How to make a connected graph properly connected

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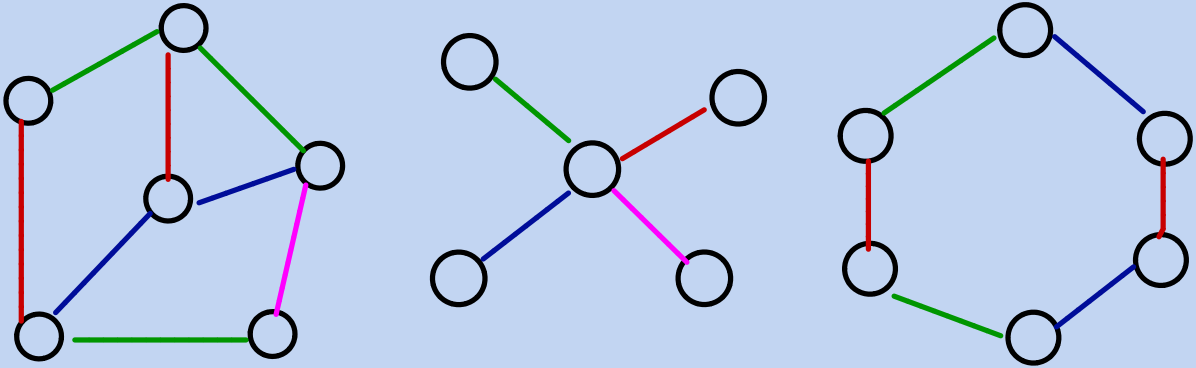


Definitions:

An edge-colored graph is properly connected

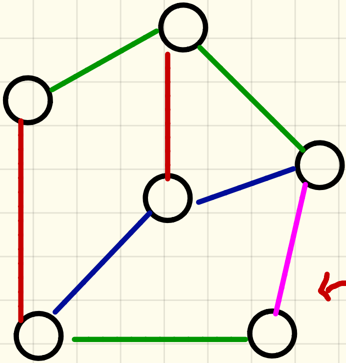
$\Leftrightarrow$ For any $u,v \in V(G)$, $\exists$ properly colored path joining them

$$\delta^c(G) := \min\{\, d^c(v) \mid v \in V(G) \,\}$$

color degree of v; i.e., the number of colors adjacent to v in G.

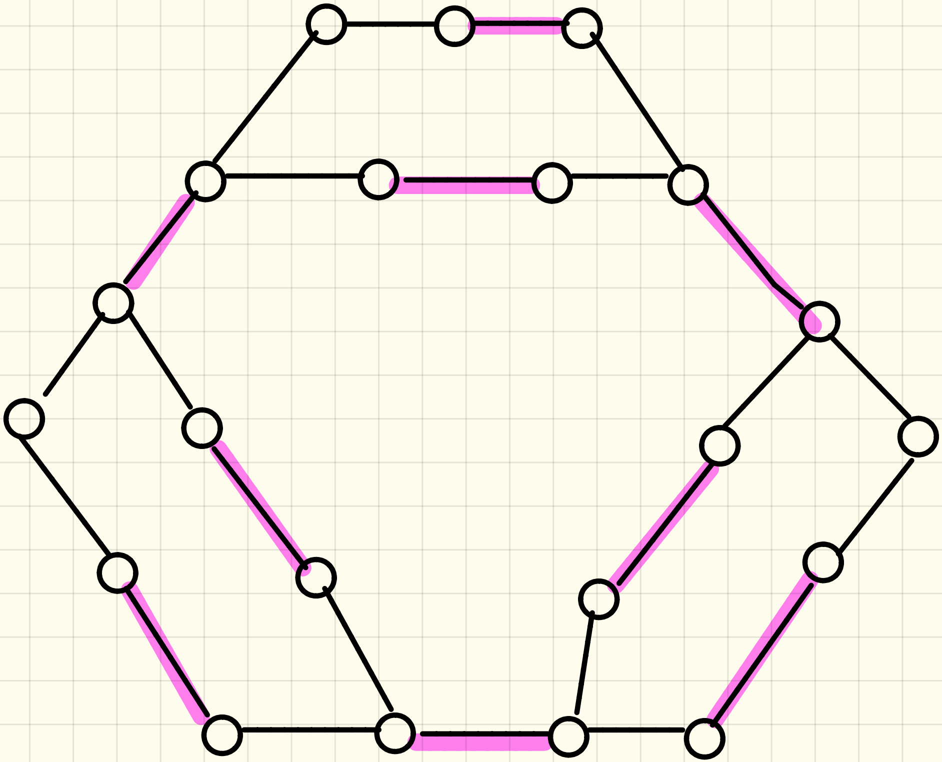
$$\boxed{\text{Note: } \delta(G) \geq \delta^c(G)}$$

Ex. G



$\delta^c(G) = 2$

Is this properly connected ??



