

# On Deza Circulants

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based on joint work in progress with

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Sun Yat-sen University, Guangzhou  
November 9th, 2018

# Notation

We consider undirected graphs without loops and multiple edges.

For a graph  $\Gamma$  and its vertex  $x$ , define the **neighbourhood** of  $x$ :

$$\Gamma(x) := \{y \mid y \in V(\Gamma), y \sim x\}.$$

A graph  $\Gamma$  is called **regular** of valency  $k$  if

$$|\Gamma(x)| = k$$

holds for all  $x \in \Gamma$ .

## Strongly regular graphs and Deza graphs

A graph  $\Delta$  is called a **Deza graph** with parameters  $(v, k, b, a)$  (usually  $a < b$ ), if  $\Delta$  has  $v$  vertices, and for any pair of vertices  $x, y \in \Delta$ :

$$|\Delta(x) \cap \Delta(y)| = \begin{cases} k, & \text{if } x = y, \\ a \text{ or } b, & \text{if } x \neq y. \end{cases}$$

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A graph  $\Gamma$  is called **strongly regular** with parameters  $(v, k, \lambda, \mu)$ , if  $\Gamma$  has  $v$  vertices, and for any pair of vertices  $x, y \in \Gamma$ :

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} k, & \text{if } x = y, \\ \lambda, & \text{if } x \sim y, \\ \mu, & \text{if } x \neq y \text{ and } x \not\sim y. \end{cases}$$

A Deza graph  $\Delta$  is called a **strictly Deza graph**, if  $\Delta$  has diameter 2, and is not SRG.

## One more

Let  $G$  be a finite group.

Let  $S \subset G$  be a nonempty subset with the following properties

- ▶  $1_G \notin S$ ;
- ▶  $\forall s \in S \Rightarrow s^{-1} \in S$ .

A graph  $\text{Cay}(G, S)$  whose vertices are the elements of  $G$ , and the adjacency is defined by the following rule

$$x \sim y \Leftrightarrow xy^{-1} \in S, \quad \forall x, y \in G$$

is called a **Cayley graph of group  $G$  with the connection set  $S$** .

A Cayley graph of a cyclic group is called a **circulant**.

# Problem and some results

**Problem.** What are Cayley graphs with given combinatorial restrictions (strongly regular graphs, distance-regular graphs,...)?

- ▶ Strongly regular circulants

(Wielandt 1935; Bridges, Mena 1979; Hughes, van Lint, Wilson 1979; Ma 1984)

- ▶ Distance-regular circulants

(Miklavic, Potocnik 2003)

- ▶ Strongly regular Cayley graphs of  $C_{p^n} \times C_{p^n}$ ,  $p$  is a prime

(Leifman, Muzychuk, 2005)

- ▶ Distance-regular Cayley graphs of dihedral groups

(Miklavic, Potocnik, 2007)

## Deza graphs on small number of vertices

$(v, k, b, a)$	Cayley
(8,4,2,0)	
(8,4,2,1)	+
(8,5,4,2)	+
(9,4,2,1)	+
(9,4,2,1)	
(10,5,4,2)	+
(12,5,2,1)	+
(12,6,3,2)	+
(12,6,3,2)	
(12,7,4,3)	+
(12,7,6,2)	+
(12,9,8,6)	
(13,8,5,4)	+

Erickson et al. (1999)

$(v, k, b, a)$	Cayley
(14,9,6,4)	+
(15,6,3,1)	
(16,5,2,1)	+
(16,7,4,2)	+
(16,7,4,2)	+
(16,8,4,2)	+
(16,9,6,4)	+
(16,9,6,4)	
(16,9,8,2)	+
(16,11,8,6)	+
(16,12,10,8)	+
(16,13,12,10)	+

Goryainov, Shalaginov (2011)

## Cayley-Deza graph

Let  $G$  be a finite group,  $|G| = v$ ,  $S$  be a connection set,  $|S| = k$ .

