Rainbow Connection in Hypergraphs

Henry Liu

Universidade Nova de Lisboa, Portugal

Joint work with Rui Carpentier, Manuel Silva and Teresa Sousa

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All graphs and hypergraphs are simple and finite.
Introduction

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Definition

The *rainbow connection number* $rc(G)$ of a connected graph $G$ is the minimum number of colours needed to colour the edges of $G$ such that, any two vertices are connected by a path with distinct colours (i.e., a *rainbow path*).
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The study of $rc(G)$ has since attracted a lot of interest. Many generalisations and variant functions have been considered. Recently, a survey by Li, Shi and Sun, and a book by Li and Sun, were published on the rainbow connection subject.
Example (Chartrand et al., 2008)

\[ rc(G) = e(G) \text{ if and only if } G \text{ is a tree.} \]
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A (Berge) **path** consists of distinct vertices $v_1, \ldots, v_{\ell + 1}$ and distinct edges $e_1, \ldots, e_\ell$ such that $v_i, v_{i+1} \in e_i$ for $1 \leq i \leq \ell$. 
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![Hypergraph Path Diagram](image)
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**Definition**

The rainbow connection number $rc(\mathcal{H})$ of a connected hypergraph $\mathcal{H}$ is the minimum number of colours needed to colour the edges of $\mathcal{H}$ such that, any two vertices are connected by a rainbow path.
Definition

For $1 \leq s < r$, an $(r, s)$-path is an $r$-uniform interval hypergraph where every two consecutive edges intersect in $s$ vertices.
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The $(r, s)$-rainbow connection number $rc(\mathcal{H}, r, s)$ of a connected hypergraph $\mathcal{H}$ is the minimum number of colours needed to colour the edges of $\mathcal{H}$ such that, any two vertices are connected by a rainbow $(r, s)$-path (if the minimum exists).
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Hence, we have two generalisations of $rc(G)$ to hypergraphs.
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Theorem 1 (CLSS, 2012)

Let $\mathcal{H}$ be a connected hypergraph with $e(\mathcal{H}) \geq 1$. Then, $rc(\mathcal{H}) = e(\mathcal{H})$ if and only if $\mathcal{H}$ is minimally connected.
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It is not true that $rc(\mathcal{H}) = e(\mathcal{H})$ if and only if $\mathcal{H}$ is a hypertree.
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\[ \text{rc}(\mathcal{H}) = 2 \]
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\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

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**Definition**

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\begin{align*}
\text{rc}(\mathcal{H}) &= 2 \\
\text{e}(\mathcal{H}) &= 3
\end{align*}
\]

Minimally connected hypergraphs and hypertrees are two rather different families, unlike in the graphs setting.
Definition

For $n > r \geq 2$, the $(n, r)$-cycle is the $r$-uniform hypergraph $C_r^n$...
Hypergraph Cycles

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$rc(C_{n}^{r}, r, s)$ well-defined for all $1 \leq s < r$. We consider $r \geq 3$. 

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Rainbow Connection in Hypergraphs
Theorem 2 (CLSS, 2012)

Let $n > r \geq 3$. Then for sufficiently large $n$,

(a) $rc(C_n^r) = rc(C_n^r, r, 1) = \lceil \frac{n}{2(r-1)} \rceil$.

(b) $rc(C_n^r, r, r - 1) = \lceil \frac{n}{2} \rceil$.

(c) $rc(C_n^r, r, s) \in \{d, d + 1\}$ for $1 \leq s \leq r - 2$, where

$$d = \lceil \frac{n+1-2s}{2(r-s)} \rceil,$$

the "$(r, s)$-diameter of $C_n^r$".

Hence, $rc(C_n^r, r, s) = \lceil \frac{n}{2(r-s)} \rceil + O(1)$ for all $1 \leq s < r$. 
**Theorem 2 (CLSS, 2012)**

Let \( n > r \geq 3 \). Then for sufficiently large \( n \),

(a) \( rc(C^r_n) = rc(C^r_n, r, 1) = \lceil \frac{n}{2(r-1)} \rceil \).