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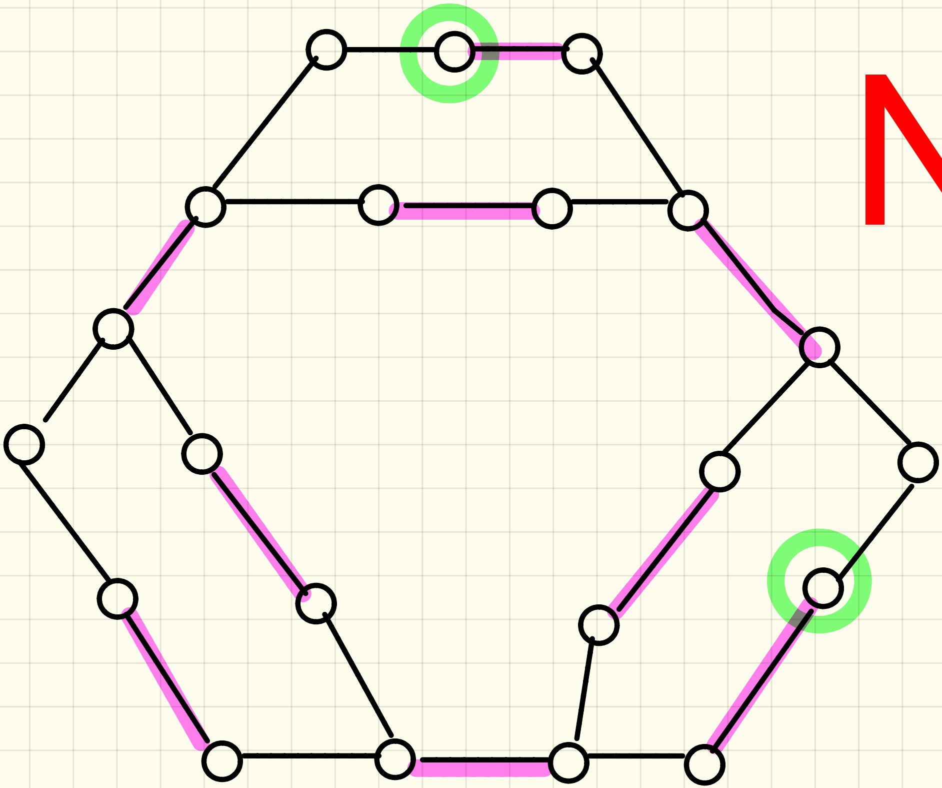
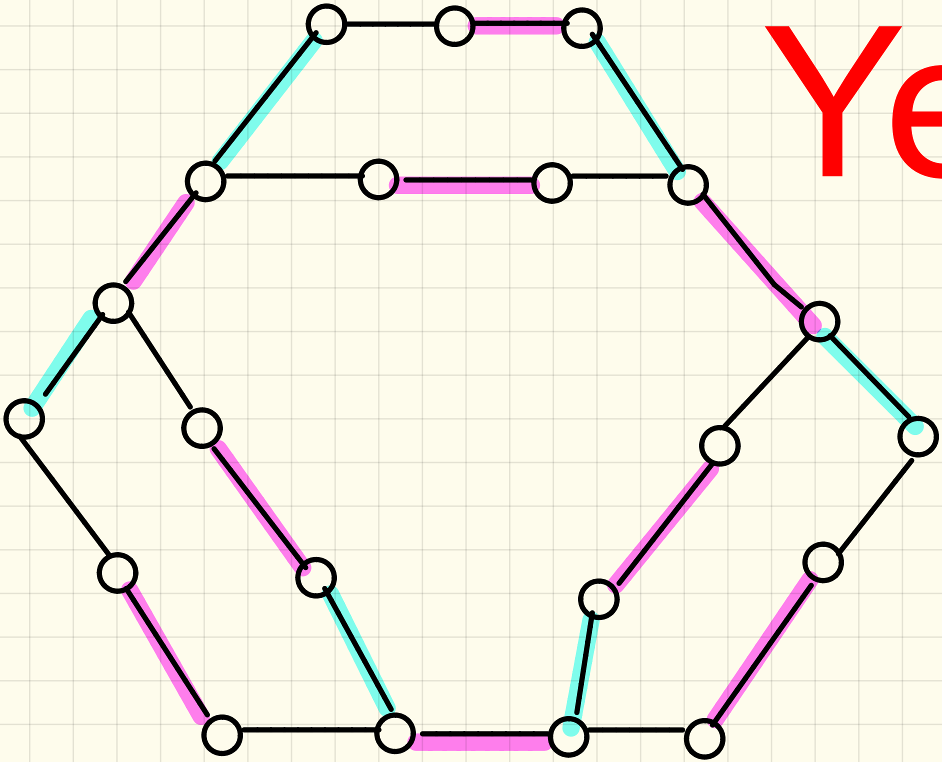


