The spectral radius of edge chromatic critical graphs

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Abstract

A connected graph G with maximum degree Δ and edge chromatic number $\chi'(G) = \Delta + 1$ is called Δ-critical if $\chi'(G - e) = \Delta$ for every edge e of G. In this paper, we consider two weaker versions of Vizing's conjecture, which concern the spectral radius $\rho(G)$ and the signless Laplacian spectral radius $\mu(G)$ of G. We obtain some lower bounds for $\rho(G)$ and $\mu(G)$, and present some cases where the conjectures are true. Finally, several open problems are also proposed.

Key Words: Critical graphs; Spectral radius; Vizing's Conjecture
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1 Introduction

We consider simple connected graphs in this paper. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$, edge set $E(G)$, with $|V(G)| = n$, $|E(G)| = m$. For a vertex x, we set $N(x) = \{v : xv \in E(G)\}$ and $d(x) = d_G(x) = |N(x)|$, the degree of x in G. The maximum and minimum degrees of G are denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$, respectively. A vertex of maximum degree in G is called a *major vertex*. We use $d_\Delta(x)$ to denote the number of major vertices of G adjacent to x.

The *adjacency matrix* of a graph G is $A(G) = (a_{ij})$, where $a_{ij} = 1$ if two vertices i and j are adjacent in G, and $a_{ij} = 0$ otherwise. Let $D(G) = (d_{ij})$ be the diagonal degree matrix of G, i.e., d_{ii} is the degree of the vertex i in G, and $d_{ij} = 0$ otherwise. We call the matrix $L(G) = D(G) - A(G)$ the *Laplacian matrix* of G, and the matrix $Q(G) = D(G) + A(G)$ the *signless Laplacian matrix* or Q-*matrix* of G. We denote the largest eigenvalues of $A(G)$ and $Q(G)$ by $\rho(G)$ and $\mu(G)$, respectively, and call them *the spectral radius* and *the signless Laplacian spectral radius* (or the Q-*spectral*

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radius) of G, respectively. For background on the matrices $A(G)$ and $Q(G)$ of G, the reader is referred to [6, 7] and the references therein.

Brualdi and Solheid [1] proposed the following problem concerning the spectral radius of graphs: *Given a set $\mathcal{G}$ of graphs, find an upper bound for the spectral radius over all graphs of $\mathcal{G}$, and characterize the graphs in which the supremum spectral radius is attained.* Inspired by this problem, the eigenvalues of special classes of graphs are well studied in the literature, such as graphs with given chromatic number [9], matching number [10], diameter [11], and domination number [28]. One can refer to a recent, comprehensive book by Stevanović [29] for more details. Also, the theory of eigenvalues of graphs has found successful applications in other disciplines such as chemistry and biology, and a typical and widely studied invariant is the energy of graphs, for which one may refer to [16, 17, 18, 19] and the references therein.

A *k-edge-coloring* of a graph G is a function $\phi : E(G) \to \{1, \dots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. That is, if we consider $\{1, \dots, k\}$ as a set of k colors, then any two adjacent edges receive different colors. The *edge chromatic number* of G, denoted by $\chi'(G)$, is the smallest integer k such that G has a k-edge-coloring. The celebrated Vizing's Theorem [31] states that $\chi'(G)$ is either Δ or $\Delta + 1$. A graph G is *class one* if $\chi'(G) = \Delta$ and *class two* if $\chi'(G) = \Delta + 1$. A class two graph G is Δ-*critical* if $\chi'(G - e) = \Delta$ for each edge e of G and it has maximum degree Δ. We simply say that G is *critical* if we do not wish to refer to its maximum degree Δ.

The theory of edge-colorings in graphs is one of the most fundamental areas in graph theory, and often appears in various scheduling problems like the file transfer problem on computer networks. The main tools in previous research of edge-coloring problems include Vizing's Adjacency Lemma, the discharging method, the Vizing Fans, the Kierstead Paths and the Tashkinov Trees. One may refer to the monograph [30] for details. Thus according to the Brualdi-Solheid problem, it would be interesting to consider the spectral properties of the class two graphs, specifically, the edge chromatic critical graphs.