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the "(r, s)-diameter of \( C^r_n \)".

Hence, \( rc(C^r_n, r, s) = \left\lceil \frac{n}{2(r-s)} \right\rceil + O(1) \) for all \( 1 \leq s < r \).

**Proof (sketch).**

Lower bound: Computation of \( d \) is easy \( \Rightarrow \) (a) (except for \( n \equiv 1 \) (mod \( 2(r-1) \))) and (c). Remaining lower bounds also easy.
Theorem 2 (CLSS, 2012)

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\( d = \lceil \frac{n+1-2s}{2(r-s)} \rceil \), the “\((r, s)\)-diameter of \( C^r_n \)”.

Hence, \( \text{rc}(C^r_n, r, s) = \lceil \frac{n}{2(r-s)} \rceil + O(1) \) for all \( 1 \leq s < r \).

Proof (sketch).

Lower bound: Computation of \( d \) is easy \( \Rightarrow \) (a) (except for \( n \equiv 1 \) (mod 2(\( r - 1 \)))) and (c). Remaining lower bounds also easy.
Upper bound: Colour a “wraparound” \((r, s)\)-path with the required number of colours. \( \square \)
Complete Multipartite Hypergraphs

Definition

For $t \geq r \geq 2$ and $1 \leq n_1 \leq \cdots \leq n_t$, the $r$-uniform hypergraph $K_{n_1, \ldots, n_t}$ is ...
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\[
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$\mathcal{K}^{3}_{2,2,3,4}$

$\mathcal{K}^{r}_{n_1,\ldots,n_t}$ is a complete multipartite hypergraph.
\( \text{rc}(K^2_{n_1, \ldots, n_t}) \) determined by Chartrand et al. Let \( r \geq 3 \).
rc($K_{n_1,\ldots,n_t}^2$) determined by Chartrand et al. Let $r \geq 3$.

**Theorem 3 (CLSS, 2012)**

Let $t \geq r \geq 3$ and $1 \leq n_1 \leq \cdots \leq n_t$. Then,

$$rc(K_{n_1,\ldots,n_t}^r) = \begin{cases} 
1 & \text{if } n_t = 1, \\
2 & \text{if } n_{t-1} \geq 2, \text{ or } t > r, n_{t-1} = 1 \text{ and } n_t \geq 2, \\
n_t & \text{if } t = r \text{ and } n_{t-1} = 1.
\end{cases}$$

Proof not too difficult. For the last case, $K_{n_1,\ldots,n_t}^r$ is minimally connected.
rc($\mathcal{K}^2_{n_1,...,n_t}$) determined by Chartrand et al. Let $r \geq 3$.

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\[
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\end{cases}
\]

Proof not too difficult. For the last case, \( K_{n_1,\ldots,n_t}^r \) is minimally connected.

More interesting to consider \( rc(K_{n_1,\ldots,n_t}^r, r, s) \).
Theorem 4 (CLSS, 2012)

Let $t \geq r \geq 3$, $s \leq r - 2$ and $1 \leq n_1 \leq \cdots \leq n_t$ be such that $n_t = 1$ or $n_{2(t-r)+s+1} \geq 2$. Then,

$$rc(K_{n_1,\ldots,n_t}, r, s) = \begin{cases} 1 & \text{if } n_t = 1, \\ 2 & \text{if } n_t \geq 2. \end{cases}$$
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$$rc(K_{n_1,\ldots,n_t}^r, r, s) = \begin{cases} 1 & \text{if } n_t = 1, \\ 2 & \text{if } n_t \geq 2. \end{cases}$$

Proof still not difficult. But surprisingly, $rc(K_{n_1,\ldots,n_t}^r, r, r - 1)$ is a lot more difficult to determine ...
Theorem 5 (CLSS, 2012)

