Turán function and $H$-decomposition problem for gem graphs

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Abstract

Given a graph $H$, the Turán function $\text{ex}(n, H)$ is the maximum number of edges in a graph on $n$ vertices not containing $H$ as a subgraph. For two graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a graph isomorphic to $H$. Let $\phi(n, H)$ be the smallest number $\phi$ such that any graph $G$ of order $n$ admits an $H$-decomposition with at most $\phi$ parts. Pikhurko and Sousa conjectured that $\phi(n, H) = \text{ex}(n, H)$ for $\chi(H) \geq 3$ and all sufficiently large $n$. Their conjecture
has been verified by Özkahya and Person for all edge-critical graphs $H$. In this article, we consider the gem graphs $gem_4$ and $gem_5$. The graph $gem_4$ consists of the path $P_4$ with four vertices $a, b, c, d$ and edges $ab, bc, cd$ plus a universal vertex $u$ adjacent to $a, b, c, d$, and the graph $gem_5$ is similarly defined with the path $P_5$ on five vertices. We determine the Turán functions $ex(n, gem_4)$ and $ex(n, gem_5)$, and verify the conjecture of Pikhurko and Sousa when $H$ is the graph $gem_4$ and $gem_5$.

**Keywords:** gem graph; Turán function; extremal graph; graph decomposition

## 1 Introduction

Given a graph $H$, the *Turán function* $ex(n, H)$ is the maximum number of edges in a graph on $n$ vertices, and not containing a copy of $H$ as a subgraph. The important result of Turán [13] states that when $H = K_r$ is the complete graph on $r \geq 3$ vertices, we have $ex(n, K_r) = t_{r-1}(n)$. Here $t_{r-1}(n)$ denotes the number of edges in the *Turán graph* of order $n$, $T_{r-1}(n)$, which is the unique complete $(r-1)$-partite graph on $n$ vertices where every partition class has either $\lfloor \frac{n}{r-1} \rfloor$ or $\lceil \frac{n}{r-1} \rceil$ vertices. Moreover, $T_{r-1}(n)$ is the unique extremal graph on $n$ vertices that has the maximum number of edges not containing $K_r$ as a subgraph. For general graphs $H$, the Turán function $ex(n, H)$ has been well studied by numerous researchers, which led to many important results and open problems in extremal graph theory. For example, when $H = C_{2k}$ is the even cycle of length $2k$, where $k \geq 2$, the exact determination of the function $ex(n, C_{2k})$ is still a wide open problem. It has been conjectured that $ex(n, C_{2k}) = (c_k + o(1))n^{1+1/k}$ for some constant $c_k > 0$, and this conjecture is only known to be true for $k = 2, 3, 5$. See for example [8] and the references therein. When $H = P_k$ is the path of order $k \geq 3$, Faudree and Schelp [5] have determined the function $ex(n, P_k)$ exactly. In order to obtain $ex(n, P_k)$, we can take the graph on $n$ vertices containing as many disjoint copies of $K_{k-1}$ as possible, and a smaller complete graph on the remaining vertices. For odd $k$, this graph is the unique $P_k$-free extremal graph attaining $ex(n, P_k)$, and for even $k$ and certain values of $n$, there are other such extremal graphs. Here we state the result of Faudree and Schelp as follows, which will be useful in this paper.
Theorem 1.1. [5] Let \( k \geq 3 \) and \( n = a(k - 1) + b \), where \( a \geq 0 \) and \( 0 \leq b < k - 1 \). Then \( \text{ex}(n, P_k) = a\binom{k-1}{2} + \binom{b}{2} \). Moreover, a \( P_k \)-free graph on \( n \) vertices attaining \( \text{ex}(n, P_k) \) is \( aK_{k-1} \cup K_b \), the disjoint union of a copies of \( K_{k-1} \) and one copy of \( K_b \).

For two graphs \( G \) and \( H \), an \( H \)-decomposition of \( G \) is a partition of the edge set of \( G \) such that each part is either a single edge or forms a graph isomorphic to \( H \). Let \( \phi(G, H) \) be the smallest possible number of parts in an \( H \)-decomposition of \( G \). It is easy to see that, for non-empty \( H \), we have \( \phi(G, H) = e(G) - p_H(G)(e(H) - 1) \), where \( p_H(G) \) is the maximum number of pairwise edge-disjoint copies of \( H \) that can be packed into \( G \) and \( e(G) \) denotes the number of edges in \( G \). Dor and Tarsi [3] showed that if \( H \) has a component with at least three edges, then the problem of checking whether a graph \( G \) admits a partition into \( H \)-subgraphs is NP-complete. Thus, it is NP-hard to compute the function \( \phi(G, H) \) for such \( H \). Here we study the function

\[
\phi(n, H) = \max\{\phi(G, H) \mid v(G) = n\},
\]

which is the smallest number \( \phi \) such that any graph \( G \) of order \( n \) admits an \( H \)-decomposition with at most \( \phi \) parts.

This function was first studied, in 1966, by Erdős, Goodman and Pósa [4], who were motivated by the problem of representing graphs by set intersections. They proved that \( \phi(n, K_3) = t_2(n) \). A decade later, this result was extended by Bollobás [2], who proved that \( \phi(n, K_r) = t_{r-1}(n) \), for all \( n \geq r \geq 3 \).

General graphs \( H \) were only considered recently by Pikhurko and Sousa [9]. They proved the following result.

Theorem 1.2 (See Theorem 1.1 from [9]). Let \( H \) be any fixed graph of chromatic number \( r \geq 3 \). Then,

\[
\phi(n, H) = \text{ex}(n, H) + o(n^2).
\]

Pikhurko and Sousa also made the following conjecture.

Conjecture 1.3. [9] For any graph \( H \) of chromatic number \( r \geq 3 \), there exists \( n_0 = n_0(H) \) such that \( \phi(n, H) = \text{ex}(n, H) \) for all \( n \geq n_0 \).

A graph \( H \) is edge-critical if there exists an edge \( e \in E(H) \) such that \( \chi(H) > \chi(H - e) \), where \( \chi(H) \) denotes the chromatic number of \( H \). For \( r \geq 4 \) a clique-extension of order \( r \) is a connected graph that consists of a \( K_{r-1} \) plus another vertex.
say $v$, adjacent to at most $r - 2$ vertices of $K_{r-1}$. Conjecture 1.3 has been verified by Sousa for some edge-critical graphs, namely, clique-extensions of order $r \geq 4$ ($n \geq r$) [11] and the cycles of length 5 ($n \geq 6$) and 7 ($n \geq 10$) [10, 12]. Later, Özkahya and Person [7] verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Their result is the following.

**Theorem 1.4 (See Theorem 3 from [7]).** For any edge-critical graph $H$ with chromatic number $r \geq 3$, there exists $n_0 = n_0(H)$ such that $\phi(n, H) = \text{ex}(n, H)$, for all $n \geq n_0$. Moreover, the only graph attaining $\text{ex}(n, H)$ is the Turán graph $T_{r-1}(n)$.

Recently, as an extension of Özkahya and Person’s work, Allen, Böttcher, and Person [1] improved the error term obtained by Pikhurko and Sousa in Theorem 1.2. In fact, they proved that the error term $o(n^2)$ can be replaced by $O(n^{2-\alpha})$ for some $\alpha > 0$. Furthermore, they also showed that this error term has the correct order of magnitude. Their result is indeed an extension of Theorem 1.4 since the error term $O(n^{2-\alpha})$ that they obtained vanishes for every edge-critical graph $H$.

Conjecture 1.3 has also been verified by Liu and Sousa [6] for the $k$-fan graph $F_k$, which is the graph on $2k + 1$ vertices consisting of $k$ triangles intersecting in exactly one common vertex. Observe that $\chi(F_k) = 3$ and for $k \geq 2$ the graph $F_k$ is not edge-critical. Thus, the result of Liu and Sousa is not a particular case of Theorem 1.4 by Özkahya and Person.

In this article, we consider the gem graphs gem$_4$ and gem$_5$, defined as follows. For the graph gem$_4$, we take the path $P_4$ with vertices $a, b, c, d$ and edges $ab, bc, cd$ and add a universal vertex $u$ adjacent to $a, b, c, d$. Similarly for the graph gem$_5$, we take the path $P_5$ with vertices $a, b, c, d, e$ and edges $ab, bc, cd, de$ and add a universal vertex $u$ adjacent to $a, b, c, d, e$. See Figure 1 below. For convenience, we write $abcd + u$ and $abcde + u$ for these two graphs.

![Figure 1. The graphs gem$_4$ and gem$_5$.](image-url)
In Section 2, we will determine the Turán functions \( \text{ex}(n, \text{gem}_4) \) for \( n \geq 6 \), and \( \text{ex}(n, \text{gem}_5) \) for \( n \geq 8 \). Then, in Section 3, we will prove Pikhurko and Sousa conjecture for these two gem graphs. That is, we will show that \( \phi(n, \text{gem}_4) = \text{ex}(n, \text{gem}_4) \) for \( n \geq 6 \), and \( \phi(n, \text{gem}_5) = \text{ex}(n, \text{gem}_5) \) for \( n \geq 8 \). Note that \( \chi(\text{gem}_4) = \chi(\text{gem}_5) = 3 \), and that \text{gem}_4 and \text{gem}_5 are not edge-critical graphs. Thus, our results are again not implied by Theorem 1.4.

Our notations throughout the paper are fairly standard. For a vertex \( v \) in a graph \( G \), the *neighbourhood* of \( v \), denoted by \( N(v) \), is the set of vertices in \( G \) that are adjacent to \( v \). The *degree* of \( v \) is \( \deg(v) = |N(v)| \), and the *minimum degree* and *maximum degree* of \( G \) are \( \delta(G) \) and \( \Delta(G) \), respectively. For a set \( U \subset V(G) \), let \( \deg(v, U) \) denote the number of vertices in \( U \) that are adjacent to \( v \), and let \( G[U] \) denote the subgraph of \( G \) induced by \( U \).

## 2 Turán function for the gem graphs

In this section, we will determine the Turán functions \( \text{ex}(n, \text{gem}_4) \) for \( n \geq 6 \), and \( \text{ex}(n, \text{gem}_5) \) for \( n \geq 8 \). Furthermore, we will determine the extremal graphs in each case. That is, we will determine all \text{gem}_4-free graphs on \( n \geq 6 \) vertices with \( \text{ex}(n, \text{gem}_4) \) edges, and all \text{gem}_5-free graphs on \( n \geq 8 \) vertices with \( \text{ex}(n, \text{gem}_5) \) edges.