Vizing [31] proposed the following conjecture on the size of critical graphs.

Conjecture 1.1 *If $G = (V, E)$ is a Δ-critical graph, then*

$$m \geq \frac{1}{2} \left(n(\Delta - 1) + 3 \right). \tag{1}$$

The intriguing Conjecture 1.1 has attracted much attention. It is known to be true for $\Delta \leq 6$, but a complete solution to the conjecture itself seems far from being found. Some best known lower bounds on size of critical graphs can be found in [21, 24, 33]. One may refer to [30, 35] for detailed discussions.

It is well known in spectral graph theory that

$$\rho(G) \geq \frac{2m}{n}, \qquad \mu(G) \geq \frac{4m}{n}, \tag{2}$$

with equality if and only if G is regular. While Conjecture 1.1 remains unsolved, motivated by the inequalities in (2), we consider the following spectral versions of the conjecture.

Conjecture 1.2 *If $G = (V, E)$ is a Δ-critical graph, then*

$$\rho(G) \geq \Delta - 1 + \frac{3}{n}. \tag{3}$$

Conjecture 1.3 *If $G = (V, E)$ is a Δ-critical graph, then*

$$\mu(G) \geq 2\Delta - 2 + \frac{6}{n}. \tag{4}$$

Thus, Conjectures 1.2 and 1.3 allow us to provide another view to Conjecture 1.1. These two conjectures seem to be difficult as well. It was proved in [8] that, for a connected graph G,

$$\mu(G) \geq 2\rho(G), \tag{5}$$

and equality holds if and only if G is regular. Hence Conjecture 1.2 implies Conjecture 1.3. Using known bounds on the average degree of graphs (see, for example [34]) and inequalities in (2), we have

$$\rho > \frac{8}{3} \text{ if } \Delta = 3, \quad \rho > \frac{24}{7} \text{ if } \Delta = 4, \quad \rho > \frac{30}{7} \text{ if } \Delta = 5, \quad \rho > \frac{66}{13} \text{ if } \Delta = 6.$$

Thus, roughly, if $n \geq 40$, Conjectures 1.2 and 1.3 hold for $\Delta \leq 6$.

In general, Fiorini's bound on size (see, for example, [35]) implies

$$\rho \geq \frac{\Delta + 1}{2} \text{ (if } \Delta \text{ odd)}, \qquad \rho \geq \frac{\Delta + 2}{2} \text{ (if } \Delta \text{ even)}.$$

Sanders and Zhao's bound on size [24] implies

$$\rho \geq \frac{1}{2}(\Delta + \sqrt{2\Delta - 1}) \text{ for all } \Delta \geq 2.$$

Woodall's bound on size [33] implies

$$\rho \geq \frac{2}{3}(\Delta + 1) \text{ for all } \Delta \geq 2.$$

For $\Delta \geq 3$, any Δ-critical graph is irregular [35, Page 38] and contains no cut vertices [35, Page 22]. The upper bounds for the (signless Laplacian) spectral radius of irregular graphs are extensively studied, and one may refer to [2, 5, 20, 22, 25] and [28, Lemma 4].

In this paper, inspired by Conjectures 1.2 and 1.3, we will study the lower bounds for the spectral radius of Δ-critical graphs. Although we cannot prove Conjectures 1.2 and 1.3 completely, we shall nonetheless obtain better ratios for $\frac{\rho}{\Delta}$ and $\frac{\mu}{\Delta}$ than other papers have for $\frac{2m}{n\Delta}$ and $\frac{4m}{n\Delta}$.

2 Two Examples

Although Conjectures 1.2 and 1.3 seem hard to tackle, many examples (especially those critical graphs of small order) and related results reveal that these two conjectures may be true.