Let \( t \geq r \geq 3 \), \( 1 \leq n_1 \leq \cdots \leq n_t \), \( n = n_t \) and \( b = \sum_S \prod_{i \in S} n_i \), where \( S \) ranges over all subsets of \( \{1, \ldots, t-1\} \) with size \( r-1 \). Then,

\[
rc(H_{n_1,\ldots,n_t}^{r}, r, r-1) = \begin{cases} 
1 & \text{if } n_t = 1, \\
\lceil b\sqrt{n} \rceil & \text{if } t = r \text{ and } n_1 = 1, \\
\min(\lceil b\sqrt{n} \rceil, r+2) & \text{if } t = r \text{ and } n_1 \geq 2, \\
\min(\lceil b\sqrt{n} \rceil, 3) & \text{if } t > r.
\end{cases}
\]
Proof (sketch).
Upper bound \( rc(K_{n_1, \ldots, n_t}, r, r-1) \leq \lceil \sqrt[n]{n} \rceil =: k \).
Proof (sketch).
Upper bound $rc(\mathcal{K}_{n_1,...,n_t}^r, r, r - 1) \leq \lceil \sqrt[n]{k} \rceil =: k$. 

\begin{center}
\begin{tikzpicture}
  \foreach \x in {1,2,...,t-1} {
    \node[draw,ellipse] (n\x) at (\x,0) {$n_{\x}$};
  }
  \node[draw,ellipse] (nt) at (t,0) {$n_{t-1}$};
  \node[draw,ellipse] (nt1) at (t+0.5,0) {\ldots};
  \node[draw,ellipse] (nt2) at (t+2,0) {$n_t$};
  \path (n1) edge (n2) edge (nt1); 
\end{tikzpicture}
\end{center}
Proof (sketch).

Upper bound $rc(\mathcal{H}_{n_1, \ldots, n_t, r, r-1}) \leq \lceil b\sqrt{n} \rceil = k$. 

\[ n_1 \quad n_2 \quad \ldots \quad n_{t-1} \]

\[ \begin{array}{c}
  w \\
  \downarrow \\
  (t-1)\text{-tuple of functions } \{W^{S_1}, W^{S_2}, \ldots\} \text{ s.t. } w \neq w' \text{ get distinct tuples}
\end{array} \]

\[ n_t \]
Proof (sketch).

Upper bound $\text{rc}(\mathcal{K}^r_{n_1,\ldots,n_t}, r, r - 1) \leq \lceil \sqrt{b/n} \rceil =: k$. 

\[
\begin{array}{cccc}
  n_1 & n_2 & \cdots & n_{t-1} \\
  \hline
\end{array}
\]

\[
\begin{array}{c}
  n_t \\
  \hline
\end{array}
\]

\[
({t-1\choose r-1})\text{-tuple of functions } \{W^{S_1}, W^{S_2}, \ldots\} \text{ s.t. } w \neq w' \text{ get distinct tuples}
\]

If $W^{S_j}(i_1, \ldots, i_{r-1}) = c \in \{1, \ldots, k\}$
Proof (sketch).

Upper bound $rc(K_{r,n_1,...,n_t}, r, r - 1) \leq \lceil \sqrt[n]{n} \rceil =: k$.

$(t-1)$-tuple of functions $\{W^{S_1}, W^{S_2}, \ldots \}$ s.t. $w \neq w'$ get distinct tuples

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Proof (sketch).

Upper bound $\text{rc}(\mathcal{K}^{r}_{n_1,\ldots,n_t}, r, r - 1) \leq \lceil \frac{b}{\sqrt{n}} \rceil =: k$.

$(t-1)$-tuple of functions $\{W^{S_1}, W^{S_2}, \ldots\}$ s.t. $w \neq w'$ get distinct tuples

If $W^{S_j}(i_1, \ldots, i_{r-1}) = c \in \{1, \ldots, k\}$

This is a good $k$-colouring.
Proof (sketch).

