Cops and Robbers on Certain Hypergraphs

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Story of Cops and Robbers Game

1983 NOWAKOWSKI AND WINKLER

- Introduce the game of cops and robbers on graphs
- Characterize cop-win graphs and interested in product of cop-win graphs

1984 AIGNER AND FROMME

- Consider the situation where more cops capture the robber
- For planar graph, three cops suffice to win

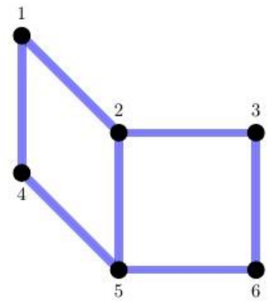
2011 WILLIAM DAVID BAIRD

- Introduce the game of cops and robbers on hypergraphs
- Investigate that hyperpath is cop-win, but hypercycle is robber-win

Cops and Robbers on Graphs

Cops and Robbers on Graphs

Start with a (reflexive) finite connected graph



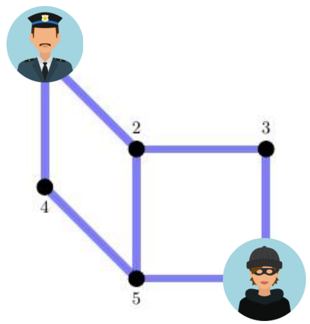
Two players: cop and robber

Cop

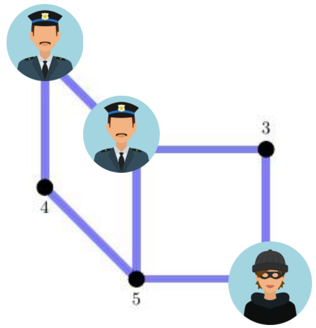
Robber

Cops and Robbers

The cop chooses a beginning vertex and then the robber chooses the other vertex to begin



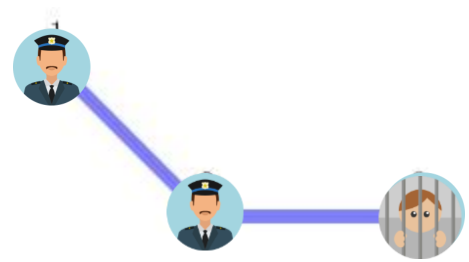
In each round, the cop and the robber take alternatively moving from their present vertex to other vertices along edges or staying put



Cop wins

Cop wins if cop can catch robber by occupying the same vertex as the robber after finite number of moves

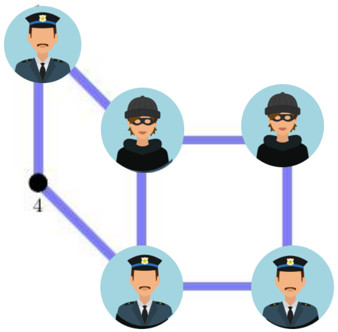
Example of cop-win graph



Robber wins

Robber wins if robber can run away (there exists an escaping way for the robber)

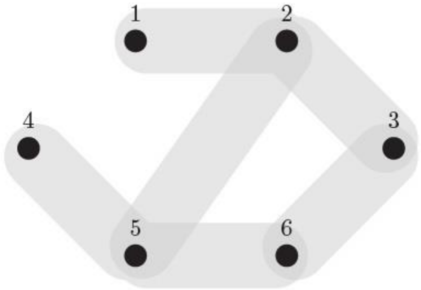
Example of robber-win graph



Cops and Robbers on Hypergraphs

Cops and Robbers on Hypergraphs

Start with a finite connected hypergraph



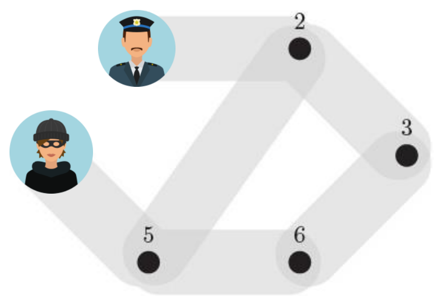
Two players: cop and robber

Cop

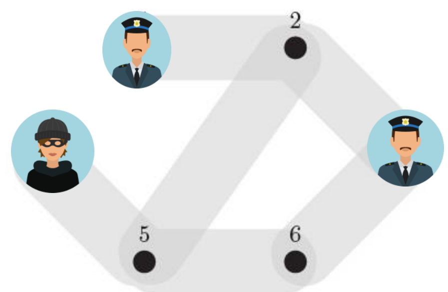
Robber

Cops and Robbers

The cop chooses a beginning vertex and then the robber chooses the other vertex to begin

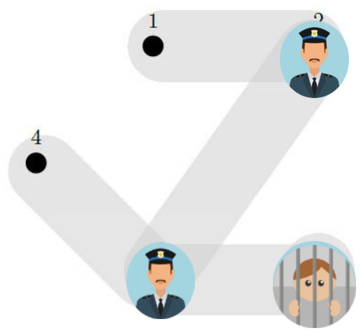


In each round, they take alternatively moving from their present vertex x to any vertex y belonging to the same hyperedge as vertex x or staying put.



