Rainbow cycles through specified vertices

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An edge-coloured graph is *rainbow* if its edges have distinct colours. For $G \in \mathcal{F}_k$, the *$k$-rainbow cycle index* of $G$, denoted by $\text{crx}_k(G)$, is the least number of colours needed to colour the edges of $G$ so that every $k$ vertices lie on a rainbow cycle.
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- Many more related problems and results …
Application: A certain country has various transport modes between certain pairs of cities. A travelling salesman wishes to visit $k$ designated cities, in no particular order, and return to his originating city. He can travel via other cities, but he does not wish to visit any city, or use the same mode of transport, more than once. How few transport modes are needed so that this is possible for any choice of $k$ cities? The answer is precisely $crx_k(G)$. 
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\( \text{crx}_k(G) \) is the “cycle version” of the rainbow index \( \text{rx}_k(G) \) (Chartrand, Okamoto, Zhang, 2010), i.e., \( \text{rx}_k(G) \) is the least number of colours needed to colour the edges of a connected graph \( G \) so that every \( k \) vertices lie in a rainbow tree.
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\( G, H \in \mathcal{F}_k, H \subset G \) spanning subgraph \( \Rightarrow \text{crx}_k(G) \leq \text{crx}_k(H) \).
Proposition 1

(a) $\mathcal{F}_1$ is the family of all graphs such that, every vertex belongs to a 2-connected block in the block decomposition.

(b) $\mathcal{F}_2$ is the family of all 2-connected graphs.
General graphs

Proposition 1

(a) $\mathcal{F}_1$ is the family of all graphs such that, every vertex belongs to a 2-connected block in the block decomposition.

(b) $\mathcal{F}_2$ is the family of all 2-connected graphs.

Theorem 2 (L. 2017+)

Let $G$ be a graph of order $n \geq 3$.

(a) Let $G \in \mathcal{F}_1$. Then $crx_1(G) = e(G)$ if and only if $G = C_n$.

(b) Let $G \in \mathcal{F}_2$. Then $crx_2(G) = e(G)$ if and only if $G$ is minimally 2-connected.

(c) Let $G \in \mathcal{F}_n$ (i.e., $G$ is Hamiltonian), and $1 \leq k \leq n$. Then $crx_k(G) = e(G)$ if and only if $G = C_n$.
Sketch proof of (b).

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- Let $G$ be minimally 2-connected, and suppose we have an edge-colouring with less than $e(G)$ colours, say $e, e' \in E(G)$ have the same colour.
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- Let $G$ be minimally 2-connected, and suppose we have an edge-colouring with less than $e(G)$ colours, say $e, e' \in E(G)$ have the same colour.
- Want to show that there are two vertices $x, y$ such that any cycle containing them must use both $e$ and $e'$. 

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$(\Leftarrow)$:

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- Can assume $e'$ lies in a 2-connected block $B$.

- Then $B - e'$ has a similar structure. Can easily find $x, y$. □
\[ k = n - 1? \]
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**Theorem 3 (L. 2017+)**

(a) If $P_{10} = \text{Petersen graph}$, then $\text{crx}_9(P_{10}) = e(P_{10}) = 15$.

(b) If $G$ is a hypohamiltonian graph of order $n$, where $11 \leq n \leq 17$, then $\text{crx}_{n-1}(G) < e(G)$. 
Sketch proof of (b).

Aldred, McKay, Wormald (1995) showed that all hypohamiltonian graphs of order at most 17 are:

\[ P_{10}, H_{13}, H_{15}, H_{16}, H'_{16}, H^1_{16}, H^2_{16} \]
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Easy to check that for \( G \in \{ H_{13}, H_{15}, H_{16}, H'_{16} \} \), \( G - v \) contains a Hamilton cycle not using both red edges, for all \( v \in V(G) \). □
Problem 4

Let $3 \leq k < n$. Characterise the graphs $G \in \mathcal{F}_k$ of order $n$ with $\text{crx}_k(G) = e(G)$.
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Let $3 \leq k < n$. Characterise the graphs $G \in \mathcal{F}_k$ of order $n$ with $\text{crx}_k(G) = e(G)$. In particular, when $k = n - 1$, does there exist $G$ such that $\text{crx}_{n-1}(G) = e(G)$, other than $G = C_n$, or $n = 10$ and $G = Petersen$ graph?
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**Theorem 5 (L. 2017+)**