No!

Is this properly connected ??



Yes!

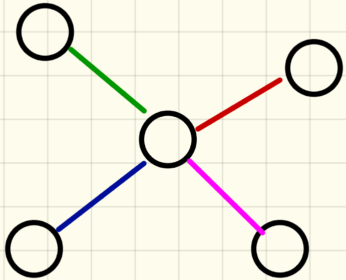
In this talk, we consider edge-colored connected graphs.

The proper connection number of G means

$pc(G) := \min\{ c \in \mathbb{N} :$ one can make G property connected by c colors$\}$

Note: If G is a tree, then

$pc(G) \geqslant \Delta(G)$



Known results:

Th. 1 (Magnant & F 2011)

If $\delta^c(G) \geq \frac{|V(G)|}{2}$ then G is properly connected.

Th. 2 (Borozan et al. 2012)

If G is 2-connected, then $pc(G) \leq 3$.

Th. 3 (Huang et al. 2017)

If G is 2-conn. and $diam(G) \leq 3$, then $pc(G) \leq 2$.

Th. 4 (Brause et al. 2017)

If G is 2-conn. and $\delta(G) > \max\{2, \frac{|G|+8}{20}\}$, then $pc(G) \leq 2$.

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I feel that the following is a challenging problem.

Pbm: Characterize 2-conn. graphs G

s.t. $pc(G) = 3$.

Th.5 (Brause et al. 2017)

For every $d \geq 3$, there exists a 2-conn. graph G

s.t. $\delta(G) = d$, $|G| = 42d$ and $pc(G) = 3$.

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In this talk, I would like to consider how to make a graph properly connected.

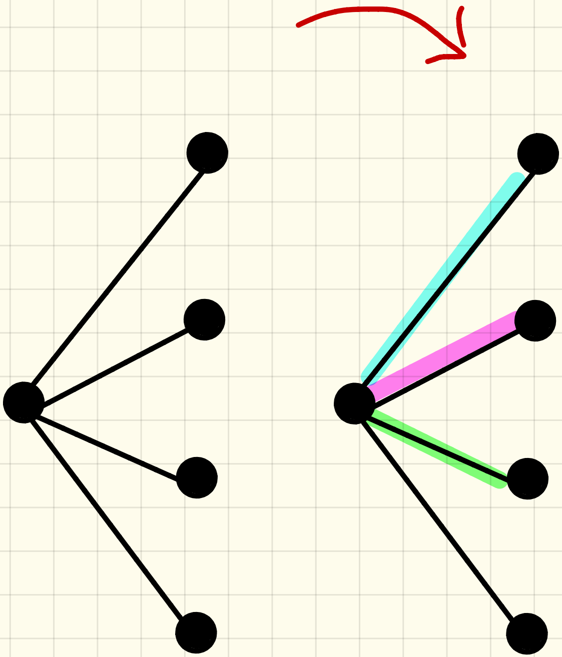
Let G be a monochromatic graph with color 1.

