Some results on partition problems of graphs

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2. *k*-partition problem
3. Our results on *k*-partition problem
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7. Almost bisection problem of graphs with constraints on the minimum degree

Some basic notations

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- Let $h(m) = \sqrt{2m + \frac{1}{4}} - \frac{1}{2}$, and let K_n denote the complete graph with n vertices.

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- [1] C. S. Edwards, [*Canadian J. Math.*, **25** (1973) 475–485.]
- [2] C. S. Edwards, [in *Proc. 2nd Czechoslovak Symposium on Graph Theory*, Prague, (1975) 167–181.]

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$$e(V_1, V_2, ..., V_k) \geq \frac{k-1}{k}m + \frac{k-1}{2k}h(m) - \frac{(k-2)^2}{8k}, \qquad (1)$$

and they also [3] proved that the vertex set of a graph with m edges can be partitioned into V_1, V_2, ..., V_k such that

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- [6] B. Xu and X. Yu, [*J. Combin. Theory Ser. B* **99** (2009) 324–337.]

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Let G be a graph with m edges, and $k \geq 2$ be an integer. If $m \geq \frac{9}{128}k^4(k-2)^2$, and G contains at most $\frac{1}{k}h(m) - \frac{1}{8}(3k^2 - 6k - 11)$ vertices with degrees being multiples of k, then $V(G)$ has a k-partition satisfying both (1) and (2).

- [9] M. Liu, B. Xu [M. Liu, B. Xu, *J. Comb. Optim.*, 31(2016),1383-1398].

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Bipatition problem with minimum degree

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- Suppose that (X, Y) is a partition of $V(G)$. If $-1 \leq |X| - |Y| \leq 1$, then (X, Y) is called a bisection of G. If $\lfloor \frac{1}{2}|V(G)| \rfloor - 2 \leq |X| \leq |Y| \leq \lceil \frac{1}{2}|V(G)| \rceil + 2$, then (X, Y) is called an almost bisection of G.

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- Suppose that $A \subseteq V(G)$ and $x \in V(G)$. Then, $d_A(x) = |N(x) \cap A|$.

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Conjecture [2]:

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- 13 years later, Stiebitz [3] confirmed Thomassen's conjecture, with an elegant argument, and proved that

Theorem [3].

Let G be a graph and $a, b : V(G) \longmapsto \mathbb{N}_0$ two functions. Suppose that $d_G(v) \geq a(v) + b(v) + 1$ for each vertex v of G. Then, there exists a partition of $V(G)$ into A and B such that

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Let G be a $(K_4 - e)$-free graph with $|V(G)| \geq 4$, and $a, b : V(G) \longmapsto \mathbb{N}$ be two functions. If $d_G(v) \geq a(v) + b(v)$ for each vertex v of G, then there exists a partition of $V(G)$ into A and B such that

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- Let $G = K_4 - e$, and let $a, b : V(G) \longmapsto \mathbb{N}$ be two functions such that $a(x) = d(x) - 1$ and $b(x) = 1$ for each vertex $x \in V(G)$. Then, $(K_4 - e)$-free is necessary in our result.

- [8] M. Liu, B. Xu [Discrete Appl. Math., 226 (2017), 87–93.]

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- We did not find the extremal graphs. But the complete bipartite graph $K_{3,3}$ shows that the restriction on the sparsity of quadrilaterals cannot be relaxed too much if we let $a(x) = b(x) = 2$ for each x.

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Problem 2:

What is the bound $g(s,t)$ for general graph G with $g(G) = k$.

-

The crucial lemma to the proof

- A pair (A, B) of disjoint subsets A and B of $V(G)$ is said to be (a, b)-*feasible* if $d_A(x) \geq a(x)$ for each $x \in A$ and $d_B(y) \geq b(y)$ for each $y \in B$. An (a, b)-feasible partition is just an (a, b)-feasible pair (A, B) with $A \cup B = V(G)$.

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Key Lemma [3]:

For any two functions $a, b : V(G) \longmapsto \mathbb{N}$ such that $d_G(v) \geq a(v) + b(v) - 1$, if G has an (a, b)-feasible pair, then it admits an (a, b)-feasible partition.

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- [3] M. Stiebitz, [*J. Graph Theory*, **23** (1996) 321–324.]

Maurer's partition problems of graphs with constraints on the minimum degree

- While studying graph colorings with some particular properties, Maurer proved an interesting result (see [9]).

Theorem (see [9]):

Let G be a connected graph with n vertices and $\delta(G) \geq 2$. Then, for any positive integer k with $2 \leq k \leq n-2$, G admits a $(1,1)$-partition (X, Y) such that $|X| = k$ and $|Y| = n-k$.

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- In 1998, Arkin and Hassin [10] proved that

Theorem [10]:

Every graph G has a bisection (X, Y) such that $\delta(X) + \delta(Y) \geq \delta(G) - 1$.

- [9] J. Sheehan, [*J. Graph Theory* **14** (1990) 673–685.]
- [10] E. M. Arkin and R. Hassin, [*Discrete Math.* **190** (1998)

Main result

- In [11], we have improved the results of both [9] and [10] via proving that

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- [10] E. M. Arkin and R. Hassin, [*Discrete Math.* **190** (1998) 55–65.]
- [11] M. Liu and B. Xu, [*Sci China Math,* 58(2015), 869–874.]

Almost bisection problem of graphs with constraints on the minimum degree

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Conjecture [10]:

Each graph G with $\delta(G) \geq 4$ admits a $(2,2)$-partition (X, Y) such that $\lfloor \frac{1}{2}|V(G)| \rfloor - 2 \leq |X| \leq |Y| \leq \lceil \frac{1}{2}|V(G)| \rceil + 2$.

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If $\bar{G}$ contains no $K_{3,r}$, where $r = \lfloor \frac{n}{2} \rfloor - 3$, then Arkin and R. Hassin's conjecture holds for graph G with n vertices.

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- [4] X. Hu, Y. Zhang and Y. Chen, [*Bull. Aust. Math. Soc.*, 91(2014),177–182.]
- [10] E. M. Arkin and R. Hassin, [*Discrete Math.* **190** (1998) 55–65.]

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