A Cayley graph  $Cay(G, S)$  is a **Deza graph** (C.-D. graph) with parameters  $(v, k, b, a)$



there are integers  $b > a > 0$  and a partition  $G = \{e\} \cup A \cup B$ , such that

the multiset  $SS^{-1} = \{s_1 s_2^{-1} \mid s_1, s_2 \in S\} = k \cdot 1_G \cup a \cdot A \cup b \cdot B$

### Example

Let  $G = C_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ ,  $S = \{1, 3, 4, 5, 7\}$ .

Then  $Cay(G, S)$  is a Deza graph with parameters  $(8, 5, 4, 2)$  and with  $A = \{1, 3, 5, 7\}$ ,  $B = \{2, 4, 6\}$  and

the multiset  $SS^{-1} = 5 \cdot 1_G \cup 2 \cdot A \cup 4 \cdot B$

# Cayley-Deza graphs over cyclic groups of order $\leq 95$

All\* small Deza circulants we found can be divided into several families:

- 1  $K_x[yK_2] \cong \text{Cay}(C_{2xy}, S_1)$  with  $S_1 = \{i \not\equiv 0 \pmod{x}\} \cup \{xy\}$ ;
- 2  $K_n \times K_4 \cong \text{Cay}(C_{4n}, S_2)$  where  $S_2 = \{4i \mid 0 < i < n\} \cup \{n, 2n, 3n\}$  and  $n$  is odd;
- 3 **divisible design graphs** from a regular graphical Hadamard  $4 \times 4$ -matrix;
- 4 ***Paley***( $p$ )[ $K_2$ ],  $p$  is prime,  $p \equiv 1(4)$ ;
- 5 from **cyclotomic schemes with 3 classes**, on  $p$  vertices,  $p$  is prime.
- 6 a **new family** of Deza graphs on  $q_1q_2$  vertices with  $q_2 - q_1 = 4$ ,  $q_1, q_2$  are prime.

\*: and two exceptions,  $(8, 4, 2, 1)$ ,  $(9, 4, 2, 1)$ .

## Problem

Show that any Deza circulant belongs to one of these families.



# Divisible design graphs

**Divisible design graph**  $\Gamma$  with parameters  $(v, k, \lambda_1, \lambda_2, m, n)$ :

- ▶ a  $k$ -regular graph on  $v = mn$  vertices
- ▶ its vertex set can be partitioned into  $m$  classes of size  $n$
- ▶ any two distinct vertices  $x, y \in \Gamma$  have exactly
  - ▶  $\lambda_1$  common neighbours, if  $x, y$  are from the same class
  - ▶  $\lambda_2$  common neighbours, if  $x, y$  are from the different classes
- ▶ proper if  $m > 1$ ,  $n > 1$  or  $\lambda_1 \neq \lambda_2$

Note that any proper **DDG**  $\Gamma$  is a **Deza** graph (unless  $\Gamma \cong mK_n$  or  $\bar{\Gamma} \cong mK_n$ ).

# Regular graphical Hadamard matrices

Let  $H$  be an  $m \times m$  Hadamard matrix.

$H$  is called

- ▶ **graphical** if  $H$  is symmetric with constant diagonal,
- ▶ **regular** if all row and column sums are equal (to  $\ell$  say).

The two smallest regular graphical Hadamard matrices of size  $m = 4$  with  $\ell = 2$  and  $\ell = -2$ , respectively:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

### (3) Divisible design graphs from Hadamard matrices

DDG from  $H$  (Haemers, Kharaghani, Meulenberg, 2011)

Let  $H$  be a regular graphical Hadamard matrix of order  $m \geq 4$  and row sum  $\ell = \pm\sqrt{m}$ . Let  $n \geq 2$ . Replace each entry of  $H$

- ▶ with value  $-1$  by  $J_n - I_n$ ,
- ▶ with value  $+1$  by  $I_n$ .

The result is the adjacency matrix of a DDG with parameters  $(mn, n(m - \ell)/2 + \ell, (n - 2)(m - \ell)/2, n(m - 2\ell)/4 + \ell, m, n)$ .

The smallest regular graphical Hadamard matrices:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$



(2)  $K_n \times K_4$



(3) another family of Cayley-Deza graphs

#### (4) Deza circulant $Paley(p)[K_2]$

Let  $q$  be a prime power,  $q \equiv 1(4)$ . Define  $S = \{x^2 \mid x \in \mathbb{F}_q^*\}$ .

$$Paley(q) = Cay(\mathbb{F}_q^+, S).$$

It is an SRG with parameters  $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$ .