Define :

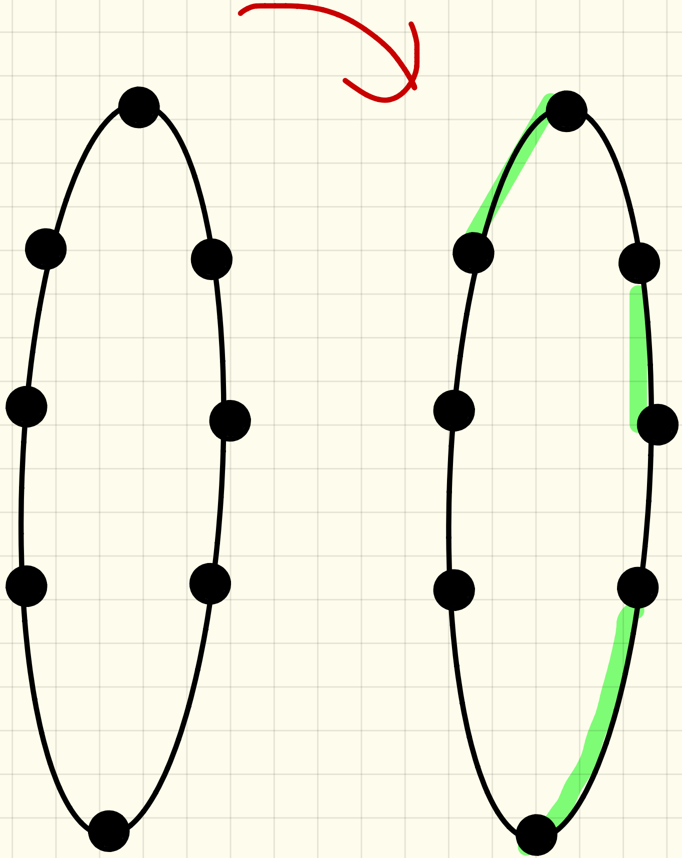
$$pc_{opt}(G) = \min\left\{ p+q \;\middle|\; \begin{array}{l} \text{We can make } G \text{ properly} \\ \text{connected by recoloring} \\ p \text{ edges with } q \text{ colors.} \end{array} \right\}$$

Easy to check :

$$pc_{opt}(K_{1,m}) = 2m-2 \,, \quad pc_{opt}(C_n) = \left\lfloor \frac{n-1}{2} \right\rfloor + 1$$

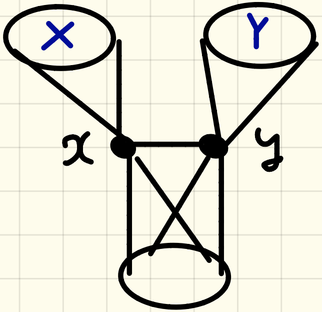


$PC_{opt}(K_{1,4}) = 3+3$



$PC_{opt}(C_8) = 3+1$

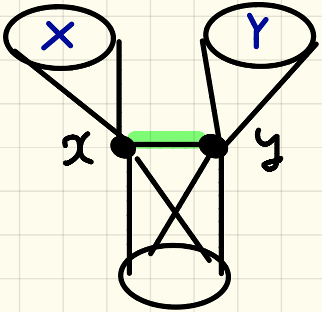
An edge $e=xy$ is good $\Leftrightarrow$ We can find a partition $V(G)-(N(x)\cap N(y))=X\,\dot\cup\,Y$ s.t. $X\subset N(x)$ and $Y\subset N(y)$, and $G[X]$ and $G[Y]$ are cliques.



Th. 6 (F)

A conn. graph G has $\mathrm{pc}_{opt}(G)\leq 2$ if and only if G contains a good edge.

An edge $e=xy$ is good $\Leftrightarrow$ We can find a partition $V(G)-(N(x)\cap N(y)) = X \dot\cup Y$ s.t. $X \subset N(x)$ and $Y \subset N(y)$, and $G[X]$ and $G[Y]$ are cliques.



Th. 6 (F)

A conn. graph G has $pc_{opt}(G) \leq 2$ if and only if G contains a good edge.

Th.7 (F)

If $\alpha(G) \leq 2$ then $pc_{opt}(G) \leq 3$.

Th.8 (F) Let $m \geq n \geq 2$ and $m+n \geq 9$.