### 2.1 Turán function for \text{gem}_4

We will now determine the function \( \text{ex}(n, \text{gem}_4) \). In order to state our result, we first define the family of graphs \( \mathcal{F}_{n,4} \), which will consist of all the extremal graphs. Let \( n \geq 6 \) and \( \mathcal{F}_{n,4} \) be the family of graphs on \( n \) vertices as follows. For \( n \equiv 0 \) (mod 4), let \( G_n^0 \) be the graph obtained by taking the Turán graph \( T_2(n) \) and embedding a maximum matching into a class of \( T_2(n) \). For \( n \equiv 1 \) (mod 4), let \( G_n^{11} \) and \( G_n^{12} \) be the graphs obtained by embedding a maximum matching into the smaller class and the larger class of \( T_2(n) \), respectively. For \( n \equiv 2 \) (mod 4), let \( G_n^{21} \) and \( G_n^{22} \) be the graphs obtained by embedding a maximum matching into a class of \( T_2(n) \), and into the larger class of the complete bipartite graph \( K_{n/2-1,n/2+1} \), respectively. For \( n \equiv 3 \) (mod 4), let
$G^3_n$ be the graph obtained by embedding a maximum matching into the larger class of T$_2(n)$. Let the vertex classes of $G^0_n$ be $A^0_n$ and $B^0_n$, with similar notations for the other graphs. Let $F_{n,4} = \{G^0_n\}$, $F_{n,4} = \{G^1, G^1\}$, $F_{n,4} = \{G^2, G^2\}$ and $F_{n,4} = \{G^3\}$ for $n \equiv 0, 1, 2, 3 \pmod{4}$, respectively. Figure 2 below shows the graphs of $F_{n,4}$. Note that in $G^1$, we have an unmatched vertex in the class $B^1$, and similarly for $G^2$ with the class $B^2$.

It is easy to see that every graph of $F_{n,4}$ is gem$_4$-free. Let $G \in F_{n,4}$, and suppose that there exists a copy of gem$_4$ in $G$, say $abcd + u$. We may consider in turn whether $u$ is in the independent class of $G$, or in the class containing the maximum matching. In each case, we can easily verify that no four neighbours of $u$ form a path $P_4$ in $G$, which is a contradiction. Also, for any graph of $F_{n,4}$, by adding an edge, we obtain a graph that contains a copy of gem$_4$. Indeed, let $G \in F_{n,4}$. Since $n \geq 6$, if an edge $cu$ is added to the independent class of $G$, then we may find an edge $ab$ and another vertex $d$ in the other class. If an edge $bu$ is added to the class of $G$ containing the maximum matching, then we may assume that $du$ is an edge in the matching, and choose vertices $a, c$ in the other class. In both cases, we have $abcd + u$ is a copy of gem$_4$.

We can easily check that for $n \geq 6$, all graphs of $F_{n,4}$ have the same number of
edges. Thus for \( G \in F_{n,4} \), we let \( e_n \) denote the number of edges in the graph \( G \). Then, we can easily check that the number of edges of \( G \) is

\[
e(G) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 
0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\
1 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]  

(2.1) \text{sizeeq1}

Moreover, for \( n \geq 7 \), \( G \in F_{n,4} \) and \( G' \in F_{n-1,4} \), we have

\[
e(G) - e(G') = e_n - e_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor + \begin{cases} 
0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\
1 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]  

(2.2) \text{diffeq1}

We have the following result for the Turán function \( \text{ex}(n, \text{gem}_4) \).

**Theorem 2.1.** For \( n \geq 6 \), we have

\[
\text{ex}(n, \text{gem}_4) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 
0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\
1 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

Moreover, the only \( \text{gem}_4 \)-free graphs with \( n \) vertices and \( \text{ex}(n, \text{gem}_4) \) edges are the members of \( F_{n,4} \).

We will prove Theorem 2.1 by induction on \( n \). We first prove the base case as follows.

**Lemma 2.2.** \( \text{ex}(6, \text{gem}_4) = e_6 = 10 \) and the only \( \text{gem}_4 \)-free graphs with six vertices and 10 edges are \( G_{6}^{21} \) and \( G_{6}^{22} \).

**Proof.** It suffices to prove that, for any graph \( G \) with six vertices and \( e_6 = 10 \) edges, either \( G \) contains a copy of the graph \( \text{gem}_4 \), or \( G \in F_{6,4} = \{ G_{6}^{21}, G_{6}^{22} \} \). Then for any graph \( G' \) with six vertices and \( e(G') \geq 11 \), we can take a spanning subgraph \( G \subset G' \) with \( e(G) = e_6 = 10 \), so that either \( G \) contains a copy of \( \text{gem}_4 \), or \( G \in F_{6,4} \). In either case, \( G' \) contains a copy of \( \text{gem}_4 \) and we are done.

Let \( G \) be a graph with six vertices and \( e_6 = 10 \) edges. Note that \( G \) has either a vertex of degree 5, or two vertices of degree 4. Otherwise, we have \( e(G) \leq \left\lfloor \frac{1}{2}(4 + 5 \cdot 3) \right\rfloor = 9 < 10 = e_6 \), a contradiction.

Suppose first that \( G \) has a vertex \( u \) with \( \text{deg}(u) = 5 \). By Theorem 1.1, we have \( \text{ex}(5, P_4) = \binom{5}{2} + \binom{2}{2} = 4 \). We have \( e(G - u) = 10 - 5 = 5 \geq 4 = \text{ex}(5, P_4) \), and thus
$G - u$ contains a copy of the path $P_4$, which together with $u$, form a copy of gem$_4$ in $G$.

Now, suppose that $G$ has two vertices of degree 4, say $u$ and $v$. Let $x_1, x_2, x_3, x_4$ be the remaining four vertices, and assume that $G$ does not contain a copy of gem$_4$. Suppose first that $uv \in E(G)$. If $u$ and $v$ have three common neighbours, say $x_1, x_2, x_3$, then we must have $x_i x_4 \in E(G)$ for $i = 1, 2, 3$, so that $G = G_6^{21}$. If $u$ and $v$ have two common neighbours, say $x_1, x_2$, then let $ux_3, vx_4 \in E(G)$ and $ux_4, vx_3 \not\in E(G)$. We see that only the edges $x_1x_2, x_3x_4$ can be added to avoid creating a copy of gem$_4$, so that $G$ can only have at most nine edges, a contradiction. Now, suppose that $uv \not\in E(G)$. Then $G$ contains all edges between $\{u, v\}$ and $\{x_1, x_2, x_3, x_4\}$. If $G$ does not contain a copy of gem$_4$, then the remaining two edges must be independent within $\{x_1, x_2, x_3, x_4\}$, so that $G = G_6^{22}$.

We conclude that either $G$ contains a copy of gem$_4$, or $G \in \mathcal{F}_{6,4}$, as required. □

We are now able to prove Theorem 2.1.

**Proof of Theorem 2.1.** Let $n \geq 6$. The lower bound $\text{ex}(n, \text{gem}_4) \geq e_n$ follows instantly by considering any graph of $\mathcal{F}_{n,4}$. We prove the upper bound $\text{ex}(n, \text{gem}_4) \leq e_n$ by induction on $n$. Lemma 2.2 proves the result for $n = 6$. Now suppose that $n \geq 7$, and the theorem holds for $n - 1$. We will prove that if $G$ is a graph on $n$ vertices and $e(G) = e_n$, then either $G$ contains a copy of gem$_4$, or $G$ is one of the graphs of $\mathcal{F}_{n,4}$. This clearly implies the upper bound $\text{ex}(n, \text{gem}_4) \leq e_n$, and thus the theorem for $n$. Indeed, if we have a graph $G'$ with $n$ vertices and $e(G') > e_n$, then by taking a spanning subgraph $G \subset G'$ with $e(G) = e_n$, we see that either $G$ contains a copy of gem$_4$, or $G \in \mathcal{F}_{n,4}$. In either case, $G'$ contains a copy of gem$_4$.

Firstly, suppose that $\delta(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ and let $v \in V(G)$ be a vertex of minimum degree. Then by (2.2), we have

$$e(G - v) = e(G) - \deg(v) \geq e_n - \left\lfloor \frac{n}{2} \right\rfloor \geq e_{n - 1}.$$  (2.3)

If $e(G - v) > e_{n - 1}$, then by induction, $G - v$, and thus $G$, contains a copy of gem$_4$. Next, $e(G - v) = e_{n - 1}$ holds if and only if $\deg(v) = \left\lfloor \frac{n}{2} \right\rfloor$ and $e_n - e_{n - 1} = \left\lfloor \frac{n}{2} \right\rfloor$. The latter condition holds for $n \neq 3 \mod 4$. By induction, either $G - v$, and thus $G$, contains a copy of gem$_4$ and we are done; or $G - v \in \mathcal{F}_{n - 1,4}$, and we must consider
the following cases.

**Case 1.** $n \equiv 0 \pmod{4}$.

We have $G - v = G_{n-1}^3$ with classes $A_{n-1}^3$ and $B_{n-1}^3$, where $|A_{n-1}^3| = \frac{n}{2} - 1$ and $|B_{n-1}^3| = \frac{n}{2}$, with $B_{n-1}^3$ containing a perfect matching. Since $\deg(v) = \frac{n}{2}$, if $N(v) = B_{n-1}^3$, then $G = G_n^0$. Otherwise, if $v$ has neighbours $c \in A_{n-1}^3$ and $u \in B_{n-1}^3$, then $abcv + u$ is a copy of gem$_4$ in $G$, where $a \in A_{n-1}^3 \setminus \{c\}$ and $b \in B_{n-1}^3$ is the vertex adjacent to $u$.

**Case 2.** $n \equiv 1 \pmod{4}$.

We have $G - v = G_{n-1}^0$ with classes $A_{n-1}^0$ and $B_{n-1}^0$, where $|A_{n-1}^0| = |B_{n-1}^0| = \frac{n-1}{2}$, with $B_{n-1}^0$ containing a perfect matching. Since $\deg(v) = \frac{n-1}{2}$, if $N(v) = B_{n-1}^0$ then $G = G_n^{11}$, and if $N(v) = A_{n-1}^0$ then $G = G_n^{12}$. Otherwise $v$ has a neighbour in both $A_{n-1}^0$ and $B_{n-1}^0$, so that as in Case 1, $G$ contains a copy of gem$_4$.