Example 2.1 *Chetwynd, Hilton [3] proved that if G is obtained from K_n by removing any $\frac{n-3}{2}$ edges, for odd $n \geq 5$, then G is $(n-1)$-critical. Hence in this case, $\Delta = n-1$, and $m = \frac{1}{2}(n^2-2n+3) = \frac{1}{2}(n(\Delta-1)+3)$. We have*

$$\frac{2m}{n} = \Delta - 1 + \frac{3}{n}.$$

From the inequalities in (2), we immediately have that Conjectures 1.2 and 1.3 hold for G.

Suppose that $x \in V(G)$ with $d(x) > 2$. We say that the graph H is obtained from G by *splitting* x into two vertices u and v ($u, v \notin V(G)$) if

$$\begin{aligned} V(H) &= V(G-x) \cup \{u, v\}, \\ E(H) &= E(G-x) \cup \{uv\} \cup \{uy \mid y \in A\} \cup \{vz \mid z \in B\}, \end{aligned}$$

for some non-empty, disjoint sets A and B with $A \cup B = N(x)$.

Let $G = O_r^t$ be the complete t-partite graph having r vertices in each class.

Example 2.2 *Yap [35, Theorem 4.4, Page 28] obtained the following: Let r and t be two positive integers such that $rt \geq 4$ is even. Let H be a graph obtained from $G = O_r^t$ by splitting a vertex x into two vertices u and v. Then H is critical.*

We can easily see that $\Delta(H) = tr - r$, $|V(H)| = |V(G)| + 1 = tr + 1$, $|E(H)| = |E(G)| + 1 = \frac{1}{2}tr(tr-r) + 1$. Thus, we have

$$\frac{2|E(H)|}{|V(H)|} = \frac{tr(tr-r)+2}{tr+1} > (tr-r) - 1 + \frac{3}{tr+1} = \Delta(H) - 1 + \frac{3}{|V(H)|}.$$

From the inequalities in (2), Conjectures 1.2 and 1.3 hold for H.

3 Lower Bounds for $\rho(G)$ and $\mu(G)$

In this section, we present two lower bounds for $\rho(G)$ and $\mu(G)$. As Conjectures 1.2 and 1.3 hold for $\Delta \leq 6$, our results will be new contributions to the conjectures for $\Delta \geq 7$.

Lemma 3.1 *[32] (Vizing's Adjacency Lemma) Suppose G is a Δ-critical graph and $vw \in E(G)$. Then $d_\Delta(v) \geq \Delta - d(w) + 1$.*

Lemma 3.2 *[15] Let $M = (m_{ij})$ be an $n \times n$ irreducible non-negative matrix with spectral radius (i.e., largest eigenvalue) $\rho(M)$, and let $s_i(M)$ be the ith row sum of M, i.e., $s_i(M) = \sum_{j=1}^{n} m_{ij}$. Let P be any polynomial. Then*

$$\min\{s_i(P(M)) \mid 1 \le i \le n\} \le P(\rho(M)) \le \max\{s_i(P(M)) \mid 1 \le i \le n\}.$$

Moreover, if the row sums of M are not all equal, then both of the inequalities above are strict.

Now we are ready to present the main results of this paper. We first consider the adjacency spectral radius.

Theorem 3.3 *Let G be a Δ-critical graph with $\Delta \ge 7$. Then*

$$\rho(G) \ge \frac{1}{8}\Big(3\Delta + 1 + \sqrt{(3\Delta+1)^2 + 16(\Delta-1)}\Big).$$