(a) $\text{crx}_1(W_n) = 3$ for $n \geq 3$.
(b) $\text{crx}_2(W_3) = 3$, and $\text{crx}_2(W_n) = \lceil \frac{n}{2} \rceil + 2$ for $n \geq 4$.
(c) $\text{crx}_3(W_n) = \begin{cases} n & \text{if } 3 \leq n \leq 7, \\ n - 1 & \text{if } 8 \leq n \leq 11, \\ n - 2 & \text{if } n \geq 12. \end{cases}$
(d) For $k \geq 4$ and $n + 1 \geq k$, we have

$$crx_k(W_n) = \begin{cases} 
  n + 1 & \text{if } n < 2k, \\
  n & \text{if } n \geq 2k.
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- Obviously $\text{crx}_k(W_n) \leq n + 1$ since $W_n$ is Hamiltonian.
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- Obviously $crx_k(W_n) \leq n + 1$ since $W_n$ is Hamiltonian.
- For $n \geq 2k$, not hard to define an edge-colouring which gives $crx_k(W_n) \leq n.$
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- Obviously $crx_k(W_n) \leq n + 1$ since $W_n$ is Hamiltonian.
- For $n \geq 2k$, not hard to define an edge-colouring which gives $crx_k(W_n) \leq n$.
- $crx_k(W_n) \geq n$ for all $n \geq k + 1$: If we use $\leq n - 1$ colours in an edge-colouring, then $\exists e, e' \in E(C_n)$ with the same colour. Pick $S$ with $|S| = k$ containing the end-vertices of $e, e'$. Then no rainbow cycle can contain $S$. 

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- \( \text{crx}_k(W_n) \geq n + 1 \) for all \( n < 2k \): Suppose we have a good edge-colouring with \( n \) colours. The \( C_n \) must be rainbow coloured. We consider cycles \( C \) containing \( v \) of two types – either \( |V(C)| \geq \lceil \frac{n}{2} \rceil + 2 \) or not.
\[ \text{crx}_k(W_n) \geq n + 1 \text{ for all } n < 2k: \] Suppose we have a good edge-colouring with \( n \) colours. The \( C_n \) must be rainbow coloured. We consider cycles \( C \) containing \( v \) of two types – either \( |V(C)| \geq \lceil \frac{n}{2} \rceil + 2 \) or not. We can choose a set \( S \) of \( k \) vertices containing \( v \) such that, no rainbow cycle of either type can contain \( S \), a contradiction. □
Proposition 6 (L. 2017+)

\( crx_1(K_n) = crx_2(K_n) = 3 \) for \( n \geq 3 \).
Complete, complete bipartite and multipartite graphs

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\[ \text{cr}_1(K_n) = \text{cr}_2(K_n) = 3 \text{ for } n \geq 3. \]

Proof.

Suffices to show: \( \text{cr}_2(K_n) \leq 3 \). Use induction on \( n \).
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Theorem 7 (L. 2017+)

For $k \geq 3$, $\text{cr}_k(K_n) = 2k - 1$ for $n \geq N(k)$ sufficiently large.
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($\geq$): Let $n \geq R_{2k-2}(k)$, the Ramsey number for $2k - 2$ $K_k$'s.
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$\Rightarrow$ If $E(K_n)$ is $r$-coloured, $r \leq 2k - 2$, $\exists$ monochromatic $K_k$ on some $\{v_1, \ldots, v_k\}$. 
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$\Rightarrow$ $\text{cr}_{x_k}(K_n) \geq 2k - 1$. 
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For \(A \subset V(K_n), |A| = k\), let \(E_A = \text{event that no rainbow cycle } \supset A\). Done if \(P(E_A) = o(n^{-k})\), since then \(P(\bigcup_A E_A) < 1\).
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![Diagram of complete multipartite graph]

$A \bigcup k$

$B_1 \bigcup (k-1)$ \hspace{1cm} $B_2 \bigcup (k-1)$ \hspace{1cm} $\cdots$ \hspace{1cm} $(k-1) \bigcup B_t$
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$E_A$ occurs $\Rightarrow$ no $A \cup B_i$ contains a rainbow cycle $\supset A$. 
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$\mathbb{P}(E_A) \leq d_k c^t (2k - 1)^{\binom{n-k}{2}} / (2k - 1)^{\binom{n}{2}} = o(n^{-k})$, where $d_k$ and $c < (2k - 1)^{k(k-1)}$ are constants.
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For $A \subset V(K_n)$, $|A| = k$, let $E_A = \text{event that no rainbow cycle } \supset A$. Done if $\mathbb{P}(E_A) = o(n^{-k})$, since then $\mathbb{P}(\bigcup_A E_A) < 1$.

