends on the sphere radius R, the spike height h, and the sphere center position (x, z).

The discrepancy function $D(x, z, R, h)$ with respect to the variables x, z, R, h → Claim $\min D(x, z, R, h)$ for obtaining the appropriate x, z, R, h

The parameters for obtaining the best fit spherical Voronoi diagram

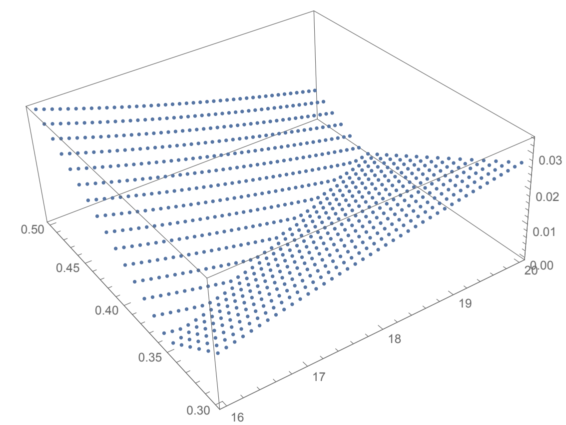
We consider the optimization problem by constructing an iterated (decreasing) sequence tending to the minimum.

Fix R, h and optimize $D(x, z)$



The Method of Steepest Descent

Fix x, z and optimize $D(R, h)$



The Circular Search

Weight Approximation

Tessellation Fitting using spherical Laguerre Voronoi diagram

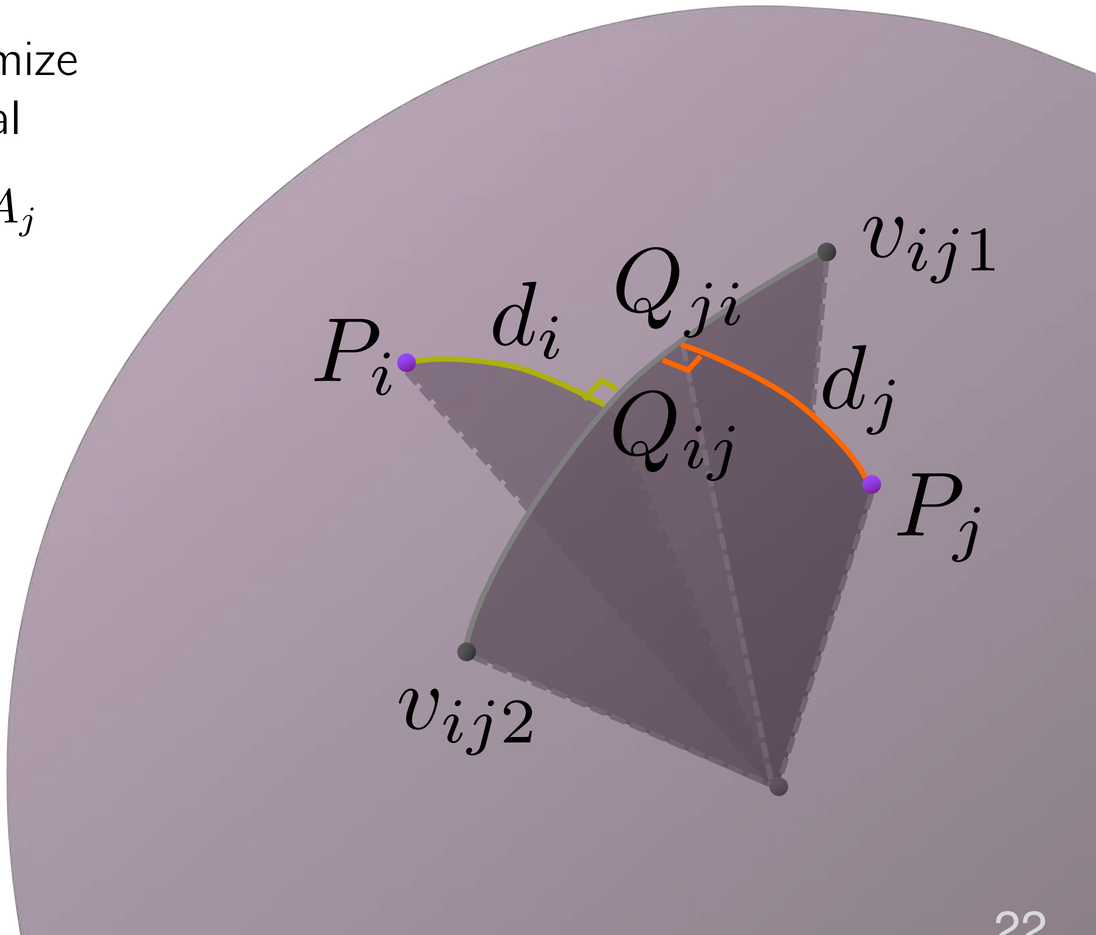
From the fitting result using an ordinary spherical Voronoi diagram, we approximate weight of each generator. The tessellation edges of the given tessellation on the plane are projected onto the sphere.