Cop wins and Robber wins

Example of cop-win hypergraph



Example of robber-win hypergraph

Cops and Robbers on Products of Hypergraphs

Generalization of cops and robbers

1983 (Strong) product of cop-win graphs is cop-win.

Consider products of cop-win hypergraphs

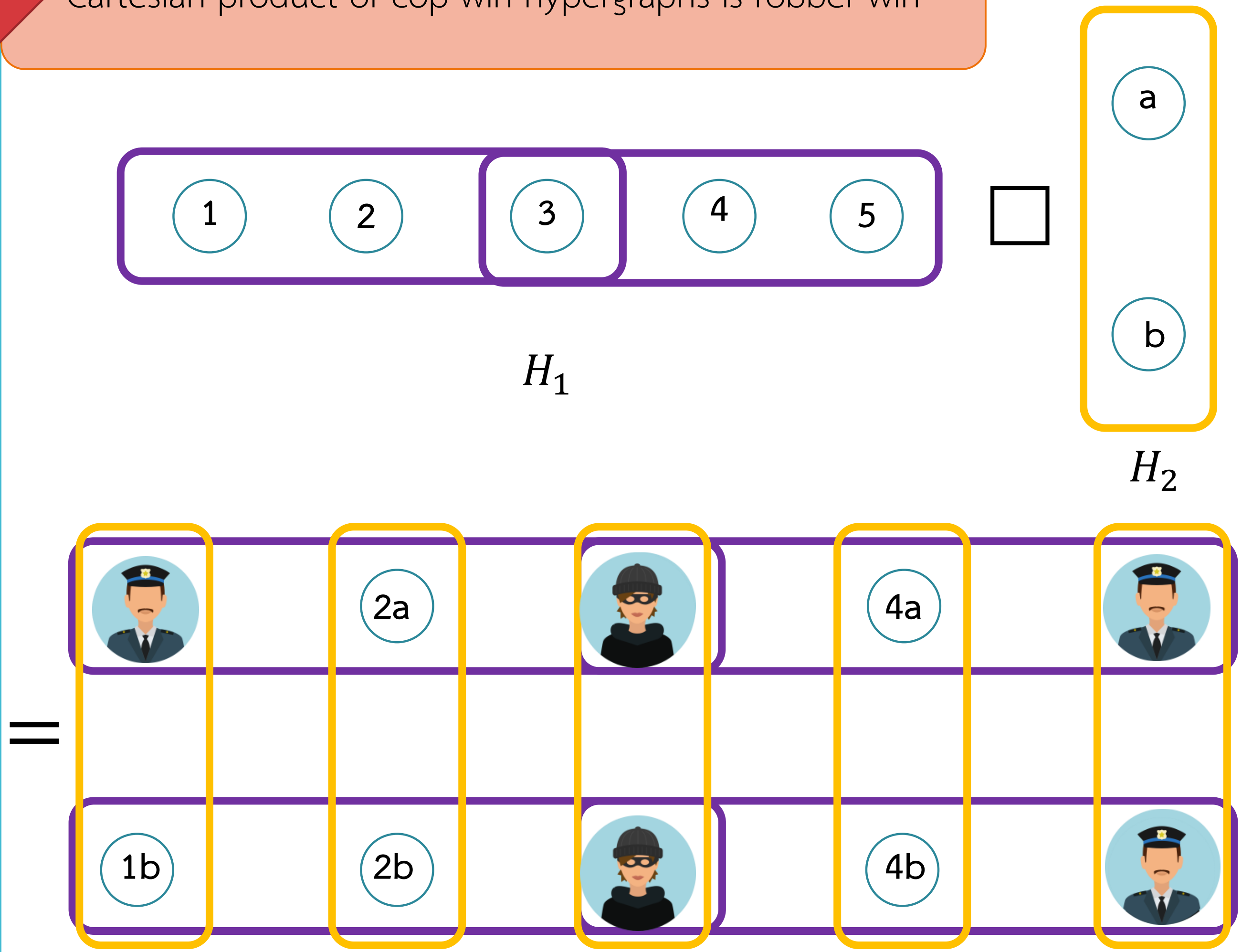
Cartesian product of cop-win hypergraphs is robber-win