Theorem (Wielandt, 1935)

A **strongly regular circulant** is  $Paley(p)$ ,  $p$  is prime.

$Paley(q)[K_2]$  is a **Deza** graph with parameters

$$(2q, q, q-1, \frac{1}{2}(q-1)),$$

and it is a circulant if and only if  $q$  is prime (not prime power).

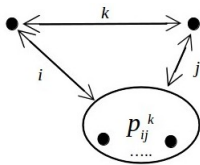
Theorem

Let  $p$  be a prime, and  $\Delta$  be a **strictly Cayley-Deza graph over  $C_{2p}$** . Then  $p \equiv 1(4)$  and  $\Delta \cong Paley(p)[K_2]$ .

## Association schemes $\mathcal{S} = (V, \mathcal{R})$

$V$  — a set of  $v$  elements,  $\mathcal{R}$  — a partition of  $V \times V$  into  $d + 1$  binary relations  $R_0, R_1, \dots, R_d$ , which satisfy:

- ▶  $R_0 = \{(x, x) \mid x \in V\}$ , the identity relation,
- ▶  $\forall i: R_i^\top = \{(y, x) \mid (x, y) \in R_i\}$  is a member of  $\mathcal{R}$ ,
- ▶ if  $(x, y) \in R_k$ , then the number of  $z$  such that



$$(x, z) \in R_i$$

$$(z, y) \in R_j$$

is a constant denoted by  $p_{ij}^k$ .

An association scheme  $\mathcal{S}$  is

- ▶ **commutative** if  $p_{ij}^k = p_{ji}^k$ , for  $\forall i, j, k$ .

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- ▶ **symmetric** if  $R_i = R_i^\top$ , for  $\forall i$ .

# Cyclotomic scheme

- ▶ Let  $q$  be a prime power, and  $e$  be a divisor of  $q - 1$ .
- ▶ Fix a primitive element  $\alpha$  of the multiplicative group of  $\mathbb{F}_q$ .
- ▶  $\langle \alpha^e \rangle$  is a subgroup of  $\mathbb{F}_q^*$  of index  $e$  and its cosets are  $\alpha^i \langle \alpha^e \rangle$ , ( $0 \leq i \leq e - 1$ ).

Define  $R_0 = \{(x, x) \mid x \in \mathbb{F}_q\}$  and

$$R_i = \{(x, y) \mid x - y \in \alpha^i \langle \alpha^e \rangle, x, y \in \mathbb{F}_q\} \quad (1 \leq i \leq e).$$

Then  $(V, \mathcal{R}) = (\mathbb{F}_q, \{R_i\}_{i=0}^e)$  forms an association scheme and it is called the **cyclotomic scheme of class  $e$  on  $\mathbb{F}_q$** .

The cyclotomic scheme of class  $e$  on  $\mathbb{F}_q$  is symmetric if and only if  $q$  or  $(q - 1)/e$  is even.

## (5) Deza graphs from cyclotomic schemes

### Theorem

Let  $q$  be a prime power, and  $\mathcal{S}$  be the **cyclotomic scheme of class 3 on  $\mathbb{F}_q$** . Let  $F \subset \{1, 2, 3\}$ .

Then a graph with adjacency matrix  $A_F = \sum_{f \in F} A_f$  is a **Deza** if and only if  $q$  is a prime and one of the following holds:

- ▶  $|F| = 1$  and  $q = x^2 + 3$  for some integer  $x$ ,
- ▶  $|F| = 2$  and  $q = x^2 + 12$  for some integer  $x$ .