- For each pair, compute that geodesic lengths d_i, d_j and minimize the sum of square of the residual

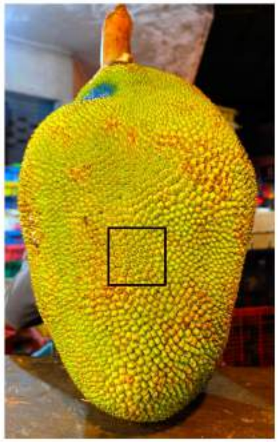
$$\cos\left(\frac{R_j}{R}\right) A_i - \cos\left(\frac{R_i}{R}\right) A_j$$

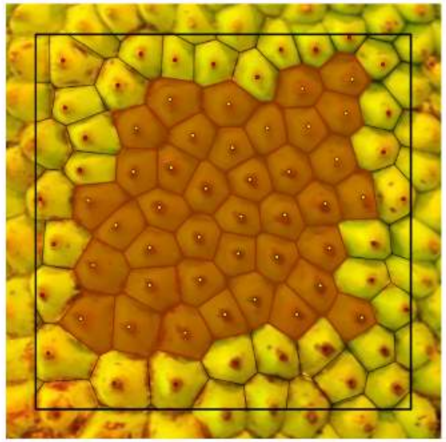
where $A_i = \cos\left(\frac{\tilde{d}(P_i, Q_i)}{R}\right)$

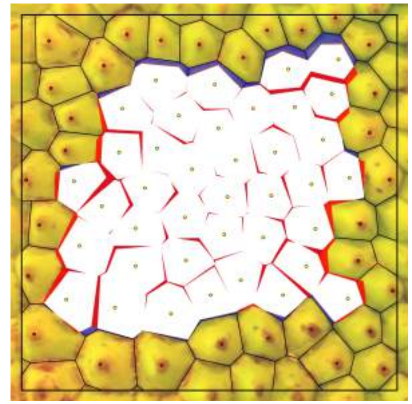
The approximation is done using the fact of SLVD



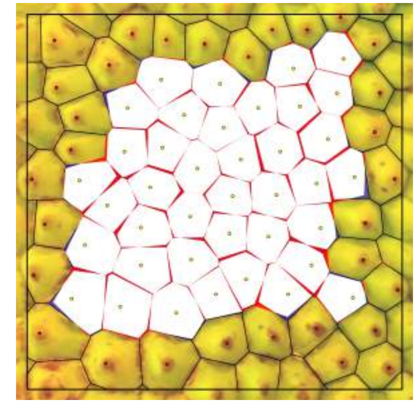
Experimental Results



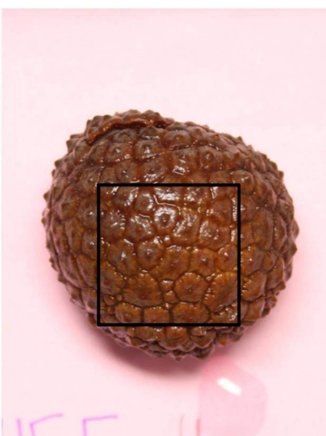


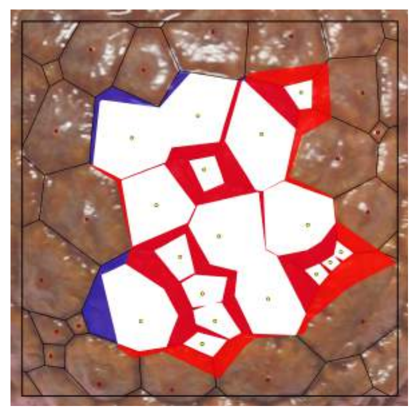


Fitting with the ordinary spherical Voronoi diagram

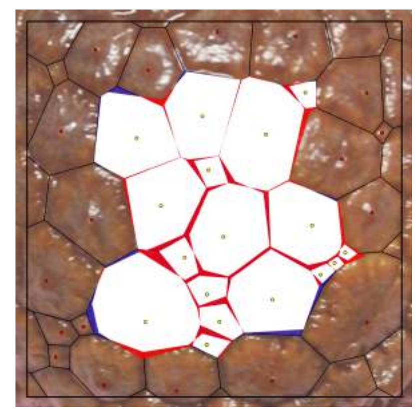


Fitting with the spherical Laguerre Voronoi diagram





Fitting with the ordinary spherical Voronoi diagram



Fitting with the spherical Laguerre Voronoi diagram

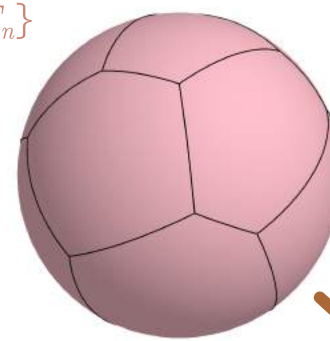
Spherical Laguerre Voronoi Diagram Approximation Problem

(Objects without Spikes)