\[ E_A \text{ occurs } \Rightarrow \text{ no } A \cup B_i \text{ contains a rainbow cycle } \supset A. \]

\[ \mathbb{P}(E_A) \leq d_k c^t (2k - 1)^{\binom{n-k}{2}} / (2k - 1)^{\binom{n}{2}} = o(n^{-k}), \text{ where } d_k \text{ and } c < (2k - 1)^{k(k-1)} \text{ are constants } \Rightarrow \text{crx}_k(K_n) \leq 2k - 1. \]
Theorem 8 (L. 2017+)

(a) For $k \geq 3$, $\text{cr}_{x_k}(K_{n,n}) = 2k$ for $n \geq N(k)$ sufficiently large.

(b) For $k, t \geq 2$, $\text{cr}_{x_k}(K_{t \times n}) = 2k$ for $n \geq N(k, t)$ sufficiently large.
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($\geq$): Easy; consider $k$ vertices in one class.
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Problem 9

What is $crx_k(K_{m,n})$ for $1 \leq k \leq m \leq n$?
Discrete Cube

Recall: The *discrete cube* $Q_n$ consists of $2^n$ vertices labelled by all $(0, 1)$-vectors of length $n$, and $x \sim y$ iff $x$ and $y$ differ in exactly one coordinate.
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Aim: To determine $crx_k(Q_n)$.
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- Faudree, Gyárfás, Lesniak, Schelp (1993) proved that for $n \geq 4$, $n \neq 5$, there exists an edge-colouring of $Q_n$, using $n$ colours, such that every $C_4$ is rainbow.
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- Mubayi and Stading (2013) proved that for $\ell \equiv 0 \pmod{4}$, there exists an edge-colouring of $Q_n$ with $\Theta_n(n^{\ell/4})$ colours such that, every copy of $C_\ell$ is rainbow, and moreover, $\Theta_n(n^{\ell/4})$ colours are also necessary.
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- Many others ...
Theorem 10 (L. 2017+)

For $n \geq 2,$
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For $n \geq 2$,

(a) $\text{cr}_{x_1}(Q_n) = 4$, 

(b) $\text{cr}_{x_2}(Q_n) = \text{cr}_{x_3}(Q_n) = 2$, 

(c) $\text{cr}_{x_k}(Q_n) = 2n$ for $2n - 1 \leq k \leq 2n$. 

Proof of (b).

$\text{cr}_{x_2}(Q_n) \geq 2n$: Shortest cycle containing $(0, \ldots, 0)$ and $(1, \ldots, 1)$ has length $2n$. 

Henry Liu
Rainbow cycles through specified vertices
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\[ \text{crx}_3(Q_n) \leq 2n: \text{ Induction on } n. \]
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n \geq 3:
$crx_3(Q_n) \leq 2n$: Induction on $n$. $n = 2$: Rainbow colour $Q_2$.

$n \geq 3$:

Each $Q_{n-1}$ coloured with colours $1, \ldots, 2n - 2$. 

$Q_{n-1}$ $Q_{n-1}$

$u$ $u'$
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Each $Q_{n-1}$ coloured with colours $1, \ldots, 2n - 2$. Colour $uu'$ with colour $2n - 1$ if $\sum_{i<n} u_i = \sum_{i<n} u'_i$ is odd, and with colour $2n$ if even.
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\[ n \geq 3: \]

Each \( Q_{n-1} \) coloured with colours 1, \ldots, 2n – 2. Colour \( uu' \) with colour 2n – 1 if \( \sum_{i<n} u_i = \sum_{i<n} u'_i \) is odd, and with colour 2n if even. Easy to check that this is a good colouring for \( Q_n \).  \( \square \)
Theorem 11 (L. 2017+)

Let $k \geq 4$ and $n \geq 4k^2$. Then there exist constants $c_k, C_k > 0$ (depending only on $k$) such that

$$c_k n \leq \text{crx}_k(Q_n) \leq C_k n$$

Thus, we have $\text{crx}_k(Q_n) = \Theta_n(n)$. 
Sketch of proof.