J. H. E. Cohn, The Diophantine equation  $x^2 + C = y^n$ . *Acta Arith.* **65** (1993) 367–381.

### Conjecture (Bunyakovsky, 1857)

Let  $f(x)$  be a polynomial in one variable satisfying:

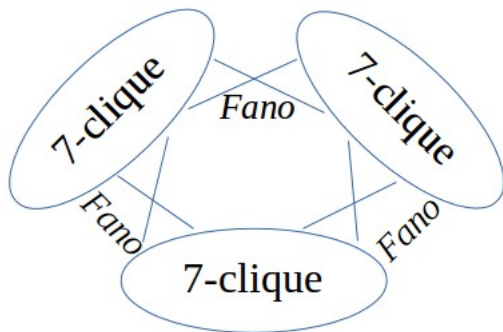
- ▶ the leading coefficient of  $f(x)$  is positive,
- ▶ the polynomial is irreducible over the integers,
- ▶ the coefficients of  $f(x)$  are relatively prime.

Then  $f(n)$  is prime for infinitely many positive integers  $n$ .

## New family (6)

Among all Cayley-Deza graphs on  $\leq 95$  vertices, we found two examples with parameters  $(21, 12, 7, 6)$  and  $(77, 40, 21, 20)$ . These parameters sets did not satisfy any previously known construction.

The graph on 21 vertices has two systems of imprimitivity ( $3 \times 7$  and  $7 \times 3$ ), one of which gives the following picture.





## New family (6)

We first described a connection set  $S$  for the graph on 21 vertices, and then generalized it.

Let  $q_1, q_2$  be two prime powers with  $q_2 - q_1 = 4$ .

Define

- ▶  $S_{q_1} := \{x^2 \mid x \in \mathbb{F}_{q_1}^*\}$  and, similarly,  $S_{q_2}$ ,
- ▶  $\overline{S}_{q_1} = \mathbb{F}_{q_1}^* \setminus S_{q_1}$  and, similarly,  $\overline{S}_{q_2}$ .

Let

$$S_0 = \{(0, x) \mid x \in \mathbb{F}_{q_2}^*\},$$
$$S_1 = S_{q_1} \times \overline{S}_{q_2}, \quad S_2 = \overline{S}_{q_1} \times S_{q_2}.$$

Theorem (Joint with Galina Isakova)

$\text{Cay}(\mathbb{F}_{q_1}^+ \times \mathbb{F}_{q_2}^+, S_0 \cup S_1 \cup S_2)$  is a **Deza** graph with parameters  $(v, \frac{1}{2}(v+3), \frac{1}{4}(v+7), \frac{1}{4}(v+3))$ ,  $v = q_1 q_2$ .

If  $q_1$  and  $q_2$  are prime it is a **circulant** because  $\mathbb{Z}_{q_1 q_2} \simeq \mathbb{Z}_{q_1} \times \mathbb{Z}_{q_2}$   
It is only conjectured that there exist infinitely many such pairs of prime numbers  $q_1, q_2$ .

# Strictly Deza circulants having $\leq 95$ vertices

C8	■(8,4,2,1)■ (8,5,4,2)
C9	■(9,4,2,1)■
C10	(10,5,4,2)
C12	(12,5,2,1) (12,7,4,3) (12,7,6,2) (12,9,8,6)
C13	(13,8,5,4)
C16	(16,9,8,2) (16,13,12,10)
C18	(18,13,12,8)
C19	(19,6,2,1)
C20	(20,7,3,2) (20,11,10,2) (20,13,9,8) (20,17,16,14)
C21	(21,12,7,6)
C24	(24,13,12,2) (24,17,16,10) (24,19,18,14) (24,21,20,18)
C26	(26,13,12,6)
C28	(28,9,5,2) (28,15,14,2) (28,19,15,12) (28,25,24,22)
C30	(30,21,20,12) (30,25,24,20)
C32	(32,17,16,2) (32,25,24,18) (32,29,28,26)
C34	(34,17,16,8)
C36	(36,11,7,2) (36,19,18,2) (36,25,21,16) (36,25,24,14) (36,31,30,26) (36,33,32,30)
C37	(37,24,16,15)
C40	(40,21,20,2) (40,31,30,22) (40,33,32,26) (40,37,36,34)
C42	(42,29,28,16) (42,37,36,32)
C44	(44,13,9,2) (44,23,22,2) (44,31,27,20) (44,41,40,38)
C48	(48,25,24,2) (48,33,32,18) (48,37,36,26) (48,41,40,34) (48,43,42,38) (48,45,44,42)

$K_x[yK_2]$ ,  $K_n \times K_4$ , Divisible design graphs,  $Paley(p)[K_2]$ , From cyclotomic schemes with 3 classes, New family, ■ — ?Sporadic?