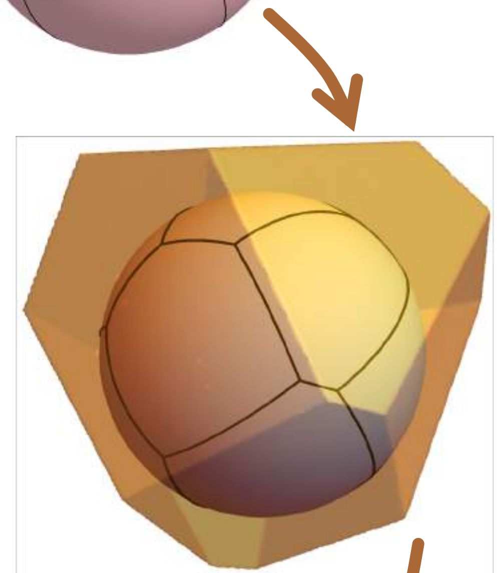
S. Chaidee, K. Sugihara (2018), Graphical Models
Spherical Laguerre Voronoi Diagram Approximation to Tessellations without Generators

Tessellation Comparison

Given spherical tessellation $\mathcal{T}$
$\mathcal{T}=\{T_1, \ldots, T_n\}$



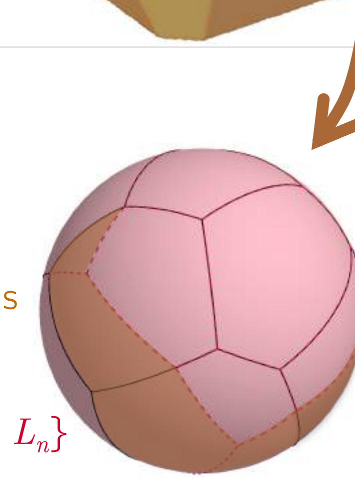
Suppose that the given tessellation $\mathcal{T}$ is not SLVD.
We will find the SLVD that approximates the tessellation $\mathcal{T}$.



From Algorithms, the polyhedron can be constructed

but the SLVD will not coincide with the given tessellation.

The difference between two tessellations occurs.



Discrepancy

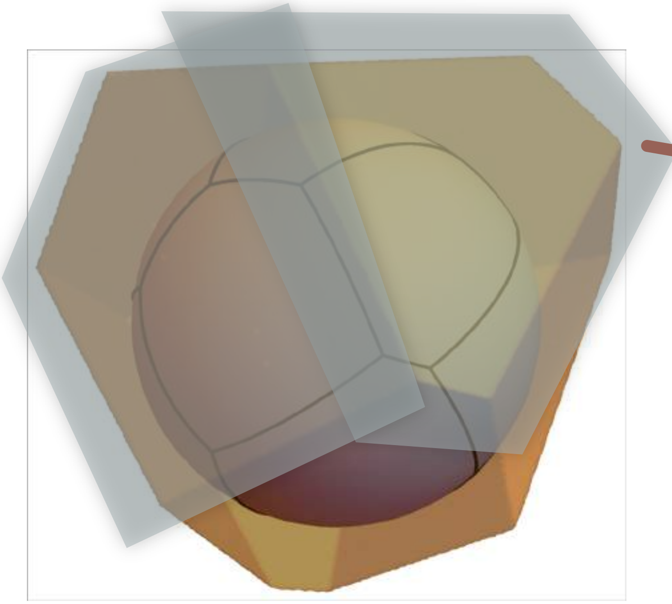
The ratio between difference area and total area

$$\Delta_{\mathcal{T},\mathcal{L}} = 1 - \frac{1}{4\pi}\sum_{i=1}^{n}\left(\sum_{j=1}^{m_i}\alpha_{i,j} - (m_i - 2)\pi\right)$$

To find the best fit SLVD, we
minimize the discrepancy

Tessellation Fitting

SLVD ⟷ polyhedron ⟷ halfspaces ⟷ planes



To decrease the discrepancy, we adjust the SLVD.

This implies that we adjust the planes.

Plane equation $P_i : A_i x + B_i y + C_i z = 1$

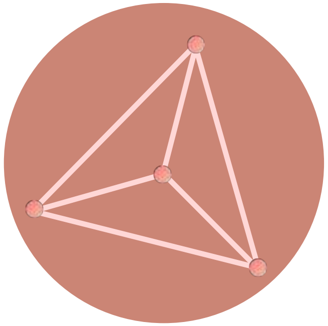
The discrepancy depends on plane parameters A_i, B_i, C_i

For n tessellation cells, define the discrepancy as a function of $\mathbf{x} = (A_1, \ldots, A_n, B_1, \ldots, B_n, C_1, \ldots, C_n)$ by

