Connected subgraphs in edge-coloured graphs

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Based on a joint survey with Shinya Fujita\textsuperscript{2} and Colton Magnant\textsuperscript{3}

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Folkloric Observation (Erdős and Rado)

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What happens when we use \( r \geq 2 \) colours? Let \( m(n, r) \) be the maximum integer \( m \) such that, whenever we have an \( r \)-colouring of \( K_n \), there exists a monochromatic connected subgraph on at least \( m \) vertices.
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A graph is either connected, or its complement is connected. Equivalently, in any 2-colouring of the edges of a complete graph, there exists a monochromatic connected spanning subgraph (or, a monochromatic spanning tree).

What happens when we use $r \geq 2$ colours? Let $m(n, r)$ be the maximum integer $m$ such that, whenever we have an $r$-colouring of $K_n$, there exists a monochromatic connected subgraph on at least $m$ vertices. Thus, $m(n, 2) = n$. 
Upper bound:
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Affine plane $AG(q)$ over $\mathbb{F}_q$, where $q$ is a prime power. E.g. $AG(2)$:
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- Parallel lines classes are $L_\infty = \{x = c : c \in \mathbb{F}_q\}$, and $L_m = \{y = mx + c : c \in \mathbb{F}_q\}$ for $m \in \mathbb{F}_q$. 

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- There are $q^2$ points, and each line contains $q$ points.
- Implies that, if $r - 1$ is a prime power, then there is an $r$-colouring of $K_{(r-1)^2}$ such that the largest monochromatic connected subgraph has $r - 1$ vertices.
If $r - 1$ is a prime power, take a blow-up of $AG(r - 1)$ to $K_n$. 

E.g. $r = 3$: 

\[\left\lceil n \left( r - 1 \right)^2 \right\rceil \text{ or } \left\lfloor n \left( r - 1 \right)^2 \right\rfloor \] 

The largest monochromatic subgraph has at most $(r - 1) \left\lceil n \left( r - 1 \right)^2 \right\rceil < n r - 1 + r$ vertices, i.e. $m(n, r) < n r - 1 + r$. 

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$$\left( r - 1 \right) \left\lfloor \frac{n}{(r - 1)^2} \right\rfloor < \frac{n}{r - 1} + r$$

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**Theorem 1 (Gyárfás 1977; Füredi 1981)**

*For* \( r \geq 2 \) *and any* r-coloring of \( K_n \), *there is a monochromatic connected subgraph on at least* \( \frac{n}{r-1} \) *vertices.*
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**Theorem 1 (Gyárfás 1977; Füredi 1981)**

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Hence if $r - 1$ is a prime power, then $m(n, r) \approx \frac{n}{r-1}$ (if $n$ is large).
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Theorem 1 follows from:

**Lemma 2 (Mubayi 2002; L., Morris, Prince 2004)**

*For* $r \geq 2$ *and any* $r$-*colouring of* $K_{m,n}$, *there is a monochromatic double star on at least* $m + n r$ *vertices.*

A double star is a graph obtained by taking two vertex-disjoint stars and connecting their centres by an edge.

Gyárfás had proved Lemma 2 with "tree" in place of "double star."
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Proof of Theorem 1 (assuming Lemma 2).

Take an $r$-colouring of $K_n$. 

Let $U$ = vertex set of a monochromatic component. $|U| < n \Rightarrow$ complete bipartite graph with classes $U$ and $V$ ($K_n \setminus U$) is $(r - 1)$-coloured.

Lemma 2 $\Rightarrow$ there is a monochromatic tree on at least $n r - 1$ vertices. □

Proof of Lemma 2.

Take an $r$-colouring of $K_m, n$. Let $H$ = bipartite subgraph with most frequent colour. For $xy \in E(H)$, let $Z(xy) = d(x) + d(y)$.

$E Z = 1 e(H) \sum_{xy \in E(H)} (d(x) + d(y)) = 1 e(H) \sum_v d(v) \geq 1 e(H) (1 m + 1 n) e(H) \geq m + n r$. □
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\mathbb{E}Z = \frac{1}{e(H)} \sum_{xy \in E(H)} (d(x) + d(y)) = \frac{1}{e(H)} \sum_{v \in V(H)} d(v)^2 \\
\geq \frac{1}{e(H)} \left( \frac{1}{m} + \frac{1}{n} \right) e(H)^2 \geq \frac{m + n}{r}.
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(a) *tree of height at most 2 (Bialostocki, Dierker, Voxman 1992)*;
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(c) *broom (i.e. a path with a star at one end)* (Burr 1992).
Inspired by Lemma 2 and the affine plane construction, Gyárfás and Sárközy asked:
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**Question 4 (Gyárfás, Sárközy 2008)**

*For* $r \geq 3$ *and any* $r$-*colouring of* $K_n$, *is it true that there is a monochromatic double star on at least* $\frac{n}{r-1}$ *vertices?*
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**Theorem 5 (Gyárfás, Sárközy 2008)**

*For $r \geq 2$ and any $r$-colouring of $K_n$, there is a monochromatic double star on at least $\frac{(r+1)n+r-1}{r^2}$ vertices.*
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For \( r = 2 \), we have a monochromatic double star on at least \( \frac{3n+1}{4} \) vertices in any 2-colouring of \( K_n \).
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For \( r = 2 \), we have a monochromatic double star on at least \( \frac{3n+1}{4} \) vertices in any 2-colouring of \( K_n \). By considering Paley graphs or random graphs, the value \( \frac{3n}{4} + O(1) \) is tight.
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**Question 4 (Gyárfás, Sárközy 2008)**