Direct product of cop-win hypergraphs is robber-win

Strong product of cop-win hypergraphs is cop-win

Cartesian Product

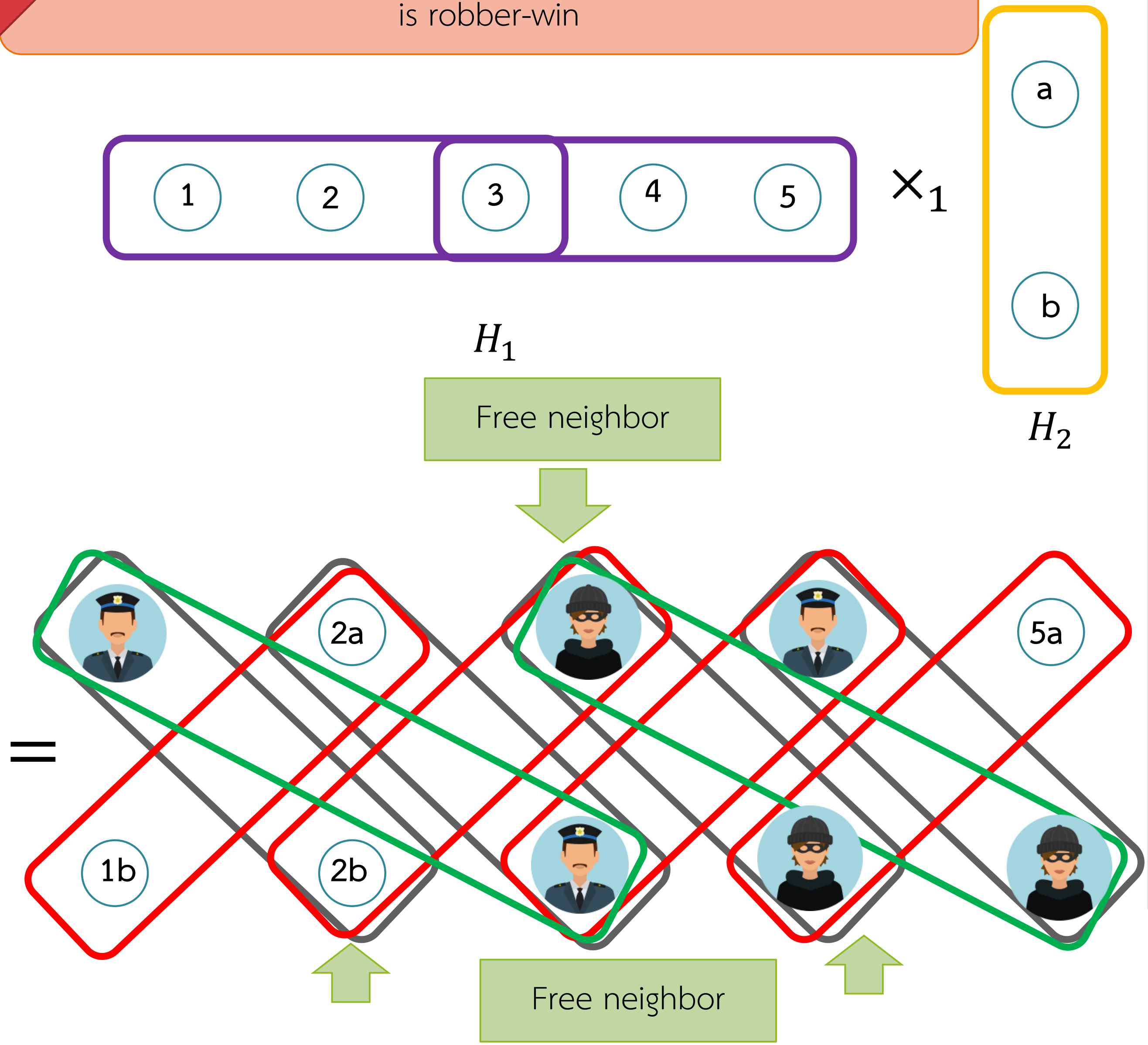
Cartesian product of cop-win hypergraphs is robber-win



The robber always find the vertex to stay far from the cop.