$$\text{minimize} \quad D(\mathbf{x}) := \Delta_{\mathcal{T},\, \mathcal{L}}.$$

$D(\mathbf{x})$

Discrepancy function value computed pointwisely

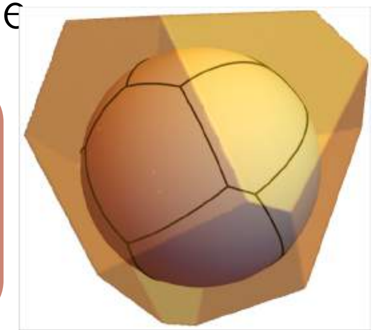


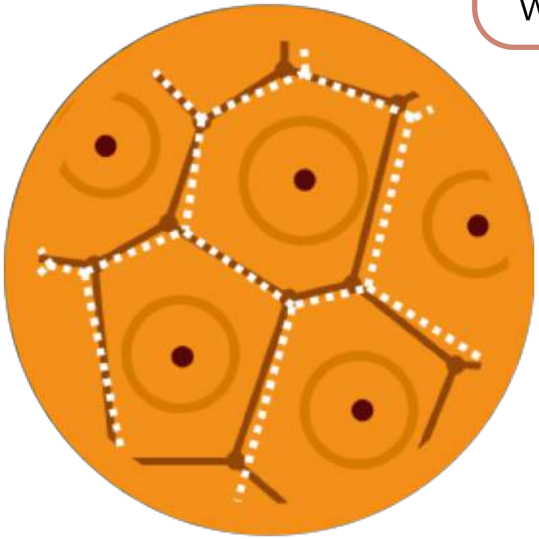
Nelder-Mead Method

for finding the local minimum

Interpretation of SLVD

To interpret the meaning of fitted Voronoi diagram, the following goals are preferable





We use four degrees of freedom Theorem for adjusting the polyhedron satisfying the real-world desired properties.

Each generator should be close to the center of the cell.

- Find the satisfied polyhedron

The generators should lay inside the cell as much as possible.

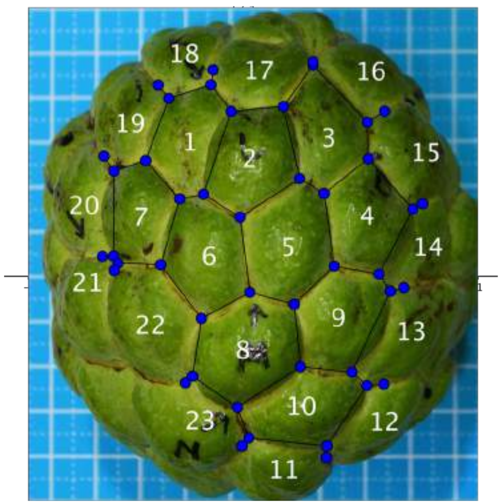
- Expected result from the first goal

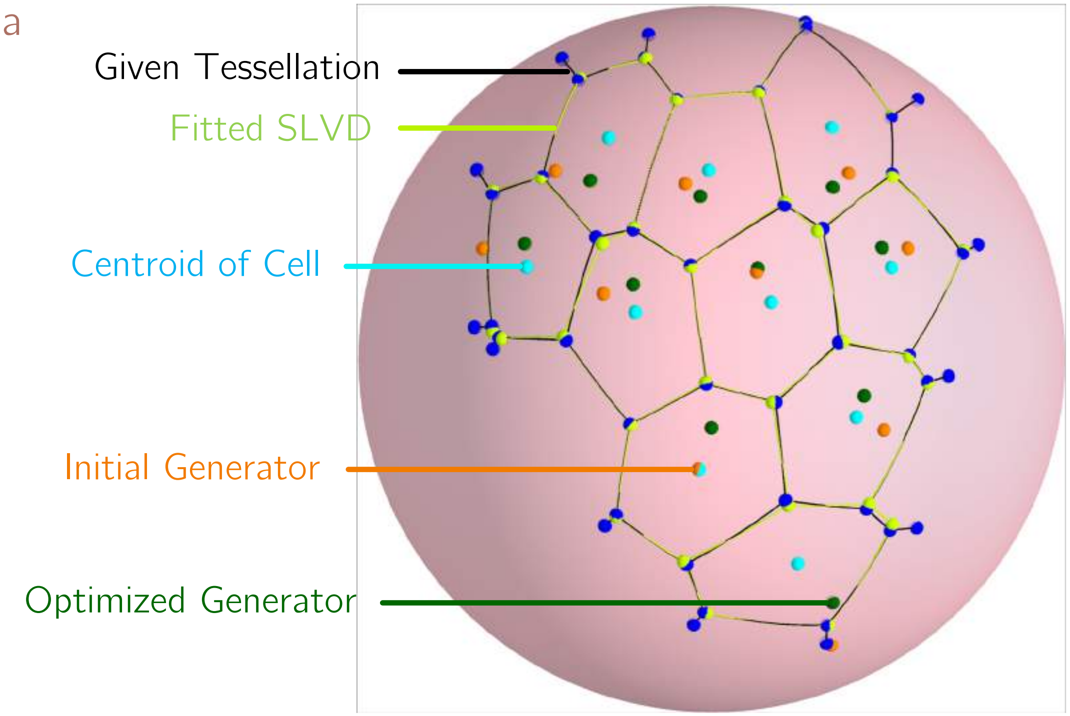
The radii of spherical circles should be a non-negative number.