**Case 3.** $n \equiv 2 \pmod{4}$.

We have $G - v \in \{G_{n-1}^{11}, G_{n-1}^{12}\}$. Suppose first that $G - v = G_{n-1}^{11}$. Then the classes of $G - v$ are $A_{n-1}^{11}$ and $B_{n-1}^{11}$, where $|A_{n-1}^{11}| = \frac{n}{2} - 1$ and $|B_{n-1}^{11}| = \frac{n}{2}$, with $A_{n-1}^{11}$ containing a perfect matching. Since $\deg(v) = \frac{n}{2}$, if $N(v) = B_{n-1}^{11}$, then $G = G_n^{21}$. Otherwise, $v$ has a neighbour in both $A_{n-1}^{11}$ and $B_{n-1}^{11}$, and $G$ contains a copy of gem$_4$ as in Case 1. Now suppose that $G - v = G_{n-1}^{12}$. Then the classes are $A_{n-1}^{12}$ and $B_{n-1}^{12}$, where $|A_{n-1}^{12}| = \frac{n}{2} - 1$ and $|B_{n-1}^{12}| = \frac{n}{2}$, with $B_{n-1}^{12}$ containing a maximum matching with one unmatched vertex, say $w$. Since $\deg(v) = \frac{n}{2}$, if $N(v) = B_{n-1}^{12}$ then again $G = G_n^{21}$, and if $N(v) = A_{n-1}^{12} \cup \{w\}$ then $G = G_n^{22}$. Otherwise, $v$ has a neighbour in both $A_{n-1}^{12}$ and $B_{n-1}^{12} \setminus \{w\}$, and again as in Case 1, $G$ contains a copy of gem$_4$.

Secondly, suppose that $\delta(G) \geq \left[\frac{n}{2}\right] + 1$. In view of (2.1), if $n$ is even, we have $e(G) \geq \frac{n}{2}(\frac{n}{2} + 1) > e_n$. If $n \equiv 1 \pmod{4}$, then $e(G) \geq \left[\frac{n}{2}\right] \left[\left[\frac{n}{2}\right] + 1\right] = \left[\frac{n^2}{4}\right] + \left[\frac{n}{2}\right] + 1 > e_n$. We have a contradiction in these cases. Now let $n \equiv 3 \pmod{4}$. We have $e(G) \geq \left[\frac{n}{2}\right] \left[\left[\frac{n}{2}\right] + 1\right] = \left[\frac{n^2}{4}\right] + \left[\frac{n}{2}\right] + 1 = e_n$. We must have equality, and thus $G$ is a $(\left[\frac{n}{2}\right] + 1)$-regular graph. Let $v \in V(G)$, so that by (2.2)

$$e(G - v) = e(G) - \deg(v) = e_n - \left(\left[\frac{n}{2}\right] + 1\right) = e_{n-1}. \quad (2.4)$$

By induction, either $G - v$, and thus $G$, contains a copy of gem$_4$; or $G - v \in F_{n-1,4}$. If the latter holds, then $G - v \in \{G_{n-1}^{21}, G_{n-1}^{22}\}$. Suppose first that $G - v = G_{n-1}^{21}$. The classes are $A_{n-1}^{21}$ and $B_{n-1}^{21}$, where $|A_{n-1}^{21}| = |B_{n-1}^{21}| = \frac{n-1}{2}$, with $B_{n-1}^{21}$ containing
a maximum matching with one unmatched vertex, say \( w \). Since \( \deg(v) = \frac{n-1}{2} + 1 \), in order for \( G \) to be \( (\lfloor \frac{n}{2} \rfloor + 1) \)-regular, we must have \( N(v) = A_{n-1}^{21} \cup \{w\} \). This gives \( G = G_n^3 \). Now, suppose that \( G - v = G_{n-1}^{22} \). The classes are \( A_{n-1}^{22} \) and \( B_{n-1}^{22} \), where \( |A_{n-1}^{22}| = \frac{n+1}{2} - 1 \) and \( |B_{n-1}^{22}| = \frac{n+1}{2} + 1 \), with \( B_{n-1}^{22} \) containing a perfect matching. Again since \( G \) is \( (\lceil \frac{n}{2} \rceil + 1) \)-regular, we must have \( N(v) = B_{n-1}^{22} \), and this also implies \( G = G_n^3 \).

This completes the proof of Theorem 2.1. \( \square \)

### 2.2 Turán function for \( \text{gem}_5 \)

We will next determine the function \( \text{ex}(n, \text{gem}_5) \). Analogously, we first define the family of graphs \( \mathcal{F}_{n,5} \), which will consist of all the extremal graphs. Let \( n \geq 8 \) and \( \mathcal{F}_{n,5} \) be the family of graphs on \( n \) vertices as follows. For \( n \geq 11 \), we let \( \mathcal{F}_{n,5} = \mathcal{F}_{n,4} \). For \( n = 8, 9, 10 \), the family \( \mathcal{F}_{n,5} \) will consist of all graphs of \( \mathcal{F}_{n,4} \) and some additional graphs. Let \( G_n' \) be the graph obtained by adding one edge into each class of \( T_2(n) \). Also for \( n = 8 \), let \( G_n'' \) be the graph obtained by embedding two vertex-disjoint triangles into the larger class of the complete bipartite graph \( K_{2,6} \). For \( n = 9 \), let \( G_n''' \) be the graph obtained by taking \( G_n' \) and joining another vertex to the four unmatched vertices within the classes of \( G_n' \). As before, let \( A_n' \) and \( B_n' \) be the classes of \( G_n' \), with similar notations for the other graphs. Figure 3 below shows these additional graphs. Let \( \mathcal{F}_{8,5} = \{G_8^0, G_8', G_8''\} \), \( \mathcal{F}_{9,5} = \{G_9^{11}, G_9^{12}, G_9', G_9''\} \), and \( \mathcal{F}_{10,5} = \{G_{10}^{21}, G_{10}^{22}, G_{10}'\} \).

![Figure 3. The additional graphs in \( \mathcal{F}_{n,5} \) for \( n = 8, 9, 10 \)](image-url)
Note that every graph of $F_{n,5}$ is gem$_5$-free. Indeed, let $G \in F_{n,5}$. If $G \not\in \{G_8', G_8'', G_9', G_9'', G_{10}'\}$, then $G$ is gem$_4$-free as before, so that $G$ is gem$_5$-free. Suppose that $G \in \{G_8', G_8'', G_9', G_9'', G_{10}'\}$ and $G$ contains a copy of gem$_5$, say $abcde + u$. It is easy to check that in each choice for $G$, whichever vertex of $G$ is chosen for $u$, we have that $u$ does not have five neighbours that form a path $P_5$ in $G$. This is a contradiction.

Also, by adding an edge to any graph of $F_{n,5}$, we obtain a graph that contains a copy of gem$_5$. To see this, let $G \in F_{n,5}$. Suppose first that $G \not\in \{G_8', G_8'', G_9', G_9'', G_{10}'\}$. Then similar to before, since $n \geq 8$, if an edge $cu$ is added to the independent class of $G$, then we can find two independent edges $ab, de$ in the other class. If an edge $bu$ is added to the class of $G$ containing the maximum matching, then we may assume that $du$ is an edge in the matching, and choose vertices $a, c, e$ in the other class. In both cases, we have $abcde + u$ is a copy of gem$_5$. Next, the case $G \in \{G_8', G_9', G_{10}'\}$ can be considered similarly, according to whether or not the added edge is incident with an edge within a class of $G$. Now, consider $G = G_8''$. If the edge $bu$ is added into $A_9''$, then let $cde$ be a triangle and $a$ be another vertex in $B_8''$. If an edge is added into $B_8''$, then there exists a path $abcde$ of order 5 in $B_8''$, and we let $u \in A_9''$. In both cases, $abcde + u$ is a copy of gem$_5$. Finally, consider $G = G_9'$. Since $G_9''$ contains $G_8'$ as a subgraph on $A_9'' \cup B_9''$, it follows that if an edge is added into $A_9''$ or $B_9''$, then we have a copy of gem$_5$. Thus, we may assume that the edge $au$ is added to $G_9'$, where $a$ is an end-vertex of the edge in $A_9''$, and $u$ is the vertex outside of $A_9'' \cup B_9''$. Then if $c, e \in A_9''$ and $b, d \in B_9''$ are the neighbours of $u$ in $G_9''$, we have $abcde + u$ is a copy of gem$_5$.

We can easily check that for $n \geq 8$, all graphs of $F_{n,5}$ have the same number of edges, which is also the same as the number of edges in any graph of $F_{n,4}$. Thus, we may also let $e_n$ denote the number of edges in any graph of $F_{n,5}$. Then, equations (2.1) and (2.2) remain true. That is, for $G \in F_{n,5}$, we have

$$e(G) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\ 1 & \text{if } n \equiv 3 \pmod{4}, \end{cases} \quad (2.5)$$

and for $n \geq 9$, $G \in F_{n,5}$ and $G' \in F_{n-1,5}$, we have

$$e(G) - e(G') = e_n - e_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor + \begin{cases} 0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\ 1 & \text{if } n \equiv 3 \pmod{4}. \end{cases} \quad (2.6)$$
We have the following result for the Turán function \( \text{ex}(n, \text{gem}_5) \).

**Theorem 2.3.** For \( n \geq 8 \), we have

\[
\text{ex}(n, \text{gem}_5) = e_n = \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \begin{cases} 
0 & \text{if } n \equiv 0, 1, 2 \pmod{4}, \\
1 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

Moreover, the only \( \text{gem}_5 \)-free graphs with \( n \) vertices and \( \text{ex}(n, \text{gem}_5) \) edges are the members of \( F_{n,5} \).

As before, Theorem 2.3 will be proved by induction on \( n \). We first prove the base case, which will involve a bit more of case analysis than in Lemma 2.2.

**Lemma 2.4.** \( \text{ex}(8, \text{gem}_5) = e_8 = 18 \) and the only \( \text{gem}_5 \)-free graphs with eight vertices and 18 edges are \( G_0^8, G'_8 \) and \( G''_8 \).

To prove Lemma 2.4, the following lemma will be useful.

**Lemma 2.5.** Let \( H \) be a graph with vertex set \( A \cup B \), where \( A = \{x, y\} \) and \( B = \{z_1, z_2, z_3, z_4\} \). Suppose that \( xy, xz_4 \in E(H) \), and \( H \) also contains all edges between \( \{x, y\} \) and \( \{z_1, z_2, z_3\} \). Suppose that \( H[B] \) contains two edges \( f_1, f_2 \), and either \( z_4 \) belongs to at least one of \( f_1, f_2 \), or \( yz_4 \in E(H) \). Then \( H \) contains a copy of \( \text{gem}_5 \).