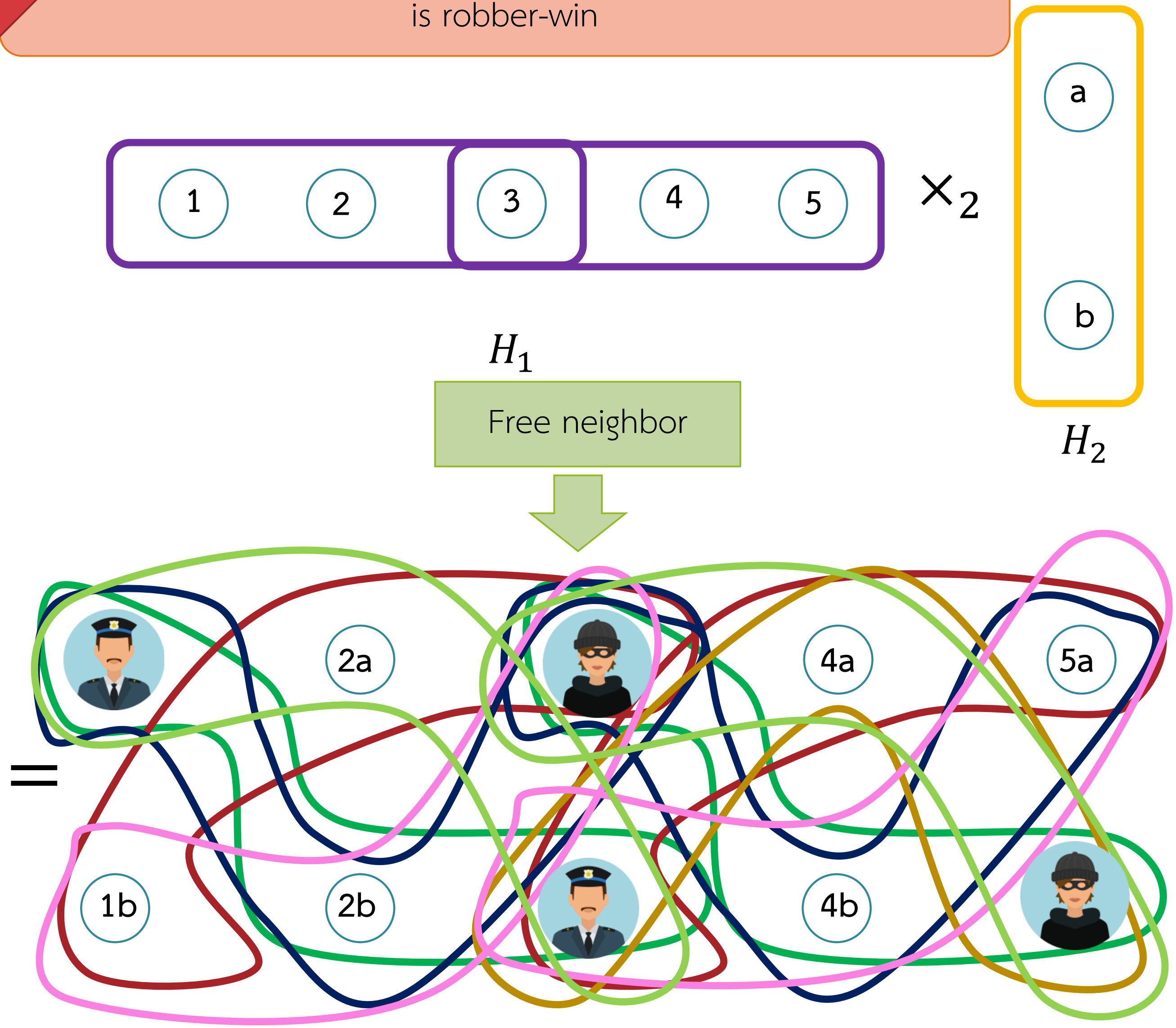
Minimal rank preserving direct product of cop-win hypergraphs is robber-win