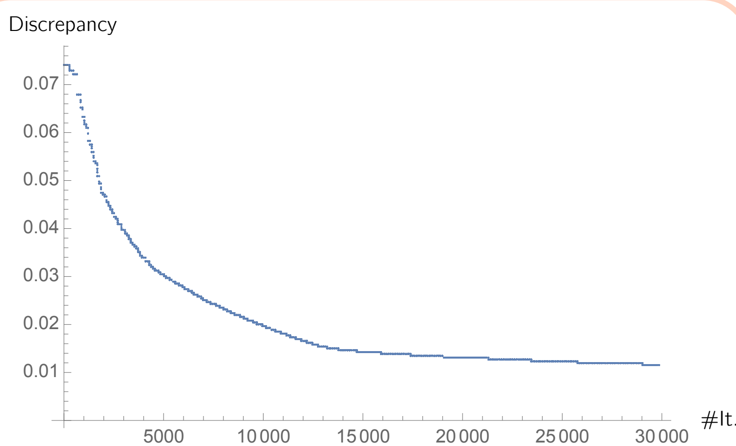
- Shrink the polyhedron until all weights are non-negative

Experimental Results

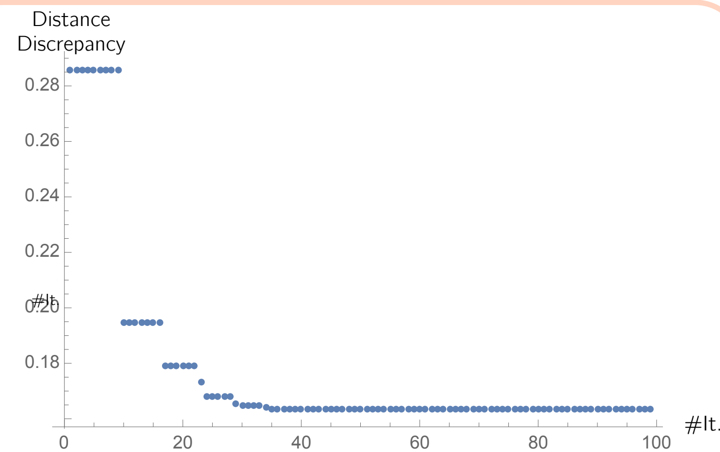
Experiments with real data



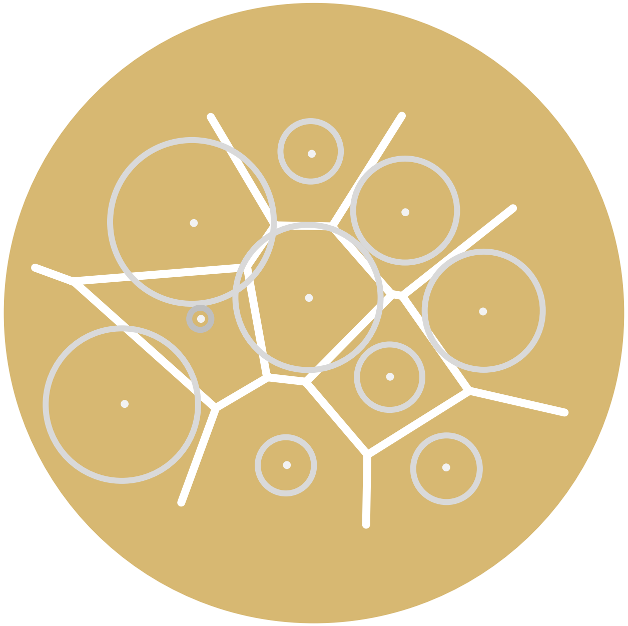




Discrepancy



Distance Discrepancy



Modeling using
Spherical Laguerre Voronoi Diagram

S. Chaidee, K. Sugihara, (2019) Graphs and Combinatorics
Laguerre Voronoi Diagram as a Model for Generating the Tessellation Patterns on the Sphere

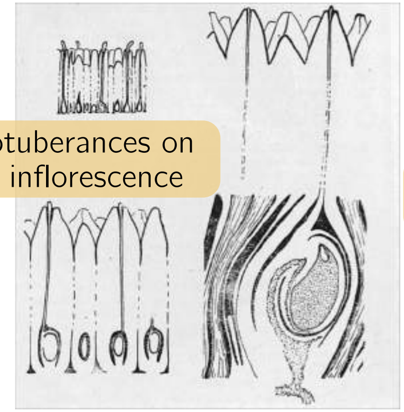
Characteristics of Real-World Patterns

Jackfruit
Multiple fruit

Lychee
Single fruit

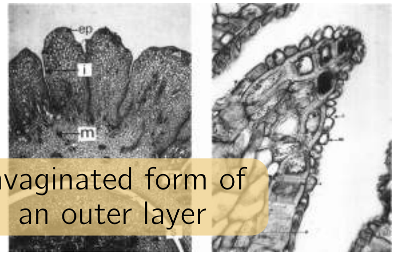
Raspberry
Aggregate fruit

Sugar apple
Aggregate fruit



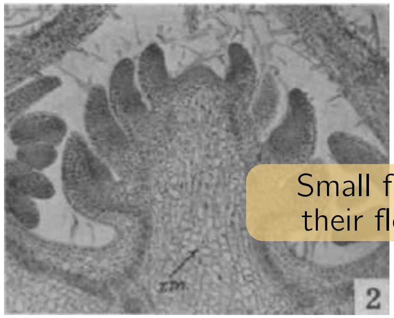
Protuberances on an inflorescence

Figure from [21]



Invaginated form of an outer layer

Figure from [94]



Small fruits attach on their flower structure

Figure from [56]