**Proof.** Firstly, if \( z_4 \) belongs to one of \( f_1, f_2 \), then we may assume that either \( f_1 = z_1z_2, f_2 = z_3z_4 \); or \( f_1 = z_1z_2, f_2 = z_2z_4 \); or \( f_1 = z_1z_4, f_2 = z_2z_4 \). Then \( z_1z_2y z_4 + x \); or \( z_3y z_1 z_4 z_2 + x \); or \( z_3y z_1 z_4 z_2 + x \) is a copy of \( \text{gem}_5 \) in \( H \), respectively.

Secondly, if \( yz_4 \in E(H) \) and \( z_4 \) does not belong to \( f_1 \) and \( f_2 \), then we may assume that \( f_1 = z_1z_2 \) and \( f_2 = z_2z_3 \). Then \( z_1z_2z_3y z_4 + x \) is a copy of \( \text{gem}_5 \) in \( H \). \( \square \)

**Proof of Lemma 2.4.** Let \( G \) be a graph with eight vertices and \( e_8 = 18 \) edges. As in Lemma 2.2, it suffices to prove that either \( G \) contains a copy of \( \text{gem}_5 \), or \( G \in F_{8,5} = \{G_0^8, G'_8, G''_8\} \). Let \( \Delta = \Delta(G) \) be the maximum degree of \( G \). Note that \( 5 \leq \Delta \leq 7 \), otherwise if \( \Delta \leq 4 \), then \( e(G) \leq \left\lfloor \frac{1}{2} \cdot 8 \cdot 4 \right\rfloor = 16 < 18 = e_8 \), a contradiction. Let \( d_1 \geq d_2 \geq \cdots \geq d_8 \) be the degree sequence of \( G \). Let \( u \in V(G) \) be a vertex of maximum degree, so that \( \deg(u) = \Delta = d_1 \). We consider three cases according to the value of \( \Delta \).
Case 1. $\Delta = 7$.

By Theorem 1.1, we have $\text{ex}(7, P_5) = \binom{4}{2} + \binom{3}{2} = 9$. Thus $e(G - u) = 18 - 7 = 11 > 9 = \text{ex}(7, P_5)$, and there exists a copy of the path $P_5$ in $G - u$, which together with $u$, form a copy of gem$_5$ in $G$.

Case 2. $\Delta = 6$.

Let $v \in V(G) \setminus \{u\}$ be a vertex with $\deg(v) = d_2$. Note that $\deg(v) = 6$ or $\deg(v) = 5$, otherwise $e(G) \leq \left\lfloor \frac{1}{2} (6 + 7 \cdot 4) \right\rfloor = 17 < 18 = \epsilon_8$, a contradiction.

Subcase 2.1. $\deg(v) = 6$.

Suppose first that $uv \notin E(G)$. We have $e(G - \{u, v\}) = 18 - 2 \cdot 6 = 6$. If there exists $x \in V(G) \setminus \{u, v\}$ with at least three neighbours in $V(G) \setminus \{u, v, x\}$, say $x_1, x_2, x_3$, then $x_1x_2v_3x_3 + x$ is a copy of gem$_5$ in $G$. Otherwise, since $e(G - \{u, v\}) = 6$, we see that every vertex of $V(G) \setminus \{u, v\}$ must have exactly two neighbours in $V(G) \setminus \{u, v\}$, and thus, the subgraph $G - \{u, v\}$ must be either $C_6$ or two vertex-disjoint copies of $C_3$. If the former, then there is a copy of $P_5$ in $G - \{u, v\}$, which together with $u$, form a copy of gem$_5$. If the latter, then $G = G''_8$.

Now, suppose that $uv \in E(G)$. Observe first that $u$ and $v$ have at least four common neighbours in $V(G) \setminus \{u, v\}$. If $G[N(u) \setminus \{v\}]$ contains two edges then Lemma 2.5 implies that $G$ contains a copy of gem$_5$. Otherwise, we may assume that $G[N(u) \setminus \{v\}]$ contains at most one edge. If $y$ is the vertex not adjacent to $u$ in $G$, then $y$ has at most five neighbours in $N(u) \setminus \{v\}$. Therefore, we have $e(G - \{u, v\}) \leq 1 + 5 = 6$. This is a contradiction, since we have $e(G - \{u, v\}) = 18 - 1 - 2 \cdot 5 = 7$.

Subcase 2.2. $\deg(v) = 5$.

Let $w \in V(G) \setminus \{u, v\}$ be a vertex with $\deg(w) = d_3$. Note that $\deg(w) = 5$, otherwise, $e(G) \leq \left\lfloor \frac{1}{2} (6 + 5 + 6 \cdot 4) \right\rfloor = 17 < 18 = \epsilon_8$. Thus, without loss of generality, we may assume $uv \in E(G)$, so that $e(G - \{u, v\}) = 18 - 1 - 5 - 4 = 8$. Let $y$ be the vertex not adjacent to $u$. Suppose that $G$ does not contain a copy of gem$_5$.

Let $vy \notin E(G)$. Then $v$ has exactly four neighbours in $N(u) \setminus \{v\}$, and by Lemma 2.5, $G[N(u) \setminus \{v\}]$ contains at most one edge, so that $e(G - \{u, v\}) \leq 6$, a contradiction.

Now let $vy \in E(G)$. Let $x_1, x_2, x_3$ be the common neighbours of $u$ and $v$, and $z_1, z_2$ be the remaining two vertices, so that $uz_1, uz_2 \in E(G)$ and $vz_1, vz_2 \notin E(G)$.
Again by Lemma 2.5, each of \(y, z_1, z_2\) has at most one neighbour in \(\{x_1, x_2, x_3\}\). If there are no edges between \(\{y, z_1, z_2\}\) and \(\{x_1, x_2, x_3\}\), then \(e(G - \{u, v\}) \leq 6\), a contradiction. Otherwise, if there exists an edge between \(\{y, z_1, z_2\}\) and \(\{x_1, x_2, x_3\}\), then by Lemma 2.5, there are no edges in \(G[\{x_1, x_2, x_3\}]\). Since there are at most three edges in \(G[y, z_1, z_2]\) and at most three edges between \(\{y, z_1, z_2\}\) and \(\{x_1, x_2, x_3\}\), we have \(e(G - \{u, v\}) \leq 6\), another contradiction.

**Case 3.** \(\Delta = 5\).

We have \(d_1 = d_2 = d_3 = d_4 = \Delta = 5\), otherwise, \(e(G) \leq \lfloor \frac{1}{2}(3 \cdot 5 + 5 \cdot 4) \rfloor = 17 < 18 = e_8\). This means that, we may assume there exists \(v \in V(G) \setminus \{u\}\) with \(\deg(v) = 5\) and \(uv \in E(G)\), so that \(e(G - \{u, v\}) = 18 - 1 - 2 \cdot 4 = 9\). If \(G\) contains a copy of \(\text{gem}_5\) then we are done, so assume otherwise.

Suppose first that \(u\) and \(v\) have four common neighbours, say \(x_1, x_2, x_3, x_4\). Let \(y_1, y_2\) be the remaining two vertices. By Lemma 2.5, \(G[\{x_1, x_2, x_3, x_4\}]\) contains at most one edge. If there is exactly one edge, say \(x_1x_2 \in E(G)\), then there are 10 edges already in \(G\). The edges between \(\{y_1, y_2\}\) and \(\{x_1, x_2, x_3, x_4\}\), as well as \(y_1y_2\), may possibly be present, and since \(e(G) = 18\), exactly one of these nine edges is not present. Suppose first that \(y_1y_2 \in E(G)\). We may assume that \(y_1x_1, y_1x_2, y_2x_1 \in E(G)\), but then \(uvx_2y_1y_2 + x_1\) is a copy of \(\text{gem}_5\). Otherwise if \(y_1y_2 \not\in E(G)\), then we have \(G = G^0_8\). Finally, if there does not exist an edge in \(G[\{x_1, x_2, x_3, x_4\}]\), then a similar edge count shows that \(G\) contains all edges between \(\{y_1, y_2\}\) and \(\{x_1, x_2, x_3, x_4\}\), as well as \(y_1y_2\). This gives \(G = G^0_8\).

Next, suppose that \(u\) and \(v\) have three common neighbours, say \(x_1, x_2, x_3\). Let \(y, z_1, z_2\) be the remaining vertices, where \(uz_1, vz_2 \in E(G)\) and \(uv, vy, uz_2, vz_1 \not\in E(G)\). By Lemma 2.5, each of \(z_1, z_2\) has at most one neighbour in \(\{x_1, x_2, x_3\}\). If there exists an edge between \(\{z_1, z_2\}\) and \(\{x_1, x_2, x_3\}\), then again by Lemma 2.5, there are no edges in \(G[\{x_1, x_2, x_3\}]\). Since there are at most three edges in \(G[y, z_1, z_2]\) and at most five edges between \(\{y, z_1, z_2\}\) and \(\{x_1, x_2, x_3\}\), we have \(e(G - \{u, v\}) \leq 8\), a contradiction. Otherwise, suppose that there are no edges between \(\{z_1, z_2\}\) and \(\{x_1, x_2, x_3\}\). Then we have \(\deg(z_i) \leq 3\) for \(i = 1, 2\). This implies that the remaining six vertices must each have degree 5, otherwise \(e(G) \leq \lfloor \frac{1}{2}(5 \cdot 5 + 4 \cdot 3) \rfloor = 17 < 18 = e_8\). In particular, we have \(x_ix_j \in E(G)\) for \(1 \leq i, j \leq 3\) and \(yx_i \in E(G)\) for \(i = 1, 2, 3\). But then \(uvx_2x_3y + x_1\) is a copy of \(\text{gem}_5\).
Finally, suppose that \( u \) and \( v \) have two common neighbours, say \( x_1, x_2 \). Let \( y_1, y_2, z_1, z_2 \) be the remaining vertices, where \( u y_1, u y_2, v z_1, v z_2 \in E(G) \) and \( u z_1, u z_2, v y_1, v y_2 \notin E(G) \). Suppose first that there are at most two edges in \( G[\{x_1, x_2, y_1, y_2\}] \), and at most two edges in \( G[\{x_1, x_2, z_1, z_2\}] \). Since there are at most four edges between \( \{y_1, y_2\} \) and \( \{z_1, z_2\} \), we have \( e(G - \{u, v\}) \leq 2 \cdot 2 + 4 = 8 \), a contradiction. Now, suppose that there are at least three edges in \( G[\{x_1, x_2, y_1, y_2\}] \). If \( x_1 y_1, y_1 y_2 \in E(G) \) or \( x_1 y_1, x_2 y_2 \in E(G) \), then \( x_2 v x_1 y_1 y_2 + u \) or \( y_1 x_1 v x_2 y_2 + u \) is a copy of \( \text{gem}_5 \). Thus, we may assume that \( x_1 x_2, x_1 y_1, x_2 y_1 \in E(G) \) and \( x_1 y_2, x_2 y_2, y_1 y_2 \notin E(G) \). If there are at most two edges in \( G[\{x_1, x_2, z_1, z_2\}] \), including \( x_1 x_2 \), then since there are at most four edges between \( \{y_1, y_2\} \) and \( \{z_1, z_2\} \), we have \( e(G - \{u, v\}) \leq 3 + 1 + 4 = 8 \), a contradiction. Thus, there are at least three edges in \( G[\{x_1, x_2, z_1, z_2\}] \), and by similarly considering the edges in \( G[\{x_1, x_2, z_1, z_2\}] \), we may assume that \( x_1 z_1, x_2 z_2 \in E(G) \) and \( x_1 z_2, x_2 z_2, z_1 z_2 \notin E(G) \). But now, \( y_1 u x_2 v z_1 + x_1 \) is a copy of \( \text{gem}_5 \).