Upper bound $rc(K^{r}_{\prod_{1}^{n_{1}}, \ldots, \prod_{t}^{n_{t}}}, r, r-1) \leq \lceil b\sqrt{n} \rceil =: k$.

\[
\binom{t-1}{r-1}-\text{tuple of functions } \{W^{S_{1}}, W^{S_{2}}, \ldots\} \text{ s.t. } w \neq w' \text{ get distinct tuples}
\]

If $W^{S_{j}}(i_{1}, \ldots, i_{r-1}) = c \in \{1, \ldots, k\}$

This is a good $k$-colouring.

$rc(K^{r}_{\prod_{1}^{n_{1}}, \ldots, \prod_{t}^{n_{t}}}, r, r-1) \leq r + 2$ if $t = r$ and $n_{1} \geq 2$; and

$rc(K^{r}_{\prod_{1}^{n_{1}}, \ldots, \prod_{t}^{n_{t}}}, r, r-1) \leq 3$ if $t > r$: similar idea.
Proof (sketch, ctd.).

Lower bound for $t > r$: $rc(\mathcal{K}_{n_1,\ldots,n_t}^r, r, r-1) \geq \min(\lceil b\sqrt{n} \rceil, 3)$. 
Proof (sketch, ctd.).

Lower bound for $t > r$: $rc(K_{n_1,\ldots,n_t}, r, r-1) \geq \min(\lceil b\sqrt{n} \rceil, 3)$. Take a 2-colouring.
Proof (sketch, ctd.).

Lower bound for $t > r$: $rc(K_{n_1,\ldots,n_t}, r, r-1) \geq \min(\lceil b\sqrt{n} \rceil, 3)$. Take a 2-colouring.

\[ \begin{align*}
    n_1 & \quad n_2 & \quad \cdots & \quad n_{t-1} \\
    n_t & \\
\end{align*} \]
Proof (sketch, ctd.).

Lower bound for $t > r$: $rc(K_{n_1, \ldots, n_t, r, r-1}) \geq \min\left(\lceil \sqrt{b} \cdot \sqrt{n} \rceil, 3 \right)$. Take a 2-colouring.

$c \in \{1, 2\}$
Proof (sketch, ctd.).

Lower bound for \( t > r \): \( \text{rc}(K_{n_1, \ldots, n_t, r, r-1}) \geq \min(\lceil b\sqrt{n} \rceil, 3) \). Take a 2-colouring.

\[
\begin{align*}
\text{c} \in \{1, 2\} \\
(t-1)\text{-tuple of functions } \{W_{S_1}, W_{S_2}, \ldots\} \text{ s.t. } W_{S_j}(i_1, \ldots, i_{r-1}) = c
\end{align*}
\]
Proof (sketch, ctd.).

Lower bound for $t > r$: $rc(K_{n_1, \ldots, n_t}, r, r - 1) \geq \min(\lceil b\sqrt{n} \rceil, 3)$. Take a 2-colouring.

$n_1$  $n_2$  $\cdots$  $n_{t-1}$

$c \in \{1, 2\}$

$(t-1)$-tuple of functions $\{W^{S_1}, W^{S_2}, \ldots\}$ s.t. $W^{S_j}(i_1, \ldots, i_{r-1}) = c$

Then $\exists w, w'$ with the same $(t-1)$-tuples
Proof (sketch, ctd.).

Lower bound for \( t > r \): \( rc(K_{n_1,...,n_t}, r, r - 1) \geq \min(\lceil b\sqrt{n} \rceil, 3) \). Take a 2-colouring.

\[
\begin{align*}
&n_1 \quad n_2 \quad \cdots \quad n_{t-1} \\
&i_1 \quad i_2 \quad \cdots \quad i_{r-1} \\
&\quad \quad \quad \quad \quad w \\
&\quad \quad \quad \quad \quad n_t
\end{align*}
\]

\( c \in \{1, 2\} \)

\((t-1)\)-tuple of functions \( \{W^{S_1}, W^{S_2}, \ldots\} \) s.t. \( W^{S_j}(i_1, \ldots, i_{r-1}) = c \)

Then \( \exists w, w' \) with the same \( (t-1)\)-tuples \( \not\exists \) rainbow \( w - w' \)
\((r, r - 1)\)-path, and the 2-colouring is bad.
Proof (sketch, ctd.).