Proof. For a vertex v in $V(G)$, let w be a vertex adjacent to v with minimum degree, say b, among all neighbors of v. Then by Lemma 3.1 we have $d_\Delta(v) \ge \Delta - d(w) + 1 = \Delta - b + 1$ and thus

$$b \ge \Delta - d_\Delta(v) + 1.$$

Writing $A = A(G)$, it follows that

$$\begin{aligned}
s_v(A^2) &= \sum_{u \in N(v)} d(u) \\
&\ge d_\Delta(v)\Delta + (d(v) - d_\Delta(v))b \\
&\ge d_\Delta(v)\Delta + (d(v) - d_\Delta(v))(\Delta - d_\Delta(v) + 1) \\
&= d_\Delta(v)^2 - (d(v)+1)d_\Delta(v) + (\Delta+1)d(v) \\
&= \left(d_\Delta(v) - \frac{1}{2}(d(v)+1)\right)^2 + \frac{1}{4}\left(4\Delta d(v) + 2d(v) - d(v)^2 - 1\right) \\
&\ge \frac{1}{4}\left(4\Delta d(v) - (d(v)-1)^2\right) \qquad (6) \\
&\ge \frac{1}{4}\left(3\Delta d(v) + d(v) + \Delta - 1\right),
\end{aligned}$$

where the last inequality follows from $(d(v)-1)^2 \le (\Delta-1)(d(v)-1)$. Since $d(v) = s_v(A)$, it follows that

$$s_v(A^2) \ge \frac{1}{4}(3\Delta+1)s_v(A) + \frac{1}{4}(\Delta-1).$$

By employing Lemma 3.2, noting that $\rho(A) = \rho(G)$, we have

$$\rho(G)^2 - \frac{1}{4}(3\Delta+1)\rho(G) - \frac{1}{4}(\Delta-1) \ge 0,$$

and hence

$$\rho(G) \geq \frac{1}{8}\Big(3\Delta + 1 + \sqrt{(3\Delta+1)^2 + 16(\Delta - 1)}\Big).$$

This implies the result. ■

It is easy to see that

$$\frac{1}{8}\Big(3\Delta + 1 + \sqrt{(3\Delta+1)^2 + 16(\Delta - 1)}\Big) \geq \frac{3\Delta + 2}{4}.$$

Therefore, one obtains the following concise bound

Corollary 3.4 *Let G be a Δ-critical graph with maximum degree $\Delta \geq 7$. Then*

$$\rho(G) \geq \frac{3\Delta + 2}{4}.$$

Next, we consider the signless Laplacian spectral radius. For a Δ-critical graph G, in view of Theorem 3.3 and Corollary 3.4, one has

$$\mu(G) \geq \frac{1}{4}\Big(3\Delta + 1 + \sqrt{(3\Delta+1)^2 + 16(\Delta - 1)}\Big) \geq \frac{3}{2}\Delta + 1. \tag{7}$$

The next theorem provides another lower bound for the signless Laplacian spectral radius of Δ-critical graphs.

Theorem 3.5 *Let G be a Δ-critical graph with maximum degree $\Delta \geq 7$, and $\delta \geq 2$ be the minimum degree of G. Then*

$$\mu(G) \geq \frac{1}{8}\Big(4\Delta + 3\delta + 2 + \sqrt{(4\Delta + 3\delta + 2)^2 - 32}\Big).$$

Proof. Write $Q = Q(G)$, $D = D(G)$ and $A = A(G)$. For a vertex $v \in V(G)$, we have $s_v(Q) = 2d(v)$, $s_v(D^2) = s_v(DA) = d(v)^2$ and $s_v(AD) = s_v(A^2) = \sum_{u \in N(v)} d(u)$. Then

$$\begin{aligned}
s_v(Q^2) &= s_v(D^2 + DA + AD + A^2) \\
&= s_v(D^2) + s_v(DA) + s_v(AD) + s_v(A^2) \\
&\geq 2d(v)^2 + \frac{1}{2}\left(4\Delta d(v) - (d(v)-1)^2\right) \qquad (8) \\
&= \frac{3}{2}d(v)^2 + (2\Delta + 1)d(v) - \frac{1}{2} \\
&= \frac{1}{4}\left(3d(v) + 4\Delta + 2\right) s_v(Q) - \frac{1}{2} \\
&\geq \frac{1}{4}\left(4\Delta + 3\delta + 2\right) s_v(Q) - \frac{1}{2},
\end{aligned}$$