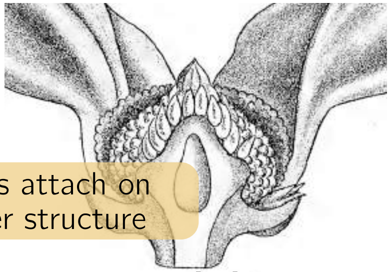
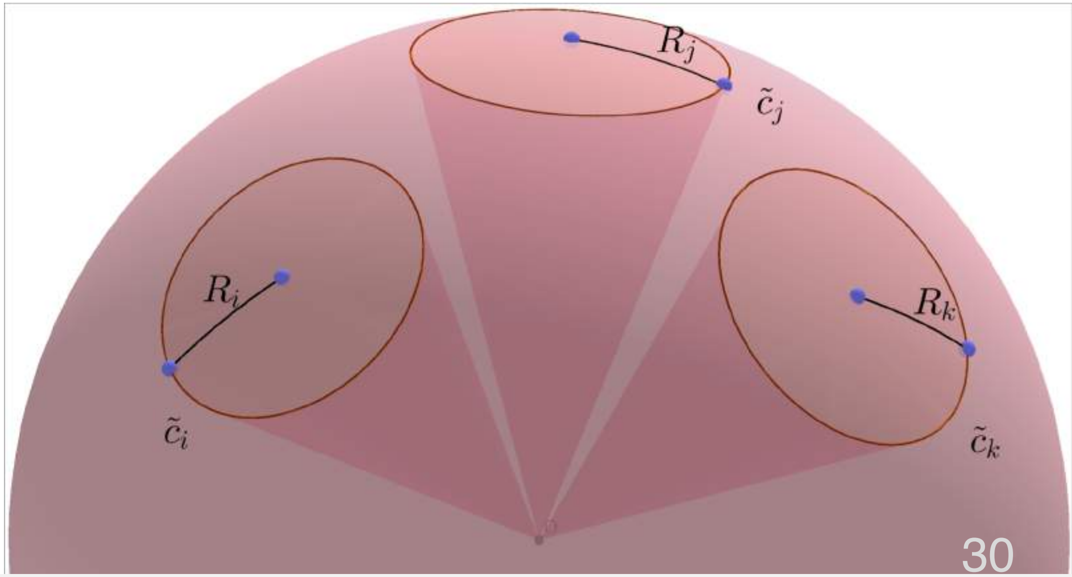


Figure from [43]

- There are microstructures attached on the large object.
- In our model, assume that each unit displays as a spherical cone whose base is a spherical circle.
- Microstructures are attached on the unit sphere.



Modeling Assumptions

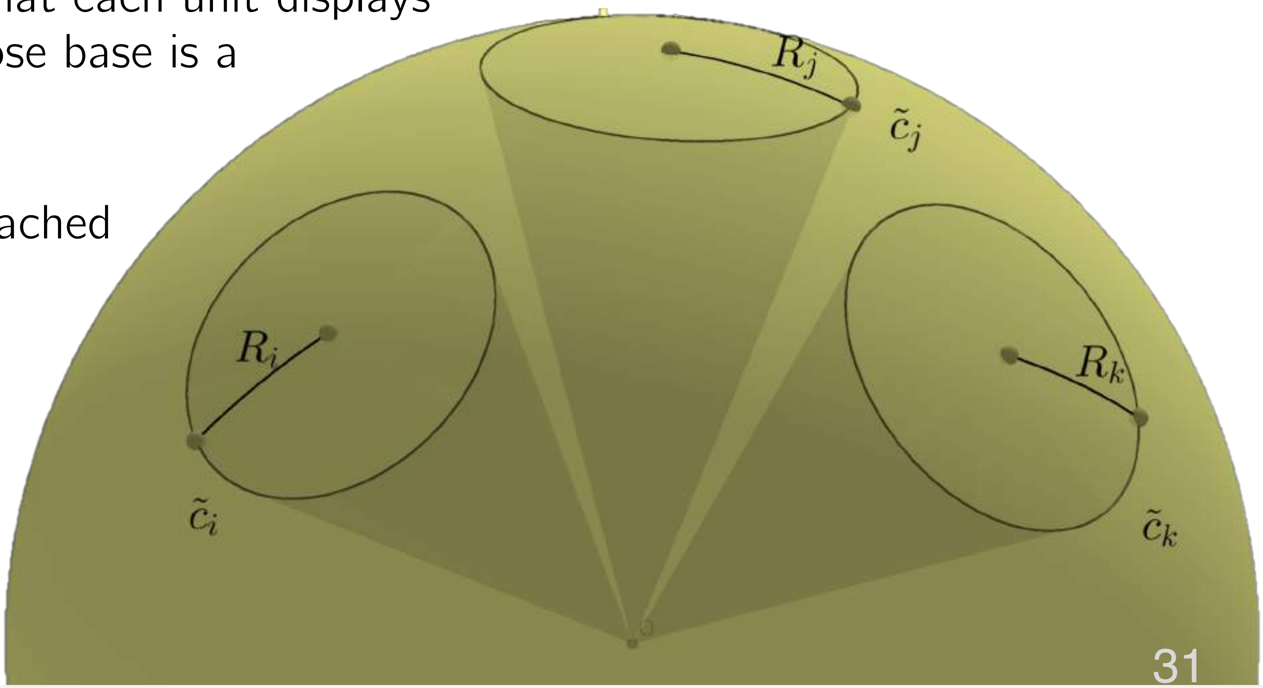
For a unit sphere U, let $\mathcal{G}(t) = \{\tilde{c}_1(t), \ldots, \tilde{c}_n(t)\}$ be a set of spherical circles at time t such that