Lower bound for $t > r$: $rc(K_{n_1, \ldots, n_t, r, r-1}) \geq \min(\lceil \sqrt[2]{n} \rceil, 3)$. Take a 2-colouring.

\[
\begin{align*}
\text{Consider a } (t-1)\text{-tuple of functions } \{W^{S_1}, W^{S_2}, \ldots\} \text{ s.t. } W^{S_j}(i_1, \ldots, i_{r-1}) = c \\
\text{Then } \exists w, w' \text{ with the same } (t-1)\text{-tuples } \Rightarrow \text{no rainbow } w - w' \text{ } (r, r-1)\text{-path, and the 2-colouring is bad.}
\end{align*}
\]

Other lower bounds: similar. □
Separation Results

**Theorem 6 (CLSS, 2012)**

Let $a > 0$, $r \geq 3$ and $1 \leq s \neq s' < r$. Then, there exists an $r$-uniform hypergraph $\mathcal{H}$ such that $rc(\mathcal{H}, r, s) - rc(\mathcal{H}) \geq a$, and there exists an $r$-uniform hypergraph $\mathcal{H}$ such that $rc(\mathcal{H}, r, s) - rc(\mathcal{H}, r, s') \geq a$. 

Proof. First part with $s > 1$, and second part with $s > s'$, take $\mathcal{H} = C_r^n$. $\Rightarrow$ Difference is at least $n^2(r - s) - n^2(r - s') + O(1) \geq \Omega(n^2)$. 

Henry Liu

Rainbow Connection in Hypergraphs
Theorem 6 (CLSS, 2012)

Let $a > 0$, $r \geq 3$ and $1 \leq s \neq s' < r$. Then, there exists an $r$-uniform hypergraph $\mathcal{H}$ such that $rc(\mathcal{H}, r, s) - rc(\mathcal{H}) \geq a$, and there exists an $r$-uniform hypergraph $\mathcal{H}$ such that $rc(\mathcal{H}, r, s) - rc(\mathcal{H}, r, s') \geq a$.

Proof.
First part with $s > 1$, and second part with $s > s'$, take $\mathcal{H} = \mathcal{C}_n^r$.

$\Rightarrow$ Difference is at least

$$\frac{n}{2(r - s)} - \frac{n}{2(r - s')} + O(1) \geq \Omega(n).$$
Proof (ctd.).

First part with $s = 1$ and second part with $s < s'$, take the following construction for $\mathcal{H}$. 
Proof (ctd.).

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![Diagram](image.png)
Proof (ctd.).

First part with $s = 1$ and second part with $s < s'$, take the following construction for $\mathcal{H}$.

\begin{align*}
&\text{Above colouring } \Rightarrow r_c(\mathcal{H}) = r_c(\mathcal{H}, r, s') = 2. \\
&\text{Considering } u \text{ and } v \Rightarrow r_c(\mathcal{H}, r, s) \geq \text{length of } (r, s)-\text{path}. \\
&\square
\end{align*}
Proof (ctd.).

First part with $s = 1$ and second part with $s < s'$, take the following construction for $\mathcal{H}$.

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Above colouring $\Rightarrow \text{rc}(\mathcal{H}) = \text{rc}(\mathcal{H}, r, s') = 2$.
Considering $u$ and $v$ $\Rightarrow \text{rc}(\mathcal{H}, r, s) \geq \text{length of } (r, s)$-path. □
References


Thank you!