Direct Product



Maximal rank preserving direct product of cop-win hypergraphs is robber-win

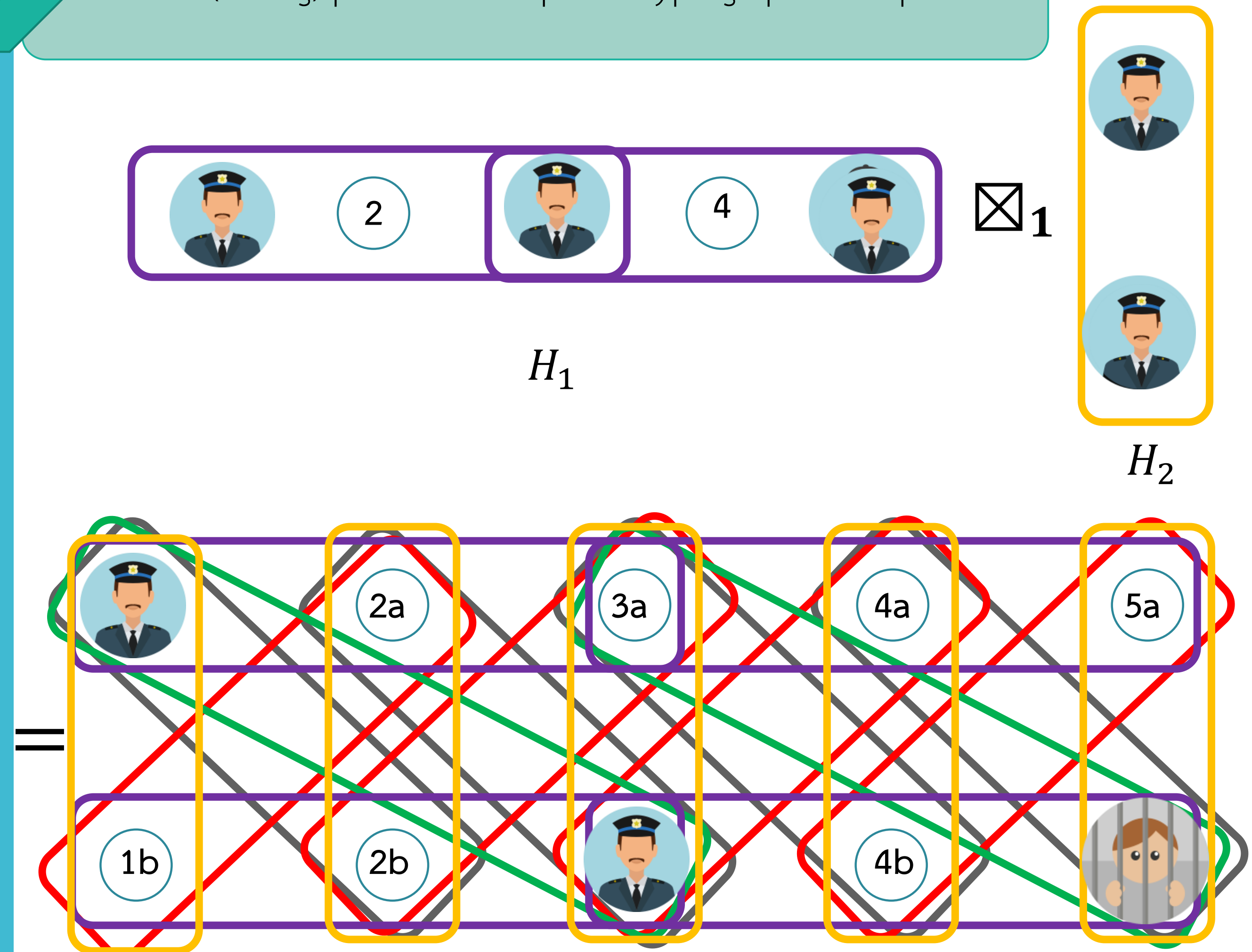
Direct Product



$$E(H_1 \boxtimes_1 H_2) = E(H_1 \square H_2) \cup E(H_1 \times_1 H_2)$$

Strong Product

Normal (strong) product of cop-win hypergraphs is cop-win



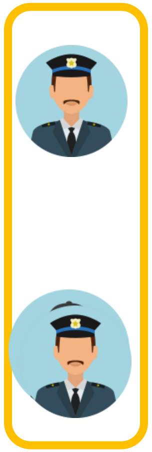
$$E(H_1 \boxtimes_2 H_2) = E(H_1 \square H_2) \cup E(H_1 \times_2 H_2)$$