where (8) follows in the same way as (6) in Theorem 3.3. By Lemma 3.2, noting that $\rho(Q) = \mu(G)$, we have

$$\mu(G)^2 - \frac{1}{4}(4\Delta + 3\delta + 2)\,\mu(G) + \frac{1}{2} \geq 0,$$

and hence

$$\mu(G) \geq \frac{1}{8}\Big(4\Delta + 3\delta + 2 + \sqrt{(4\Delta + 3\delta + 2)^2 - 32}\Big).$$

This implies the result. ■

It is easy to see from Theorem 3.5 that

$$\mu(G) \geq \frac{1}{8}\Big(4\Delta + 3\delta + 2 + \sqrt{(4\Delta + 3\delta + 2)^2 - 32}\Big) \geq \Delta + \frac{3\delta}{4} + \frac{3}{8}. \tag{9}$$

Therefore, inequalities (7) and (9) yield the following simple bound for the signless Laplacian spectral radius of Δ-critical graphs.

Corollary 3.6 *Let G be a Δ-critical graph with $\Delta \geq 7$ and minimum degree $\delta \geq 2$. Then*

$$\mu(G) \geq \begin{cases} \Delta + \dfrac{3\delta}{4} + \dfrac{3}{8} & \textit{if } \delta \geq \dfrac{4\Delta + 5}{6}, \\ \dfrac{3}{2}\Delta + 1 & \textit{if } \delta < \dfrac{4\Delta + 5}{6}. \end{cases}$$

4 Miscellaneous Results

In this section, we present some results for the spectral radius of Δ-critical graphs with a small number of major vertices. The *core* of G, denoted by G_Δ, is the subgraph of G induced by the vertices of degree $\Delta(G)$.

Lemma 4.1 *[12] Let G be a connected graph of Class 2 and $\Delta(G_\Delta) \leq 2$. Then the following statements hold:*

(i) G is critical;

(ii) $\delta(G_\Delta) = 2$;

(iii) $\delta(G) = \Delta - 1$, unless G is an odd cycle.

From Lemma 4.1, we can deduce the following.

Theorem 4.2 *Let G be a Δ-critical graph of order $n \geq 5$ with maximum degree $\Delta \geq 4$ and $\Delta(G_\Delta) \leq 2$. Then Conjectures 1.2 and 1.3 hold.*

Proof. For any connected graph G, from [13] or [36], we know $\rho(G) \geq \sqrt{\frac{1}{n}\sum d_i^2}$, where (d_i) is the degree sequence of G. Equality holds if and only if G is a regular graph or a semi-regular bipartite graph. As $\Delta \geq 4$, G is not a cycle. A consequence of Lemma 3.1 states that a Δ-critical graph G contains at least three major vertices [35, Page 24]. Together with Lemma 4.1, we have

$$\rho(G) \geq \sqrt{\frac{1}{n}\left(3\Delta^2 + (n-3)(\Delta-1)^2\right)} > \Delta - 1 + \frac{3}{n},$$

and Conjecture 1.2 holds. As $\mu(G) \geq 2\rho(G)$ from (5), we conclude that Conjecture 1.3 also holds. ■

Obviously, Theorem 4.2 contains the case for $|G_\Delta| = 3$, as any graph induced by three vertices has maximum degree at most 2.

It should be noted [4, 27] that, there does not exist any Δ-critical graph G of even order with $|G_\Delta| = 4$ or $|G_\Delta| = 5$. Thus we consider such graphs of odd order in these cases.