$$\tilde{c}_i(t) = \{p \in U : \tilde{d}(p_i(t), p(t)) = R_i(t)\}.$$

Nondecreasing bounded function

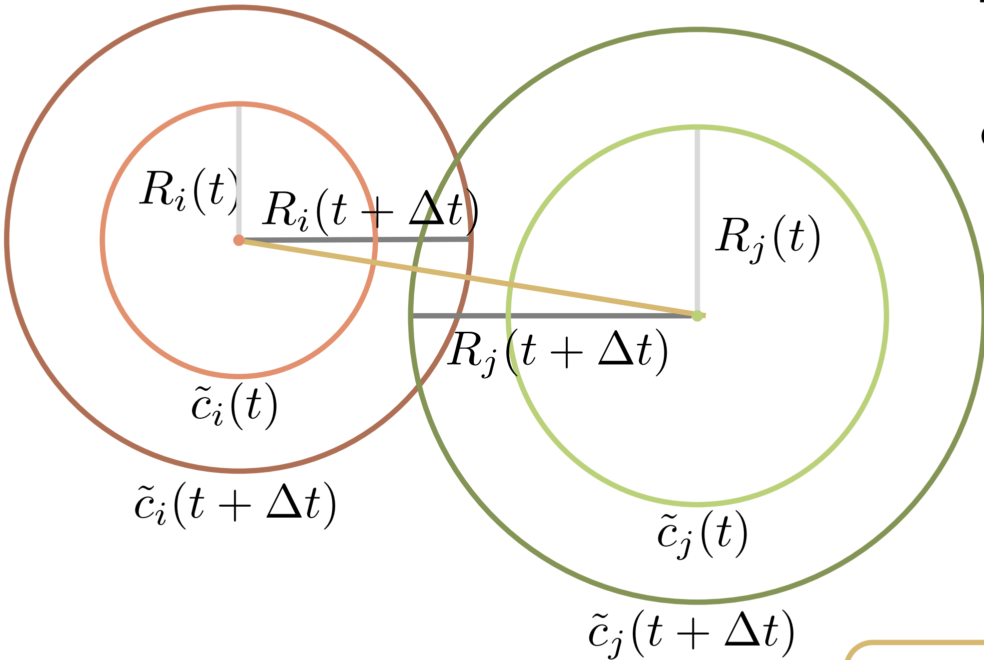
such that $0 < R_i(t) < \pi/2$, and $p_i(t)$ is the spherical circle center at time t.

- There are microstructures attached on the large object.
- In our model, assume that each unit displays as a spherical cone whose base is a spherical circle.
- Microstructures are attached on the unit sphere.



Generator Pushing Model

At time t, for each corresponding pair i, j in the spherical Laguerre Delaunay edge of time $t-1$, we consider the dynamical movement of generators.



Define the energy function

$$\Delta E = 0$$

or $\Delta E = (R_i(t) + R_j(t)) - \tilde{d}(p_i(t), p_j(t))$

$$E(\theta_1^t, \phi_1^t, ..., \theta_n^t, \phi_n^t) = \sum_{i,j} (\Delta E)^2$$

After two circles touch each other, the circles are moved.

In real world situation, the generating centers are not moved so much.

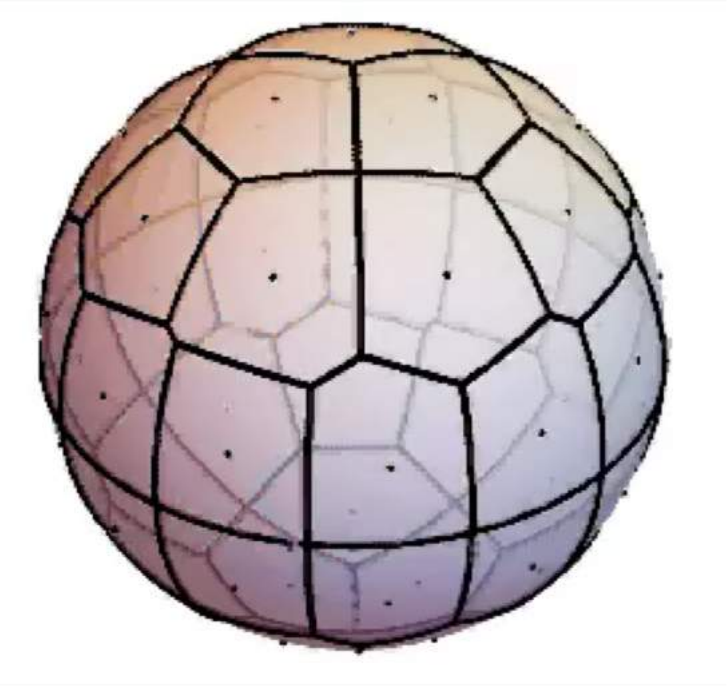
$$F(\theta_1^t, \phi_1^t, ..., \theta_n^t, \phi_n^t) = \sum_{i=1}^{n} (\tilde{d}(p_i(t-1), p(t)))^2$$