Standard strong product of cop-win hypergraphs is cop-win

Strong Product

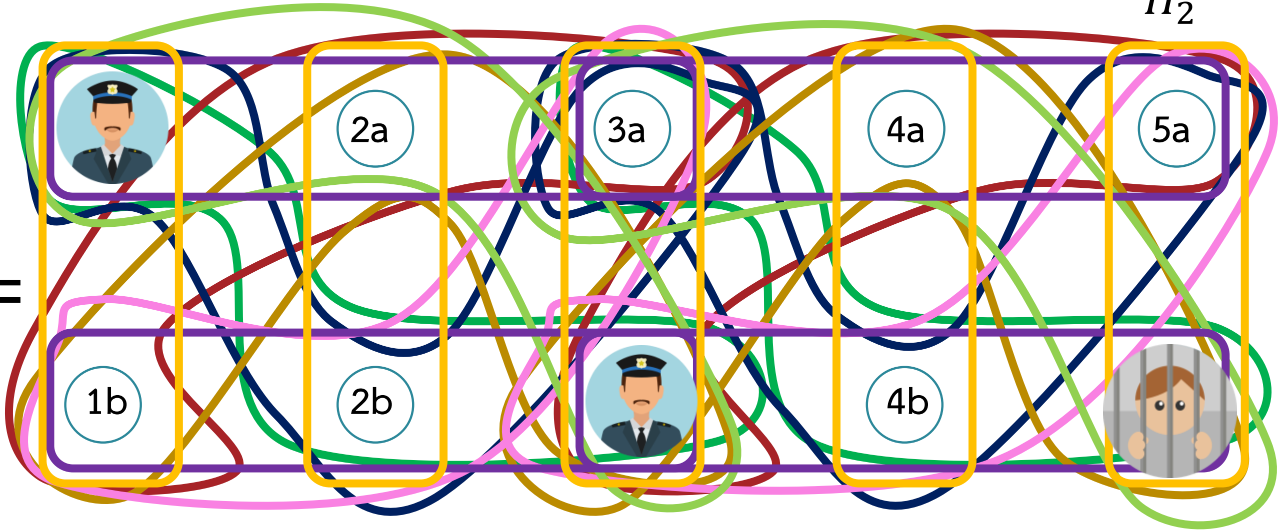
H_1

$\boxtimes_2$



H_2

=



Characterization of Cop-win Hypergraphs

Generalization of cops and robbers

by successively deletion corners (in any order), G can be reduced to K_1

1983 A finite cop-win graph if and only if it is **dismantlable**.

1984 Let x be a corner of G and $\bar{G} = G - x$. G is a cop-win graph if and only if $\bar{G}$ is a cop-win graph.

Let x be a corner of a hypergraph H. H is a cop-win hypergraph if and only if a weakly deletion $H - x$ is a cop-win hypergraph

Characterization of cops and robbers

x is a corner of a graph G.

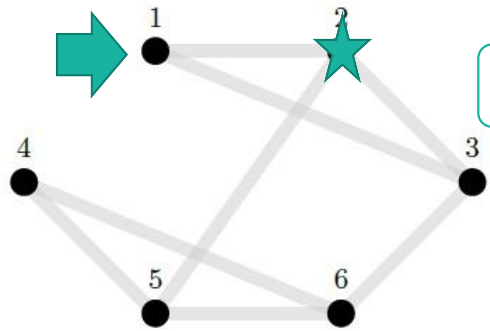
For some vertex $y \neq x$ of a graph G, $N[x] \subseteq N[y]$

x is a corner of a hypergraph H.

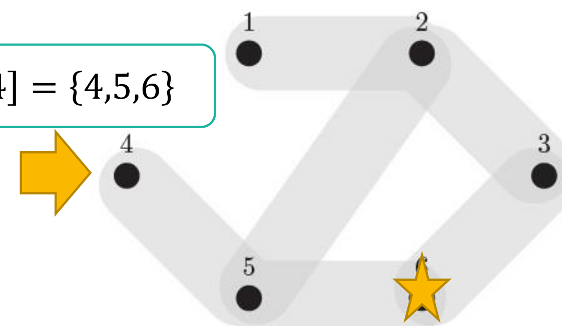
There exists a vertex y of H such that $N_H[x] \subseteq N_H[y]$

$N[1] = \{1,2,3\}$

$N[2] = \{1,2,3,5\}$



$N_H[4] = \{4,5,6\}$



$N_H[6] = \{3,4,5,6\}$

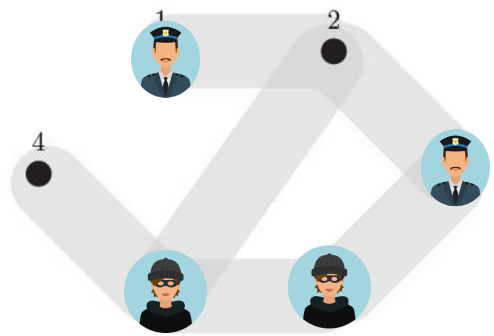
1 is a corner of a graph G

4 is a corner of a hypergraph H

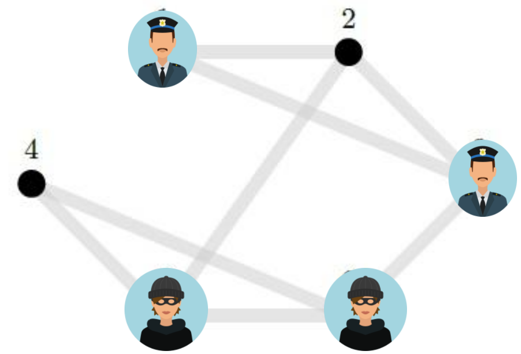
Characterization of cops and robbers

$V(G(H)) = V(H)$ and $uv \in E(G(H))$ if $\{u, v\} \subseteq e$ for some $e \in E(H)$