*For $r \geq 3$ and any $r$-colouring of $K_n$, is it true that there is a monochromatic double star on at least $\frac{n}{r-1}$ vertices?*

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For $r = 2$, we have a monochromatic double star on at least $\frac{3n+1}{4}$ vertices in any 2-colouring of $K_n$. By considering Paley graphs or random graphs, the value $\frac{3n}{4} + O(1)$ is tight. Thus, $r \geq 3$ in Question 4 is important.
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**Theorem 6 (Faudree, Lesniak, Schiermeyer 2009)**

*For any 2-colouring of $K_n$ ($n \geq 6$), there exists a monochromatic cycle with length at least $\lceil \frac{2n}{3} \rceil$.***
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For any 2-colouring of $K_n$ ($n \geq 6$), there exists a monochromatic cycle with length at least $\lceil \frac{2n}{3} \rceil$.

Clearly best possible, by taking the 2-colouring of $K_n$ where one colour induces a clique on $\lceil \frac{2n}{3} \rceil$ vertices.
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Let $f(n, r)$ be the maximum integer $\ell$ such that, every $r$-colouring of $K_n$ contains a monochromatic cycle of length at least $\ell$. The affine plane construction gives $f(n, r) < \frac{n}{r-1} + r$ if $r - 1$ is a prime power. Inspired by this, they also conjectured:
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**Conjecture 7 (Faudree, Lesniak, Schiermeyer 2009)**

*For \( r \geq 3 \) and \( n \) sufficiently large, we have \( f(n, r) \geq \frac{n}{r-1} \).*
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**Theorem 6 (Faudree, Lesniak, Schiermeyer 2009)**

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Fujita, Lesniak, Tóth (2015) showed that Conjecture 7 holds when $n$ is linear in $r$, with $r$ sufficiently large.
Recall: A graph $H$ is $k$-connected if $|V(H)| > k$, and for all $C \subset V(H)$ with $|C| < k$, the graph $H - C$ is connected.
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For \( n > 4(k - 1) \), we have \( m(n, 2, k) = n - 2k + 2 \).
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For $n > 4(k - 1)$, we have $m(n, 2, k) = n - 2k + 2$.
True for:

- $k = 1$ (Erdős and Rado observation);
- $k = 2$ (Bollobás, Gyárfás 2003);
- $k = 3$ (L., Morris, Prince 2004);
- $n \geq 13k - 15$ (L., Morris, Prince 2004);
- $n > 6(4k - 5)$ (Fujita, Magnant 2011).
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- $k = 3$ (L., Morris, Prince 2004);
- $n \geq 13k - 15$ (L., Morris, Prince 2004);
- $n > 6.5(k - 1)$ (Fujita, Magnant 2011).
For $r \geq 3$, Liu, Morris, Prince gave a construction which shows $m(n, r, k) < \frac{n-k+1}{r-1} + r$ if $r - 1$ is a prime power. They conjectured:
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**Theorem 10 (L., Morris, Prince 2004)**

(a) For $r \geq 3$, we have $m(n, r, k) \geq \frac{n}{r-1} - 11k(k - 1)r$. Hence, if $k, r$ are fixed and $r - 1$ is a prime power, then $m(n, r, k) = \frac{n}{r-1} + O(1)$.

(b) For $n \geq 480k$, we have $m(n, 3, k) \geq \frac{n-k+1}{2}$. 
Gallai colourings

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*Any Gallai colouring of a complete graph can be obtained by substituting complete graphs with Gallai colourings for the vertices of a 2-coloured complete graph on at least two vertices.*
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Theorem 11 is a “decomposition theorem”. It is widely used to prove results about Gallai colourings.
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**Theorem 12**

*In every Gallai colouring of $K_n$, there is a monochromatic ...*

Example where such an extension does not hold is when we want to find a monochromatic star. For any 2-colouring of $K_n$, there is a monochromatic star on at least about $n^2$ (sharp). But:

**Theorem 13 (Gyárfás, Simonyi 2004)**

For every Gallai colouring of $K_n$, there is a monochromatic star with at least $2n^5$ vertices. This bound is sharp.
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Let $r \geq 3$ and $k \geq 2$. If $n \geq (r+11)(k-1) + 7k \log k$. Then in any Gallai colouring of $K_n$ with $r$ colours, there is a monochromatic $k$-connected subgraph on at least $n - r(k - 1)$ vertices.

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Independence number

Now we consider: What if we colour the edges of a graph $G$, where the independence number $\alpha(G)$ is fixed?

Theorem 16 (Gyárfás, Sárközy 2010)

For every 2-colouring of a graph $G$ with $n$ vertices and $\alpha(G) = \alpha$, there exists a monochromatic connected subgraph on at least $\lceil n/\alpha \rceil$ vertices. This result is sharp.

They remarked that this can be extended to $r$-colourings, with $\alpha(r-1)$ in the role of $\alpha$.

Theorem 17 (Gyárfás, Sárközy 2010)

For every Gallai colouring of a graph $G$ with $n$ vertices and $\alpha(G) = \alpha$, there exists a monochromatic connected subgraph on at least $n\alpha^2 + \alpha - 1$ vertices. This is close to being tight.
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**Theorem 18 (L. 2011)**

Let $G$ be a graph with $n$ vertices and $\alpha(G) = \alpha$. If $n > \alpha^2 k$, then $G$ contains a $k$-connected subgraph on at least $\lceil \frac{n}{\alpha} \rceil$ vertices.
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**Problem 20**

What happens for the edge-coloured version?
References


Thank you!