Therefore, we conclude that either \( G \) contains a copy of \( \text{gem}_5 \), or \( G \in \mathcal{F}_{8,5} \). This completes the proof of Lemma 2.4.

We are now able to prove Theorem 2.3. The proof is generally similar to that of Theorem 2.1 but with a little more case analysis.

**Proof of Theorem 2.3.** Let \( n \geq 8 \). Again, the lower bound \( \text{ex}(n, \text{gem}_5) \geq e_n \) follows by considering any graph of \( \mathcal{F}_{n,5} \). We prove the upper bound \( \text{ex}(n, \text{gem}_5) \leq e_n \) by induction on \( n \). Lemma 2.4 proves the result for \( n = 8 \). Now suppose that \( n \geq 9 \), and the theorem holds for \( n - 1 \). As before, it suffices to prove that, if \( G \) is a graph on \( n \) vertices and \( e(G) = e_n \), then either \( G \) contains a copy of \( \text{gem}_5 \), or \( G \in \mathcal{F}_{n,5} \).

Firstly, suppose that \( \delta(G) \leq \lceil \frac{n}{2} \rceil \) and let \( v \in V(G) \) be a vertex of minimum degree. Then exactly as in (2.3), we have \( e(G - v) \geq e_{n-1} \). Again we are done unless \( e(G - v) = e_{n-1} \), whence \( \deg(v) = \lceil \frac{n}{2} \rceil \) and \( e_n - e_{n-1} = \lceil \frac{n}{2} \rceil \), and \( n \neq 3 \pmod{4} \). By induction, either \( G - v \), and thus \( G \), contains a copy of \( \text{gem}_5 \) and we are done; or \( G - v \in \mathcal{F}_{n-1,5} \), and we must consider the following cases.

**Case 1.** \( n \equiv 0 \pmod{4} \).

We have \( G - v = G_{n-1}^3 \) with classes \( A_{n-1}^3 \) and \( B_{n-1}^3 \), where \( |A_{n-1}^3| = \frac{n}{2} - 1 \) and \( |B_{n-1}^3| = \frac{n}{2} \), and \( B_{n-1}^3 \) containing a perfect matching. We have \( \deg(v) = \frac{n}{2} \). If \( N(v) = B_{n-1}^3 \), then \( G = G_n^0 \). Otherwise, if \( v \) has neighbours \( c, d \in A_{n-1}^3 \) and \( u \in B_{n-1}^3 \),
then $abcvd + u$ is a copy of gem$_5$ in $G$, where $a \in A^3_{n-1} \setminus \{c, d\}$ and $b \in B^3_{n-1}$ is the vertex adjacent to $u$. If $v$ has exactly one neighbour $u \in A^3_{n-1}$, then since $|B^3_{n-1}| = \frac{n}{2} > 4$, we can find $a, b, c, d \in B^3_{n-1}$ such that $ab, cd, bv, cv \in E(G)$. We have $abcvd + u$ is a copy of gem$_5$ in $G$.

Case 2. $n \equiv 1 \pmod{4}$.

If $n \geq 13$, we have $G - v = G^0_{n-1}$. If $n = 9$, we have $G - v \in \{G^0_8, G'_8, G''_8\}$.

Subcase 2.1. $n \geq 9$ and $G - v = G^0_{n-1}$.

The classes of $G - v$ are $A^0_{n-1}$ and $B^0_{n-1}$. Since $|B^0_{n-1}| = \frac{n-1}{2} \geq 4$, this subcase can be considered by combining the arguments used in Case 2 of Theorem 2.1, and in Case 1 above. We find that either $G$ contains a copy of gem$_5$, or $G \in \{G^{11}_n, G^{12}_n\}$.

Subcase 2.2. $n = 9$ and $G - v \in \{G_8', G'_8\}$.

Suppose first that $G - v = G_8'$. Then the classes of $G - v$ are $A'_8$ and $B'_8$ with $|A'_8| = |B'_8| = 4$, and each class containing one edge, say $cu$ and $ab$ are the edges in $A'_8$ and $B'_8$. We have $\deg(v) = 4$. If $N(v) = A'_8$ or $N(v) = B'_8$, then $G = G_8''$, and if $N(v) = (A'_8 \cup B'_8) \setminus \{a, b, c, u\}$, then $G = G_8''$. Otherwise, let $d \in B'_8 \setminus \{a, b\}$. We may assume that $uv \in E(G)$, and either $av \in E(G)$ or $dv \in E(G)$. Then $vabcd + u$ or $abcdv + u$ is a copy of gem$_5$.

Now, suppose that $G - v = G''_8$. Then the classes of $G - v$ are $A''_8$ and $B''_8$ with $|A''_8| = 2$, $|B''_8| = 6$, and there are two vertex-disjoint triangles embedded into $B''_8$. Let $A''_8 = \{b, d\}$ and $acu$ be one of the triangles in $B''_8$. We have $\deg(v) = 4$. If $bv, dv \in E(G)$, then we may assume that $uv \in E(G)$. We have $abcdv + u$ is a copy of gem$_5$. Otherwise, $v$ has at least three neighbours in $B''_8$, and we may assume that $av, uv \in E(G)$. Then $vabcd + u$ is a copy of gem$_5$.

Case 3. $n \equiv 2 \pmod{4}$.

If $n \geq 14$, we have $G - v \in \{G^{11}_{n-1}, G^{12}_{n-1}\}$. If $n = 10$, we have $G - v \in \{G^{11}_9, G^{12}_9, G'_9, G''_9\}$.

Subcase 3.1. $n \geq 10$ and $G - v \in \{G^{11}_{n-1}, G^{12}_{n-1}\}$.

If $G - v = G^{11}_{n-1}$, then $|A^{11}_{n-1}| = \frac{n}{2} - 1 \geq 4$. If $G - v = G^{12}_{n-1}$, then $G - v$ has the class $B^{12}_{n-1}$ which contains a maximum matching with an unmatched vertex, say $w$. We have $|B^{12}_{n-1} \setminus \{w\}| = \frac{n}{2} - 1 \geq 4$. Since $\deg(v) = \frac{n}{2}$, this subcase can be considered.
by combining the arguments used in Case 3 of Theorem 2.1, and in Case 1 above. We find that either \( G \) contains a copy of \( \text{gem}_5 \), or \( G \in \{ G_{n10}^{21}, G_{n10}^{22} \} \).

**Subcase 3.2.** \( n = 10 \) and \( G - v \in \{ G'_9, G''_9 \} \).

Suppose first that \( G - v = G'_9 \), so that the classes of \( G - v \) are \( A'_9 \) and \( B'_9 \) with \( |A'_9| = 4, |B'_9| = 5 \), and each class containing one edge. We have \( \deg(v) = 5 \). If \( N(v) = B'_9 \), then \( G = G''_{10} \). If \( v \) has a neighbour which is incident with the edge in \( A'_9 \) or the edge in \( B'_9 \), then as in the argument in the first part of Subcase 2.2, \( G \) contains a copy of \( \text{gem}_5 \). Otherwise, \( N(v) \) consists of the five vertices not incident with the two edges within \( A'_9 \) and \( B'_9 \). Therefore, if \( b, d \in A'_9 \) and \( a, c, e \in B'_9 \) are these five neighbours of \( v \), then \( abcde + v \) is a copy of \( \text{gem}_5 \).

Now, suppose that \( G - v = G''_9 \). The graph \( G - v \) consists of two sets \( A''_9 \) and \( B''_9 \) where \( |A''_9| = |B''_9| = 4 \), with one edge in each set, say \( f_1 \) in \( A''_9 \) and \( f_2 \) in \( B''_9 \); and another vertex, say \( z \), joined to the four vertices not incident with \( f_1, f_2 \). Let \( b, d \in A''_9 \) and \( a, c \in B''_9 \) be the neighbours of \( z \) in \( G - v \). We have \( \deg(v) = 5 \). Again, if \( v \) has a neighbour in each of \( A''_9 \) and \( B''_9 \) where at least one is incident with \( f_1 \) or \( f_2 \), then by the argument in Subcase 2.2, \( G \) contains a copy of \( \text{gem}_5 \). Otherwise, we may assume that \( N(v) = A''_9 \cup \{ z \} \) or \( N(v) = \{ a, b, c, d, z \} \), and \( abcdv + z \) is a copy of \( \text{gem}_5 \).

This concludes the case when \( \delta(G) \leq \lceil \frac{n}{2} \rceil \).