H is a robber-win hypergraph if and only if a graph $G(H)$ of a hypergraph H is a robber-win graph



H



$G(H)$

If H is a robber-win hypergraph, by applying winning strategy of robber in H, $G(H)$ is a robber-win graph

If $G(H)$ is a robber-win graph, by applying winning strategy of robber in $G(H)$, H is a robber-win hypergraph

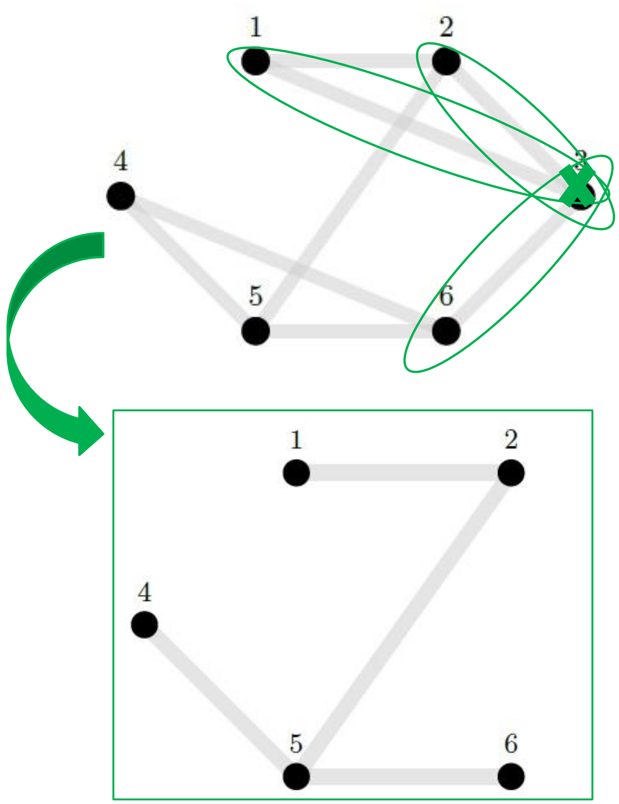
Characterization of cops and robbers

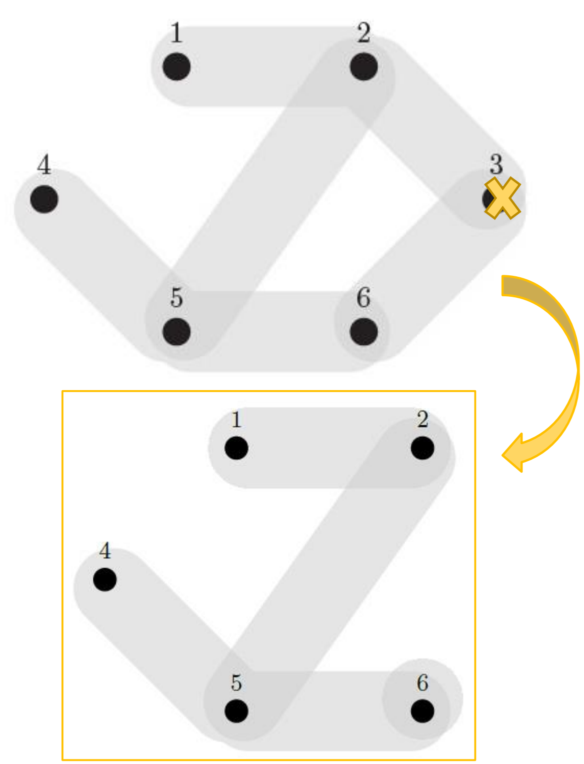
Deletion of $x \in V$ from G

Removing of x from V and removing all edges of G containing x from E

Weakly deletion of $x \in V(H)$ from H

Removing of x from $V(H)$ and from each hyperedge containing x





Characterization of cops and robbers

Let x be a corner of a hypergraph H. H is a cop-win hypergraph if and only if a weakly deletion $H - x$ is a cop-win hypergraph

If H is a cop-win hypergraph then a graph $G(H)$ of a hypergraph H is a cop-win graph.

If x is a corner in a hypergraph H, then x is a corner in a graph $G(H)$.

Thus, $G(H) - x$ is a cop-win graph, so is $G(H - x)$.

$G(H) - x = G(H - x)$.

Therefore, a weakly deletion $H - x$ is a cop-win hypergraph.