We solve the optimization problem $\min\{E(\theta_1^t, \phi_1^t, ..., \theta_n^t, \phi_n^t), F(\theta_1^t, \phi_1^t, ..., \theta_n^t, \phi_n^t)\}$

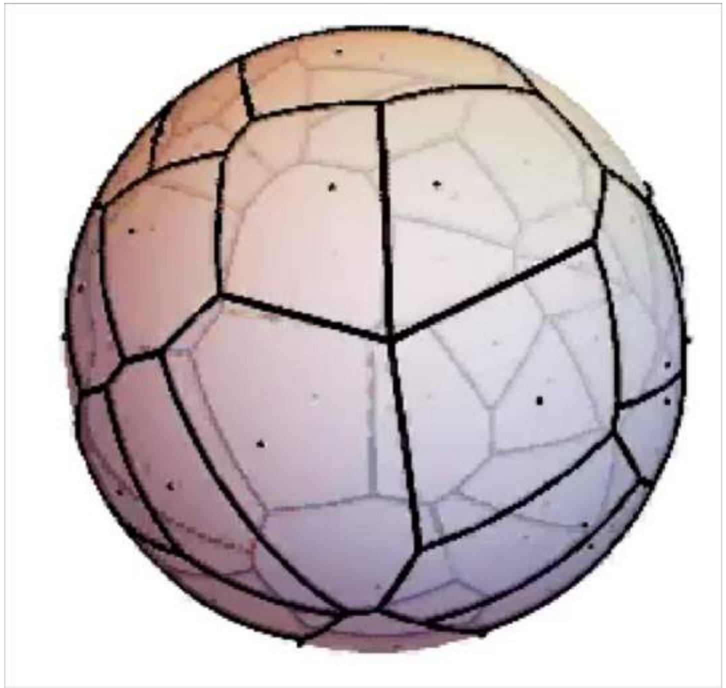
Simulation

We generate the patterns using the following parameters.

$n = 50,\ \omega = 1/8,\ \epsilon = 10^{-8},\ L_i \in [\arccos(1 - \frac{1}{n}) - \frac{\pi}{36}, \arccos(1 - \frac{1}{n}) + \frac{\pi}{36}]$
$k = 0.2,\ t_0 = 15$



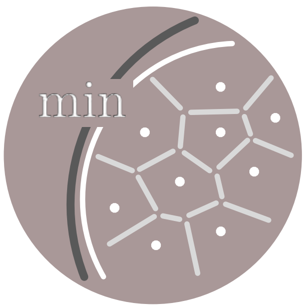
Equidistributed Points



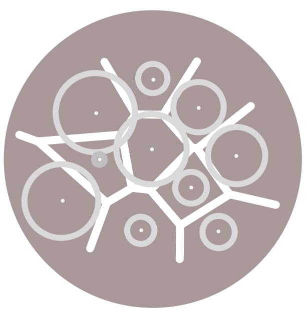
Random Points

Concluding Remarks & Future Works

The properties of SLVD based on the polyhedron help us to solve the recognition and approximation problem.

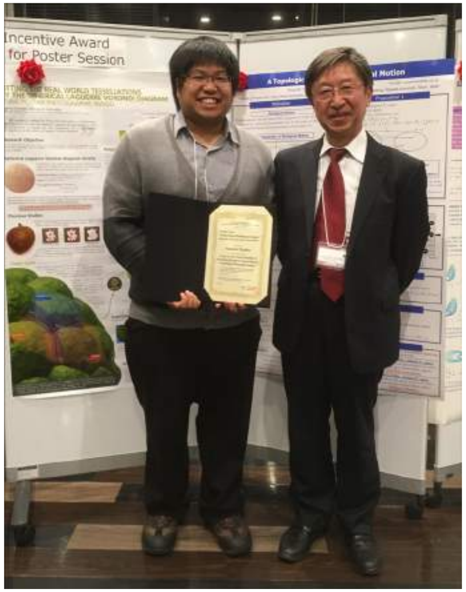


We **proposed** the models corresponding to the biological information for generating the tessellation pattern on the sphere using SLVD.



The properties of SLVD based on polyhedra may allow us to define the new kind of the Voronoi diagram.

Acknowledgements



Prof. Kokichi Sugihara
Main Supervisor

明治大学　先端数理科学研究科
Graduate School of Advanced Mathematical Sciences, Meiji University

明治大学先端数理科学インスティテュート
Meiji Institute for Advanced Study of Mathematical Sciences (MIMS)

The Development and Promotion of Science and Technology Talents Project (DPST)
under the Institute for the Promotion of Teaching Science and Technology (IPST), Ministry of Education, Thailand

Faculty of Science
Chiang Mai University, Thailand

Q&A

ありがとうございました！

謝謝 Thank You for Your Kind Attention

ポリヒードロン君

Additional information at http://www.schaidee.com/publication