Secondly, suppose that \( \delta(G) \geq \lceil \frac{n}{2} \rceil + 1 \). Then exactly as in Theorem 2.1, we must have \( n \equiv 3 \pmod{4} \), and that \( G \) is a \( (\lceil \frac{n}{2} \rceil + 1) \)-regular graph. Again for \( v \in V(G) \), we have \( e(G - v) = e_{n-1} \), using exactly the same argument as in (2.4). By induction, either \( G - v \), and thus \( G \), contains a copy of \( \text{gem}_5 \); or \( G - v \in \mathcal{F}_{n-1,5} \). If the latter holds, then for \( n \geq 15 \) we have \( G - v \in \{ G_{n-1}^{21}, G_{n-1}^{22} \} \), and for \( n = 11 \) we have \( G - v \in \{ G_{10}^{21}, G_{10}^{22}, G''_{10} \} \). If \( n \geq 11 \) and \( G - v \in \{ G_{n-1}^{21}, G_{n-1}^{22} \} \), then as in Theorem 2.1, the fact that \( G \) is a \( (\lceil \frac{n}{2} \rceil + 1) \)-regular graph implies that \( G = G_n^3 \). Otherwise, we have \( n = 11 \) and \( G - v = G'_{10} \). Then \( G \) is a 6-regular graph, which means that \( N(v) \) consists of the six vertices not incident with the two edges within \( A'_{10} \) and \( B'_{10} \). Therefore, if \( a, c, e \in A'_{10} \) and \( b, d \in B'_{10} \) are neighbours of \( v \), then \( abcde + v \) is a copy of \( \text{gem}_5 \).

This completes the proof Theorem 2.3. \( \square \)
3 Decompositions of graphs into gem graphs and single edges

Recall that for a fixed graph \( H \), \( \phi(n, H) \) denotes the smallest integer \( \phi \) such that any graph on \( n \) vertices admits an \( H \)-decomposition with at most \( \phi \) parts. In this section we will verify Pikhurko and Sousa conjecture (Conjecture 1.3) for the gem graphs \( \text{gem}_4 \) and \( \text{gem}_5 \). That is, we will show that \( \phi(n, \text{gem}_4) = \text{ex}(n, \text{gem}_4) \) for \( n \geq 6 \), and \( \phi(n, \text{gem}_5) = \text{ex}(n, \text{gem}_5) \) for \( n \geq 8 \).

3.1 \( \text{gem}_4 \)-decompositions

We begin by considering \( \text{gem}_4 \)-decompositions, and prove the following result.

**Theorem 3.1.** For \( n \geq 6 \) we have

\[
\phi(n, \text{gem}_4) = \text{ex}(n, \text{gem}_4).
\]

Moreover, the only graphs attaining \( \text{ex}(n, \text{gem}_4) \) are the members of \( \mathcal{F}_{n,4} \).

**Proof.** Let \( n \geq 6 \). The lower bound \( \phi(n, \text{gem}_4) \geq \text{ex}(n, \text{gem}_4) \) holds by considering any graph of \( \mathcal{F}_{n,4} \). We prove the matching upper bound. By Theorem 2.1, we know that \( \text{ex}(n, \text{gem}_4) = e_n \) for \( n \geq 6 \). Let \( G \) be a graph on \( n \geq 6 \) vertices. We must prove that \( \phi(G, \text{gem}_4) \leq \text{ex}(n, \text{gem}_4) = e_n \), with equality if and only if \( G \in \mathcal{F}_{n,4} \).

We proceed by induction on \( n \). For \( n = 6 \), if \( e(G) < e_6 = 10 \), then we can simply decompose \( G \) into single edges to obtain \( \phi(G, \text{gem}_4) < e_6 \). Otherwise, let \( 10 = e_6 \leq e(G) \leq 15 \). By Theorem 2.1, we either have \( G \in \mathcal{F}_{6,4} \), or \( G \) contains a copy of \( \text{gem}_4 \). If \( G \in \mathcal{F}_{6,4} \) then \( e(G) = e_6 = 10 \) and we must decompose \( G \) into single edges, thus, \( \phi(G, \text{gem}_4) = e_6 \) as required. If \( G \) contains a copy of \( \text{gem}_4 \), then \( \phi(G, \text{gem}_4) \leq 1 + e(G) - e(\text{gem}_4) \leq 9 < 10 = e_6 \). Thus, the theorem holds for \( n = 6 \).

Now, let \( n \geq 7 \), and suppose that the theorem holds for \( n - 1 \). Let \( G \) be a graph on \( n \) vertices. As before, if \( e(G) < e_n \), then \( \phi(G, \text{gem}_4) < e_n \), simply by decomposing \( G \) into single edges. If \( e(G) = e_n \), then by Theorem 2.1, either \( G \) contains a copy of \( \text{gem}_4 \), in which case \( \phi(G, \text{gem}_4) \leq 1 + e(G) - e(\text{gem}_4) = e_n - 6 < e_n \); or \( G \in \mathcal{F}_{n,4} \), in
which case we can only decompose $G$ into $e_n$ single edges for a gem$_4$-decomposition, and $\phi(G, \text{gem}_4) = e_n$ as required.

Now, suppose that $e(G) > e_n$, and let $v \in V(G)$ be a vertex of minimum degree. If $\deg(v) \leq \lfloor \frac{n}{2} \rfloor$ then by equation (2.2) we have $e(G - v) = e(G) - \deg(v) > e_n - \lfloor \frac{n}{2} \rfloor \geq e_{n-1}$, that is, $G - v \not\in \mathcal{F}_{n-1,4}$ and by the induction hypothesis we have

$$\phi(G - v, \text{gem}_4) < \text{ex}(n - 1, \text{gem}_4) = e_{n-1}.$$ 

Therefore, when going from $G - v$ to $G$ we only need to use the edges joining $v$ to the other vertices of $G$ and there are at most $\lfloor \frac{n}{2} \rfloor$ of these edges at $v$. We have

$$\phi(G, \text{gem}_4) \leq \phi(G - v, \text{gem}_4) + \deg(v) < e_{n-1} + \lfloor \frac{n}{2} \rfloor \leq e_n,$$

as required.

Therefore, we may assume that $\deg(v) \geq \lfloor \frac{n}{2} \rfloor + 1$ and let $\deg(v) = \lfloor \frac{n}{2} \rfloor + m$ for some integer $m \geq 1$. For every $x \in N(v)$, we have

$$\deg(x, N(v)) \geq \lfloor \frac{n}{2} \rfloor + m - \left( n - \lfloor \frac{n}{2} \rfloor - m \right)$$

$$= 2 \lfloor \frac{n}{2} \rfloor + 2m - n$$

$$\geq 2m - 1. \quad (3.1)$$

This means that $G[N(v)]$ must contain a path $P_{2m}$ on $2m$ vertices. Otherwise, if the longest path in $G[N(v)]$ has at most $2m - 1$ vertices, say with an end-vertex $y$, then all neighbours of $y$ in $N(v)$ must lie in the path, so that $\deg(y, N(v)) \leq 2m - 2$, contradicting (3.1).

If $m \geq 2$, then the path $P_{2m}$ contains $\lfloor \frac{2m}{4} \rfloor = \lfloor \frac{m}{2} \rfloor$ vertex-disjoint paths of order 4. Thus, we have $\lfloor \frac{m}{2} \rfloor$ edge-disjoint copies of gem$_4$, where each copy is formed by a path of order 4, together with $v$. Let $F \subset G - v$ be the subgraph of order $n - 1$, obtained by deleting the edges of the paths of order 4 from $G - v$. By induction and (2.2), and since $m \geq 2$, we have

$$\phi(G, \text{gem}_4) \leq \phi(F, \text{gem}_4) + \lfloor \frac{m}{2} \rfloor + \deg(v) - 4 \lfloor \frac{m}{2} \rfloor$$

$$\leq e_{n-1} + \lfloor \frac{n}{2} \rfloor + m - 3 \lfloor \frac{m}{2} \rfloor$$

$$< e_{n-1} + \lfloor \frac{n}{2} \rfloor$$

$$\leq e_n.$$ 

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To complete the proof it remains to consider the case $m = 1$. For this case, we will repeatedly use the following claim.

Claim 3.2. Suppose that there exists a vertex $z \in V(G)$ with $\deg(z) = \lceil \frac{n}{2} \rceil + 1$, and $G$ has a copy of gem$_4$ with at least three edges incident to $z$. Then $\phi(G, \text{gem}_4) < e_n$.

Proof. Let $F \subset G - z$ be the subgraph on $n - 1$ vertices, obtained from $G - z$ by deleting the edges of the copy of gem$_4$. By induction and (2.2), we have

$$\phi(G, \text{gem}_4) \leq \phi(F, \text{gem}_4) + 1 + \deg(z) - 3 \leq e_{n-1} + \left\lceil \frac{n}{2} \right\rceil - 1 < e_n.$$ 

We now consider three cases. Let $N(v) = V(G) \setminus (N(v) \cup \{v\})$, and note that

$$|N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq 4 \quad \text{and} \quad |\overline{N}(v)| = \left\lceil \frac{n}{2} \right\rceil - 2 \geq 2.$$ 

Case 1. $G[N(v)]$ contains a path $P$ of order 4.

Then $P$ and $v$ form a copy of gem$_4$, and we have $\phi(G, \text{gem}_4) < e_n$ by Claim 3.2.

Case 2. The order of the longest path in $G[N(v)]$ is 3.

Let $x_1x_2$ be a path of order 3 in $G[N(v)]$.

Subcase 2.1. $x_1x_2 \in E(G)$.

We have $\deg(x, N(v)) = 2$, for otherwise $G[N(v)]$ would contain a $P_4$. We must have $\deg(x, \overline{N}(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 3 \geq |\overline{N}(v)| - 1$. Similarly for $x_1, x_2$. This implies that two of $x, x_1, x_2$ have a common neighbour in $\overline{N}(v)$, say $y \in \overline{N}(v)$ is a common neighbour of $x, x_1$. Then $x_2vx_1y + x$ is a copy of gem$_4$, and by Claim 3.2 with $z = v$, we have $\phi(G, \text{gem}_4) < e_n$.

Subcase 2.2. $x_1x_2 \notin E(G)$.

Let $N(v) = \{x, x_1, x_2, \ldots, x_{\lfloor n/2 \rfloor}\}$. For $i = 1, 2$, we have $\deg(x_i, N(v)) = 1$, and

$$\deg(x_i, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rceil + 1 - 2 \geq \left\lfloor \frac{n}{2} \right\rceil - 2 = |\overline{N}(v)|. \tag{3.2}$$

We must have equality to hold throughout, whence $n$ is odd, $\deg(x_1) = \deg(x_2) = \left\lceil \frac{n}{2} \right\rceil + 1$, and both $x_1, x_2$ are adjacent to all vertices of $\overline{N}(v)$. If $x$ has a neighbour
$y \in \overline{N}(v)$, then $x_1v x_2y + x$ is a copy of gem$_4$, and again $\phi(G, \text{gem}) < e_n$ by Claim 3.2 with $z = v$.