Characterization of cops and robbers

Let x be a corner of a hypergraph H. H is a cop-win hypergraph if and only if a weakly deletion $H - x$ is a cop-win hypergraph

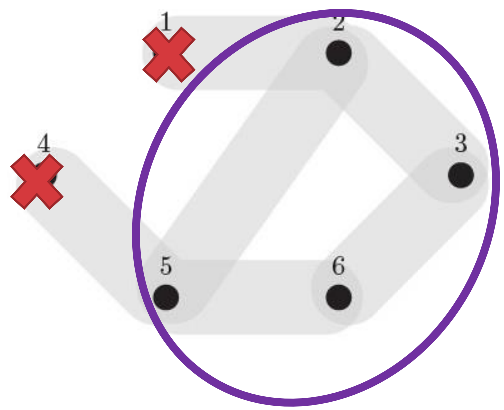
If H is a robber-win hypergraph then a graph $G(H)$ of a hypergraph H is a robber-win graph.

Thus, $G(H) - x$ is a robber-win graph, so is $G(H - x)$.

Therefore, a weakly deletion $H - x$ is a robber-win hypergraph.

Characterization of cops and robbers

A hypergraph H is a cop-win hypergraph if and only if by successively weakly deletion corners (in any order), H can be reduced to a single vertex.

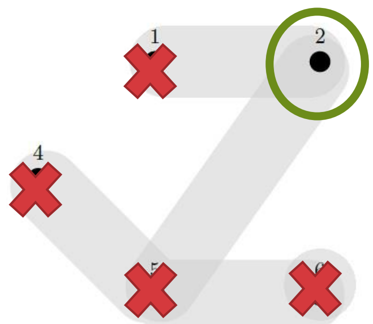


No corner

Robber-win hypergraph

Characterization of cops and robbers

A hypergraph H is a cop-win hypergraph if and only if by successively weakly deletion corners (in any order), H can be reduced to a single vertex.



Reduced to a single vertex

Cop-win hypergraph


Thank you

Hyperpath and Hpercycle

A hypergraph is t-joined if each intersection of hyperedges contains exactly t vertices

A hyperpath is a sequence of hyperedges $E_1, E_2, E_3, \ldots, E_k$, such that E_i and E_{i+1} are t-joined for $t > 0$ and for $1 \leq i \leq k-1$ and $E_i \cap E_j = \emptyset$ when $j \neq i + 1(mod\ k)$

For an integer $k > 2$, a k-hypercycle is a collection of k hyperedges $E_1, E_2, E_3, \ldots, E_k$, with two hyperedges E_i and E_j are incident if $j = i + 1(mod\ k)$

The products of $H_1(V_1, E_1)$ and $H_2(V_2, E_2)$

The Cartesian Product $H_1 \square H_2$

- Vertex-set: $V_1 \times V_2$
- Edge-set: $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_r, y_r)\}$ is an edge

if	
	1. $\{x_1, x_2, x_3, \dots, x_r\} \in E_1$ and $y_1 = y_2 = y_3 = \cdots = y_r \in V_2$
	2. $\{y_1, y_2, y_3, \dots, y_r\} \in E_2$ and $x_1 = x_2 = x_3 = \cdots = x_r \in V_1$

The products of $H_1(V_1, E_1)$ and $H_2(V_2, E_2)$

The Minimal Rank Preserving Direct Product $\boldsymbol{H_1 \times_1 H_2}$

- Vertex-set: $V_1 \times V_2$
- Edge-set: $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_r, y_r)\}$ is an edge

if

1. $\{x_1, x_2, x_3, \dots, x_r\} \in E_1$ and $\{y_1, y_2, y_3, \dots, y_r\} \subseteq e_2$ for some $e_2 \in E_2$
2. $\{x_1, x_2, x_3, \dots, x_r\} \subseteq e_1$ for some $e_1 \in E_1$ and $\{y_1, y_2, y_3, \dots, y_r\} \in E_2$

The Maximal Rank Preserving Direct Product $\boldsymbol{H_1 \times_2 H_2}$

The products of $H_1(V_1, E_1)$ and $H_2(V_2, E_2)$

- Vertex-set: $V_1 \times V_2$
- Edge-set: $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_r, y_r)\}$ is an edge

if

1. $\{x_1, x_2, x_3, \dots, x_r\} \in E_1$ and there is an edge $e_2 \in E_2$ such that $\{y_1, y_2, y_3, \dots, y_r\}$ is a multiset of elements of e_2, and $e_2 \subseteq \{y_1, y_2, y_3, \dots, y_r\}$
2. $\{y_1, y_2, y_3, \dots, y_r\} \in E_2$ and there is an edge $e_1 \in E_1$ such that $\{x_1, x_2, x_3, \dots, x_r\}$ is a multiset of elements of e_1, and $e_1 \subseteq \{x_1, x_2, x_3, \dots, x_r\}$