Lower bound.
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Lower bound.

- We prove that $\text{cr}x_k(Q_n) > \frac{kn}{2}$ for $n \geq 4k^2$. We will find $k$ vertices in $Q_n$ such that every pair is at distance $> \frac{n}{2}$.
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- Recall that a Hadamard matrix is a \((-1, 1)\) matrix where every two column vectors are orthogonal, i.e., number of agreeing coordinates \( = \) number of differing coordinates.
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- Let \( k' \) be the smallest power of 2 with \( k' \geq k \), and \( H_{k'} \) be a \( k' \times k' \) Hadamard matrix (exists by Sylvester’s construction).
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- Delete the all 1’s row, replace all $-1$ with 0, and take $k$ of the column $(k' - 1)$-vectors.
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- Delete the all 1’s row, replace all $-1$ with 0, and take $k$ of the column $(k' - 1)$-vectors.
- “Blow up” the $(k' - 1)$-vectors to $n$-vectors.
Upper bound.
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▶ We will prove a stronger assertion: *For \( n \geq 11k \) and \( C_k = 2^{2k-1} \), there is an edge-colouring with at most \( C_k n \) colours such that, for any \( S = (v_1, \ldots, v_k) \in V(Q_n)^k \), there exists a closed walk containing \( S \) such that if \( P_i \) connects \( v_i \) and \( v_{i+1} \), then \( P_i \) has length at least 2 or is trivial (if \( v_i = v_{i+1} \)), with the \( P_i \) disjoint.*
Upper bound.

- We will prove a stronger assertion: For \( n \geq 11k \) and \( C_k = 2^{22k-1} \), there is an edge-colouring with at most \( C_k n \) colours such that, for any \( S = (v_1, \ldots, v_k) \in V(Q_n)^k \), there exists a closed walk containing \( S \) such that if \( P_i \) connects \( v_i \) and \( v_{i+1} \), then \( P_i \) has length at least 2 or is trivial (if \( v_i = v_{i+1} \)), with the \( P_i \) disjoint.

- Recall: A graph is \textit{k-linked} if it has at least \( 2k \) vertices, and for any sequence \( s_1, \ldots, s_k, t_1, \ldots, t_k \) of distinct vertices, there are disjoint paths \( P_1, \ldots, P_k \) where \( P_i \) connects \( s_i \) and \( t_i \).
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- Thomas and Wollan (2005) proved that if \( G \) is \( 10k \)-connected, then \( G \) is \( k \)-linked.
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- Using this, we can show that for \( S \in V(Q_n)^k \), there exists such a closed walk in \( Q_n \) containing \( S \).
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- We will prove a stronger assertion: For $n \geq 11k$ and $C_k = 2^{22k-1}$, there is an edge-colouring with at most $C_k n$ colours such that, for any $S = (v_1, \ldots, v_k) \in V(Q_n)^k$, there exists a closed walk containing $S$ such that if $P_i$ connects $v_i$ and $v_{i+1}$, then $P_i$ has length at least 2 or is trivial (if $v_i = v_{i+1}$), with the $P_i$ disjoint.

- Recall: A graph is $k$-linked if it has at least $2k$ vertices, and for any sequence $s_1, \ldots, s_k, t_1, \ldots, t_k$ of distinct vertices, there are disjoint paths $P_1, \ldots, P_k$ where $P_i$ connects $s_i$ and $t_i$.

- Thomas and Wollan (2005) proved that if $G$ is 10$k$-connected, then $G$ is $k$-linked.

- Using this, we can show that for $S \in V(Q_n)^k$, there exists such a closed walk in $Q_n$ containing $S$.

- For $n = 11ak$, $Q_n = Q_{11(a-1)k} \oplus Q_{11k}$. Use induction on $a$. \qed
We proved Theorem 11 with $c_k = \frac{k}{2}$ and $C_k = 2^{2k-1}$. 

Problem 12

For $n \geq 4$, determine $c_{rk}(Q_n)$ for every $4 \leq k < 2^{n-1}$. Or, improve the constants $c_k$, $C_k$ in Theorem 11.
We proved Theorem 11 with $c_k = \frac{k}{2}$ and $C_k = 2^{22k-1}$.

**Problem 12**

For $n \geq 4$, determine $crx_k(Q_n)$ for every $4 \leq k < 2^{n-1}$. Or, improve the constants $c_k, C_k$ in Theorem 11.
Thank you!