Otherwise, suppose that $x$ does not have a neighbour in $\overline{N}(v)$. Then $\text{deg}(x) \leq |N(v)\cup\{v\}| - 1 = \left\lceil \frac{n}{2} \right\rceil + 1$, so that $\text{deg}(x) = \left\lceil \frac{n}{2} \right\rceil + 1$ and $xx_i \in E(G)$ for all $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. Moreover, we have $x_ix_j \notin E(G)$ for all $i \neq j$, otherwise there would exist a copy of $P_4$ in $G[N(v)]$. By a similar argument as in (3.2), we have $\text{deg}(x_i) = \left\lceil \frac{n}{2} \right\rceil + 1$, and $x_i$ is adjacent to all vertices of $\overline{N}(v)$ for all $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. In order to get a contradiction, suppose that there does not exist a path of order 3 in $G[\overline{N}(v)]$. Then the maximum number of edges in $G[\overline{N}(v)]$ is $\left\lfloor \frac{1}{2} |\overline{N}(v)| \right\rfloor$. Recall that $n$ is odd. We have

$$e(G) \leq 2|N(v)| - 1 + (|N(v)| - 1)|\overline{N}(v)| + \left\lfloor \frac{1}{2} |\overline{N}(v)| \right\rfloor$$

$$= 2 \left\lceil \frac{n}{2} \right\rceil + 1 + \left\lfloor \frac{n}{2} \right\rfloor \left( \left\lceil \frac{n}{2} \right\rceil - 2 \right) + \left\lfloor \frac{1}{2} \left( \left\lceil \frac{n}{2} \right\rceil - 2 \right) \right\rfloor$$

$$= \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n + 1}{4} \right\rfloor$$

by (2.1), which contradicts the assumption $e(G) > e_n$. Therefore, $G[\overline{N}(v)]$ must have a path of order 3, say $y_1y_2y_3$. Note that $|\overline{N}(v)| = \left\lceil \frac{n}{2} \right\rceil - 2 \geq 3$ and thus we must have $n$ odd and $n \geq 9$. Then, $x_1y_1x_2y_2 + y_2$ is a copy of gem$_4$, and by Claim 3.2 with $z = x_1$, we have $\phi(G, \text{gem}) < e_n$.

**Case 3.** The longest path in $G[N(v)]$ has order 2.

Note that this is indeed the remaining case, since $\text{deg}(x, N(v)) \geq 2m - 1 = 1$ for all $x \in N(v)$ by (3.1). Moreover $N(v)$ induces a perfect matching in $G$. Now by a similar argument as in (3.2), we must have $n$ odd, and for every $x \in N(v)$, we have $\text{deg}(x) = \left\lceil \frac{n}{2} \right\rceil + 1$ and $x$ is adjacent to all vertices of $\overline{N}(v)$. Thus, we can find an edge $x_1x_2$ in $G[N(v)]$ and a common neighbour $y \in \overline{N}(v)$ of $x_1, x_2$. Now since $vx_2y$ is a path of order 3 in $G[N(x_1)]$, we are done by applying Case 1 or Case 2 with $x_1$ in place of $v$.

The induction step is complete, and this completes the proof of Theorem 3.1. □
3.2 gem_5-decompositions

By using the same ideas as in Theorem 3.1, but with more case analysis, we will be able to prove a similar result for gem_5-decompositions. That is, we will prove the following theorem.

**Theorem 3.3.** For \( n \geq 8 \) we have

\[
\phi(n, \text{gem}_5) = \text{ex}(n, \text{gem}_5).
\]

Moreover, the only graphs attaining \( \text{ex}(n, \text{gem}_5) \) are the members of \( \mathcal{F}_{n,5} \).

**Proof.** Let \( n \geq 8 \). As before, we have \( \phi(n, \text{gem}_5) \geq \text{ex}(n, \text{gem}_5) \) by considering any graph of \( \mathcal{F}_{n,5} \). By Theorem 2.3, to prove the matching upper bound, we must prove that if \( G \) is a graph on \( n \geq 8 \) vertices, then \( \phi(G, \text{gem}_5) \leq \text{ex}(n, \text{gem}_5) = e_n \), with equality if and only if \( G \in \mathcal{F}_{n,5} \).

We proceed by induction on \( n \). For \( n = 8 \), if \( e(G) < e_8 = 18 \), then we can simply decompose \( G \) into single edges to obtain \( \phi(G, \text{gem}_5) < e_8 \). Next, suppose that \( 18 = e_8 \leq e(G) \leq 25 \). By Theorem 2.3, we either have \( G \in \mathcal{F}_{8,5} \), or \( G \) contains a copy of \( \text{gem}_5 \). If \( G \in \mathcal{F}_{8,5} \) then \( e(G) = e_8 = 18 \) and we must decompose \( G \) into single edges, and \( \phi(G, \text{gem}_5) = e_8 \). If \( G \) contains a copy of \( \text{gem}_5 \), then \( \phi(G, \text{gem}_5) \leq 1 + e(G) - e(\text{gem}_5) \leq 17 < 18 = e_8 \). Finally, suppose that \( 26 \leq e(G) \leq 28 \). Clearly, there exist two vertices \( x, y \in V(G) \) of degree 7, so that \( e(G - \{x, y\}) \geq 26 - 1 - 2 \cdot 6 = 13 \). Since \( \text{ex}(6, P_5) = \binom{4}{2} + \binom{2}{2} = 7 \) by Theorem 1.1, this means that we can find two edge-disjoint copies of \( P_5 \) in \( G - \{x, y\} \). These two copies of \( P_5 \), together with \( x \) and \( y \), form two edge-disjoint copies of \( \text{gem}_5 \) in \( G \). Thus, \( \phi(G, \text{gem}_5) \leq 2 + e(G) - 2e(\text{gem}_5) \leq 12 < 18 = e_8 \). The theorem holds for \( n = 8 \).

Now, let \( n \geq 9 \), and suppose that the theorem holds for \( n - 1 \). Let \( G \) be a graph on \( n \) vertices. As before, if \( e(G) < e_n \), then \( \phi(G, \text{gem}_5) < e_n \), simply by decomposing \( G \) into single edges. If \( e(G) = e_n \), then by Theorem 2.3, either \( G \) contains a copy of \( \text{gem}_5 \), in which case \( \phi(G, \text{gem}_5) \leq 1 + e(G) - e(\text{gem}_5) = e_n - 8 < e_n \); or \( G \in \mathcal{F}_{n,5} \), in which case we can only decompose \( G \) into \( e_n \) single edges for a \( \text{gem}_5 \)-decomposition, and \( \phi(G, \text{gem}_5) = e_n \) as required.

Now, suppose that \( e(G) > e_n \), and let \( v \in V(G) \) be a vertex of minimum degree. If \( \deg(v) \leq \lfloor \frac{n}{2} \rfloor \) then by equation (2.6), we have \( e(G - v) = e(G) - \deg(v) > e_n - \lfloor \frac{n}{2} \rfloor \geq \).
e_{n-1}, that is, \(G-v \not\in \mathcal{F}_{n-1,5}\). By induction, we have \(\phi(G-v, \text{gem}_5) < \text{ex}(n-1, \text{gem}_5) = e_{n-1}\). Thus, when going from \(G-v\) to \(G\) we only need to use the edges joining \(v\) to the other vertices of \(G\). We have 

\[
\phi(G, \text{gem}_5) \leq \phi(G-v, \text{gem}_5) + \deg(v) < e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \leq e_n.
\]

Therefore, we may assume that \(\deg(v) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1\) and let \(\deg(v) = \left\lfloor \frac{n}{2} \right\rfloor + m\) for some integer \(m \geq 1\). As in (3.1), for every \(x \in N(v)\), we have \(\deg(x, N(v)) \geq 2m - 1\), and that \(G[N(v)]\) must contain a path \(P_{2m}\) on \(2m\) vertices.

If \(m \geq 3\), then the path \(P_{2m}\) contains \(\left\lfloor \frac{2m}{5} \right\rfloor\) vertex-disjoint paths of order 5. Thus, we have \(\left\lfloor \frac{2m}{5} \right\rfloor\) edge-disjoint copies of \(\text{gem}_5\), where each copy is formed by a path of order 5, together with \(v\). Let \(F \subset G-v\) be the subgraph of order \(n-1\), obtained by deleting the edges of the paths of order 5 from \(G-v\). By induction and (2.6), and since \(m \geq 3\), we have 

\[
\phi(G, \text{gem}_5) \leq \phi(F, \text{gem}_5) + \frac{2m}{5} + \deg(v) - 5 \left\lfloor \frac{2m}{5} \right\rfloor
\]

\[
\leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor + m - 4 \left\lfloor \frac{2m}{5} \right\rfloor
\]

\[
< e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor
\]

\[
\leq e_n.
\]

For the rest of the proof, let \(\overline{N}(v) = V(G) \setminus (N(v) \cup \{v\})\). Next, suppose that \(m = 2\), so that \(|N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 2 \geq 6\) and \(|\overline{N}(v)| = \left\lfloor \frac{n}{2} \right\rfloor - 3 \geq 2\). If \(G[N(v)]\) contains a path \(P_5\) of order 5, then this path together with \(v\) form a copy of \(\text{gem}_5\). Let \(F \subset G-v\) be the subgraph of order \(n-1\), obtained by deleting the edges of the \(P_5\). Then, 

\[
\phi(G, \text{gem}_5) \leq \phi(F, \text{gem}_5) + 1 + \deg(v) - 5
\]

\[
\leq e_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor + 2 - 4
\]

\[
< e_n.
\]

Therefore, we may assume that the longest path in \(G[N(v)]\) has order 4. Let \(x_1x_2x_3x_4\) be such a path in \(G[N(v)]\). Since \(\deg(x_1, N(v)) \geq 2 \cdot 2 - 1 = 3\), we must have \(x_1x_3, x_1x_4 \in E(G)\). Moreover, the only neighbours of \(x_1\) in \(N(v)\) are \(x_2, x_3, x_4\), so that 

\[
\deg(x_1, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 2 - 4 \geq \left\lfloor \frac{n}{2} \right\rfloor - 3 = |\overline{N}(v)|.
\]
We must have equality, so that \( n \) is odd, \( \deg(x_1) = \left\lceil \frac{n}{2} \right\rceil + 2 \), and \( x_1 \) is adjacent to every vertex of \( \overline{N}(v) \). The same argument holds for \( x_4 \), so that \( x_1, x_4 \) have a common neighbour \( y \in \overline{N}(v) \). Now since \( vx_2x_3x_4y \) is a path of order 5 in \( G[N(x_1)] \), we are done by applying the previous argument with \( x_1 \) in place of \( v \).