$pc_{opt}(K_{m,n}) = 4$ for $n = 2,3$; and,

$pc_{opt}(K_{m,n}) = 5$ for $n \geq 4$.

Th.9 (F)

size of a max. matching

If G is a tree, then $pc_{opt}(G) = |G| - 2 - \alpha'(G) + \Delta(G)$.

Th.10 (F)

If G is 2-conn. P_4-free with $|G| \geq 9$, then $pc_{opt}(G) \leq 5$.

Some observations

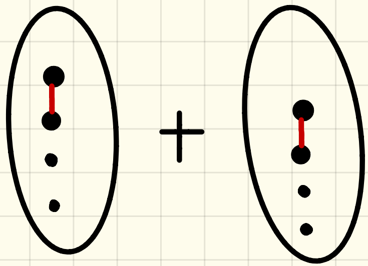
Prop. 1: If G contains a complete bip. graph H s.t.

(i) $|G|=|H|$ and (ii) each partite set contains an edge,

then $pc_{opt}(G) \leqslant 3$.

Cor. If G is a complete multipartite graph $K_{n_1,\cdots,n_\ell}$

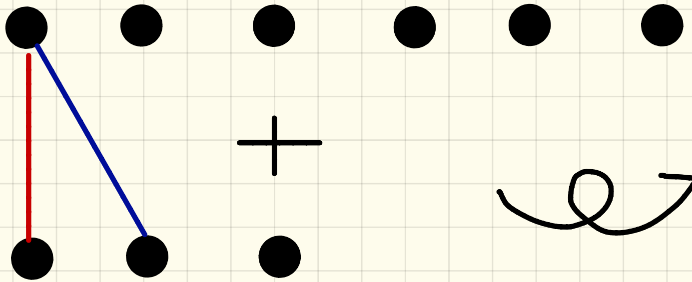
s.t. $\ell \geqslant 3$ and $2 \leqslant n_1 \leqslant \cdots \leqslant n_\ell$, then $pc_{opt}(G) \leqslant 3$.



Some observations

- Upper bound on $pc_{opt}(K_{m,n})$, where $m \geq n \geq 2$ and $m+n \geq 9$

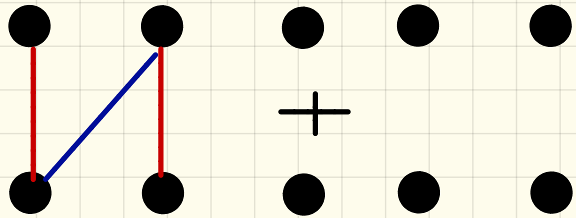
Case 1: $2 \leq n \leq 3$



$pc_{opt}(K_{m,n}) \leq 4$

Case 2: $n \geq 4$

$pc_{opt}(K_{m,n}) \leq 5$



Extension

An edge colored graph G is properly k-connected if for any pair of vertices, there exist k internally disjoint properly connected paths joining them.

We can consider the following function of a graph.

$$pc^{k}_{opt}(G) := \min\left\{ p+q \;\middle|\; \begin{array}{l} \text{we can make } G \text{ properly } k\text{-conn.} \\ \text{by recoloring } p \text{ edges with} \\ q \text{ new colors} \end{array} \right\}$$

Extension

Def: A mono. conn. graph G is (p,q)-feasible if we can make G properly connected by recoloring p edges with q new colors.

In particular, let's call (p,q)-optimal feasible if $p+q = pc_{opt}(G)$.

Remark: If G is (p,q)-feasible, then $p \geq q \geq pc(G)$.

Cor. of Th. 6 : G is (1,1)-feasible $\Leftrightarrow$ G is a complete graph

Th. 7'(F)

Any graph G s.t. $\alpha(G) \leq 2$ is (2,1)-feasible.

Th. 8' (F)

For $K_{m,n}$ with $m \geqslant n \geqslant 2$ and $m+n \geqslant 9$,

if $n=2,3$, then $K_{m,n}$ is $(2,2)$-optimal feasible;

and if $n \geqslant 4$, then $K_{m,n}$ is $(3,2)$-optimal feasible.

Th. 9' (F)

If G is a tree, then

G is $(n-1-\alpha'(G), \Delta(G)-1)$-optimal feasible.