# Strictly Deza circulants having $\leq 95$ vertices

C50	(50,41,40,32)
C52	(52,15,11,2) (52,27,26,2) (52,37,33,24) (52,49,48,46)
C54	(54,37,36,20) (54,49,48,44)
C56	(56,29,28,2) (56,43,42,30) (56,49,48,42) (56,53,52,50)
C58	(58,29,28,14)
C60	(60,17,13,2) (60,31,30,2) (60,41,40,22) (60,43,39,28) (60,49,48,38) (60,51,50,42) (60,55,54,50) (60,57,56,54)
C61	(61,40,27,24)
C64	(64,49,48,34) (64,57,56,50) (64,61,60,58)
C66	(66,45,44,24) (66,61,60,56)
C67	(67,22,9,6)
C68	(68,19,15,2) (68,35,34,2) (68,49,45,32) (68,65,64,62)
C70	(70,57,56,44) (70,61,60,52)
C72	(72,37,36,2) (72,49,48,26) (72,55,54,38) (72,61,60,50) (72,65,64,58) (72,67,66,62) (72,69,68,66)
C74	(74,37,36,18)
C76	(76,21,17,2) (76,39,38,2) (76,55,51,36) (76,73,72,70)
C77	(77,40,21,20)
C78	(78,53,52,28) (78,73,72,68)
C80	(80,41,40,2) (80,61,60,42) (80,65,64,50) (80,71,70,62) (80,73,72,66) (80,77,76,74)
C82	(82,41,40,20)
C84	(84,23,19,2) (84,43,42,2) (84,57,56,30) (84,61,57,40)
C88	(88,67,66,46) (88,81,80,74) (88,85,84,82)
C90	(90,61,60,32) (90,73,72,56) (90,81,80,72) (90,85,84,80)
C92	(92,25,21,2) (92,47,46,2) (92,67,63,44) (92,89,88,86)

$K_x[yK_2]$ ,  $K_n \times K_4$ , Divisible design graphs,  $Paley(p)[K_2]$ , From cyclotomic schemes with 3 classes, New family, ■ — ?Sporadic?

# Summary

## Results:

- ▶ It seems that there are 6 families of Deza circulants (with two exceptions on 8 and 9 vertices):
  - ▶  $K_x[yK_2]$ ;
  - ▶  $K_n \times K_4$ ;
  - ▶ from DDG based on the Hadamard  $4 \times 4$ -matrix;
  - ▶  $Paley(p)[K_2]$ ;
  - ▶ from cyclotomic schemes with 3 classes;
  - ▶ a new family of Deza graphs on  $q_1q_2$  vertices with  $q_2 - q_1 = 4$ .
  - ▶ two exceptions on 8 and 9 vertices.
- ▶ We can completely characterize Deza circulants over  $C_{2p}$ .

## Some questions:

- ▶ can we generalize the result for  $C_{2p}$  to  $C_{2q}$ ?
- ▶ what are strictly Deza circulants over  $C_{4m}$  where  $m$  is odd?
- ▶ is it true that strictly Deza circulants with prime number of vertices come from a cyclotomic scheme with of class 3?
- ▶ If  $q_1$  or  $q_2$  is not prime are there circulants in the new family (6)? (in this case  $\mathbb{F}_{q_1}^+ \times \mathbb{F}_{q_2}^+$  is not cyclic).

## Announcement

We are organizing the International Conference and PhD-Master Summer School on Groups and Graphs, Designs and Dynamics (G2D2). The main goal of this event is to bring together researchers from different fields of mathematics and its applications mainly based on group theory, graph theory, design theory and theory of dynamical systems.

All scientific activities will take place in the Three Gorges Mathematical Research Center at China Three Gorges University (Yichang, China) during August 12–25, 2019.

The scientific program consists of the G2D2-Conference and the G2D2-Summer School (with four minicourses).

Selected papers based on talks in G2D2 will be published in a special issue of The Art of Discrete and Applied Mathematics.

For more details please visit the website

<http://www.math.sjtu.edu.cn/conference/G2D2/>