Theorem 4.3 *Let G be a Δ-critical graph of order $n = 2k+1$ with $|G_\Delta| = 4$ or $|G_\Delta| = 5$. Then Conjectures 1.2 and 1.3 hold.*

Proof. In [4] for $|G_\Delta| = 4$, and in [26] for $|G_\Delta| = 5$, it was shown that under the assumption, $m = k\Delta + 1 = \frac{1}{2}((n-1)\Delta + 2)$. From the inequalities in (2), we have

$$\rho(G) \geq \frac{2m}{n} = \Delta - \frac{\Delta}{n} + \frac{2}{n} \geq \Delta - 1 + \frac{3}{n},$$

and thus Conjecture 1.2 holds. Conjecture 1.3 can be proved similarly. ■

5 Further Discussions

We conclude with some remarks and open problems in this section. For a connected graph G of order n and size m, it was obtained in [14] that

$$\rho(G) \leq \sqrt{2m - n + 1}. \tag{10}$$

By considering Conjecture 1.2, we have the following.

Corollary 5.1 *Let G be a Δ-critical graph. If Conjecture 1.2 is true, then we have*

$$m \geq \frac{1}{2}\left(\Delta^2 - 2\Delta + n\right). \tag{11}$$

In [32], it was obtained that

$$m \geq \frac{1}{8}(3\Delta^2 + 6\Delta - 1). \tag{12}$$

However, inequality (11) is stronger when compared with (12), if Δ is not too large.

Also, from [33], it was obtained that

$$m \geq \frac{n}{3}(\Delta + 1). \tag{13}$$

We find that if $n \geq \frac{3\Delta^2-6\Delta}{2\Delta-1}$, then (13) is better than the bound (11). But if $n \leq \frac{3\Delta^2-6\Delta}{2\Delta-1}$, then (11) is better than (13).

Conjectures 1.2 and 1.3 seem to be difficult to deal with. Moreover, there are also many other problems in this topic.

For $\Delta = 2$, the 2-critical graphs are odd cycles, and the only extremal graph with $\rho = \Delta - 1 + \frac{3}{n}$ is C_3. For a Δ-critical graph G with $\Delta \geq 3$, it is irregular, thus from inequalities in (2), the equality case of Conjecture 1.2 does not hold. For $\Delta \geq 3$, we propose the following problem.

Problem 5.2 *Characterize the Δ-critical graphs of order n with $\Delta \geq 3$ and minimum spectral radius (or signless Laplacian spectral radius).*

From [35, Pages 45, 49], there are no critical graphs of order $n = 4, 6, 8, 10$. For $n = 5$, there are exactly three critical graphs [35, Page 45] with different maximum degrees. For $n = 7$, there are 21 critical graphs with $\Delta \geq 3$ [35, Page 48]. It is easy to find the eigenvalues in these two cases (see [23], Graphs 727–730, 1075–1076, 1100–1106, 1225–1226, 1231–1233, 1216, 1249, 1250). For general n, the same task seems to be much more difficult.

Hajós and Jakobsen provided a construction for Δ-critical graphs from some known Δ-critical graphs of smaller order. This construction is now usually called the *HJ-Construction*, and is carried out as follows (see, for example [35, Page 30, Theorem 4.6]). Let G and H be two Δ-critical graphs of order n_1 and n_2, respectively. Let K be a graph obtained from G and H by identifying $u \in V(G)$ and $v \in V(H)$ such that $d_G(u) + d_H(v) \leq \Delta + 2$, removing edges $uz \in E(G)$ and $vz' \in E(H)$ and joining the vertices z and z'. Then K is also Δ-critical.

Problem 5.3 *Suppose that Conjecture 1.2 holds for G and H, i.e., $\rho(G) \geq \Delta - 1 + \frac{1}{n_1}$ and $\rho(H) \geq \Delta - 1 + \frac{1}{n_2}$. Is it true that $\rho(K) \geq \Delta - 1 + \frac{1}{n_1+n_2-1}$? Similarly, what happens for the analogous situation for Conjecture 1.3, i.e., the signless Laplacian version of the problem?*

We leave the above problems for further research.

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