To complete the proof it remains to consider the case \( m = 1 \). As before, we will repeatedly use the following claim which is analogous to Claim 3.2.

**Claim 3.4.** Suppose that there exists a vertex \( z \in V(G) \) with \( \deg(z) = \left\lceil \frac{n}{2} \right\rceil + 1 \), and \( G \) has a copy of \( gem_5 \) with at least three edges incident to \( z \). Then \( \phi(G, gem_5) < e_n \).

**Proof.** Exactly the same as the proof of Claim 3.2. \( \square \)

We now consider four cases. Note that we have
\[
|N(v)| = \left\lceil \frac{n}{2} \right\rceil + 1 \geq 5 \quad \text{and} \quad |\overline{N}(v)| = \left\lceil \frac{n}{2} \right\rceil - 2 \geq 3.
\]

**Case 1.** \( G[N(v)] \) contains a path \( P \) of order 5.

Then \( P \) and \( v \) form a copy of \( gem_5 \), and we have \( \phi(G, gem_5) < e_n \) by Claim 3.4.

**Case 2.** The order of the longest path in \( G[N(v)] \) is 4.

Let \( x_1x_2x_3x_4 \) be such a path in \( G[N(v)] \). It suffices to consider the following subcases.

**Subcase 2.1.** \( x_1x_3, x_1x_4 \in E(G) \).

For \( i = 1, 2, 3, 4 \), \( x_i \) does not have a neighbour in \( N(v) \setminus \{x_1, x_2, x_3, x_4\} \), so that \( \deg(x_i, N(v)) \leq 3 \). Thus,
\[
\deg(x_i, \overline{N}(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 4 \geq \left\lceil \frac{n}{2} \right\rceil - 4 = |\overline{N}(v)| - 2. \tag{3.3}
\]

If \( x_2x_4 \notin E(G) \), then we have \( \deg(x_j, N(v)) = 2 \), and \( \deg(x_j, \overline{N}(v)) \geq |\overline{N}(v)| - 1 \) for \( j = 2, 4 \). With (3.3), this implies that either \( x_1, x_2 \); or \( x_2, x_3 \); or \( x_1, x_3 \), have a common neighbour \( y \in \overline{N}(v) \). Then, either \( x_4v_2x_3x_4y + x_1 \); or \( x_4v_2x_3x_4y + x_3 \); or \( x_4v_2x_3y + x_1 \), is a copy of \( gem_5 \), respectively. By Claim 3.4 with \( z = v \), we have \( \phi(G, gem_5) < e_n \).

Now, if \( x_2x_4 \in E(G) \), then by (3.3), two of \( x_1, x_2, x_3, x_4 \) have a common neighbour in \( \overline{N}(v) \). We may assume that \( x_1, x_2 \) have a common neighbour \( y \in \overline{N}(v) \). Then we have \( \phi(G, gem_5) < e_n \) by the same argument.
Subcase 2.2. \( x_1x_3 \in E(G) \) and \( x_1x_4, x_2x_4 \not\in E(G) \).

We see that \( x_3 \) is the only neighbour of \( x_4 \) in \( N(v) \), so that

\[
\deg(x_4, N(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 2 \geq \left\lfloor \frac{n}{2} \right\rfloor - 2 = |N(v)|.
\]

We must have equality throughout, so that \( \deg(x_4) = \left\lceil \frac{n}{2} \right\rceil + 1 \) and \( n \) is odd. Moreover, \( x_4 \) is adjacent to every vertex of \( N(v) \). If \( x_3 \) has a neighbour \( y \in N(v) \), then \( x_1x_2yx_4y + x_3 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = v \). Now suppose that \( x_3 \) does not have a neighbour in \( N(v) \). Let \( x_5, x_6, \ldots, x_{|n/2|+1} \) be the remaining vertices of \( N(v) \). Then \( \deg(x_3) \geq \left\lceil \frac{n}{2} \right\rceil + 1 \) implies that \( x_3x_i \in E(G) \) for every \( i \geq 5 \). Moreover, we have \( x_1x_i, x_2x_i \not\in E(G) \) for all \( i \geq 5 \), otherwise we are in Subcase 2.1. This means that \( \deg(x_i) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( x_i \) is adjacent to every vertex of \( N(v) \) for all \( i \geq 4 \). Also, note that for \( i = 1, 2 \),

\[
\deg(x_i, N(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 3 = \left\lfloor \frac{n}{2} \right\rfloor - 3 = |N(v)| - 1.
\]

Suppose first that \( G[N(v)] \) contains a path of order 3, say \( y_1y_2y_3 \). If \( n \geq 11 \) so that \( |N(v)| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \geq 6 \), then \( x_4y_1x_3y_3x_6y_2 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = x_5 \). Now let \( n = 9 \), and suppose that \( x_1y_1, x_1y_2 \in E(G) \). Then \( x_1y_1x_4y_3x_5 + y_2 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = x_4 \). Thus, we may assume that \( x_1y_1, x_1y_3, x_2y_1, x_2y_3 \in E(G) \) and \( x_1y_2, x_2y_2 \not\in E(G) \). It is easy to check that \( G \) is the graph \( G''_9 \) with \( A''_9 = \{x_1, x_2, x_4, x_5\} \), \( B''_9 = \{v_3, y_1, y_3\} \), and \( y_2 \) is the remaining vertex, so that \( \phi(G, \text{gem}_5) = e_9 = \text{ex}(9, \text{gem}_5) \).

Now, suppose that \( G[N(v)] \) contains an edge, say \( y_1y_2 \). If \( x_1 \) is adjacent to every vertex in \( N(v) \), then we may assume that \( x_2y_1 \in E(G) \). Then \( x_3x_1x_2y_1y_2 + x_1 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = v \). Thus we may assume that \( x_1 \) and \( x_2 \) are not adjacent to exactly one vertex in \( N(v) \). Since there are at most \( |N(v)| \) edges in \( G[\overline{N}(v)] \) and at most \( \left\lfloor \frac{1}{2}|N(v)| \right\rfloor \) edges in \( G[\overline{N}(v)] \), we have

\[
e(G) \leq 2|N(v)| + 2(|N(v)| - 1) + (|N(v)| - 3)|N(v)| + \left\lfloor \frac{1}{2}|N(v)| \right\rfloor
= 2n - 4 + \left( \left\lfloor \frac{n}{2} \right\rfloor - 2 \right) \left( \left\lceil \frac{n}{2} \right\rceil - 2 \right) + \left\lfloor \frac{1}{2} \left( \left\lceil \frac{n}{2} \right\rceil - 2 \right) \right\rfloor
= \frac{n^2}{4} + \frac{n+1}{4}
= e_n,
\]
by (2.5) and since \( n \) is odd, which contradicts the assumption \( e(G) > e_n \). Finally, if \( G[\overline{N}(v)] \) does not contain an edge, then

\[
e(G) \leq 2|N(v)| + (|N(v)| - 1)|\overline{N}(v)|
\]

\[
= 2 \left( \left\lceil \frac{n}{2} \right\rceil + 1 \right) + \left\lceil \frac{n}{2} \right\rceil \left( \left\lceil \frac{n}{2} \right\rceil - 2 \right)
\]

\[
= \left\lceil \frac{n^2}{4} \right\rceil + 2 
\]

\[
\leq e_n,
\]

another contradiction.

**Subcase 2.3.** \( x_1x_4 \in E(G) \) and \( x_1x_3, x_2x_4 \notin E(G) \).

For \( i = 1, 2, 3, 4 \), \( x_i \) does not have a neighbour in \( N(v) \setminus \{x_1, x_2, x_3, x_4\} \), so that \( \deg(x_i, N(v)) = 2 \). Thus,

\[
\deg(x_i, \overline{N}(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 3 \geq \left\lceil \frac{n}{2} \right\rceil - 2 = |\overline{N}(v)| - 1. \tag{3.4} \]

If \( \deg(x_1, \overline{N}(v)) = |\overline{N}(v)| \), then we can find \( y_1, y_2 \in \overline{N}(v) \) such that, \( y_1 \) is a common neighbour of \( x_1, x_2 \), and \( y_2 \) is a common neighbour of \( x_2, x_3 \). Then \( y_1x_1vx_3y_2 + x_2 \) is a copy of \( \text{gem}_5 \), and we have \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = v \). Otherwise, we must have equality in (3.4) for \( i = 1, 2, 3, 4 \), so that \( n \) is odd, and for \( i = 1, 2, 3, 4 \), we have \( \deg(x_i) = \left\lceil \frac{n}{2} \right\rceil + 1 \), and \( x_i \) is not adjacent to exactly one vertex in \( \overline{N}(v) \). If \( n \geq 11 \) so that \( |\overline{N}(v)| = \left\lceil \frac{n}{2} \right\rceil - 2 \geq 4 \), then we can again find the vertices \( y_1, y_2 \in \overline{N}(v) \) and we are done as before. Now let \( n = 9 \), so that \( |N(v)| = 5, |\overline{N}(v)| = 3 \), and each \( x_i \) has exactly two neighbours in \( \overline{N}(v) \). If \( x_1 \) and \( x_2 \) have two common neighbours in \( \overline{N}(v) \), then we can again find \( y_1, y_2 \in \overline{N}(v) \) as before and we are done. Otherwise, we may assume that \( \overline{N}(v) = \{z_1, z_2, z_3\} \) with \( x_1z_1, x_1z_2, x_2z_1, x_2z_3 \in E(G) \). If \( z_1z_2 \in E(G) \), then \( x_4vx_2z_1z_2 + x_1 \) is a copy of \( \text{gem}_5 \), and again \( \phi(G, \text{gem}_5) < e_n \) by Claim 3.4 with \( z = v \). A similar argument holds if \( z_1z_3 \in E(G) \). Otherwise, we have at most one edge in \( G[\overline{N}(v)] \), and since there are exactly nine edges in \( G[N(v) \cup \{v\}] \) and at most \( 4 \cdot 2 + 3 = 11 \) edges between \( N(v) \) and \( \overline{N}(v) \), we have \( e(G) \leq 1 + 9 + 11 = 21 < 22 = e_9 \), which is a contradiction.

**Subcase 2.4.** \( x_1x_3, x_1x_4, x_2x_4 \notin E(G) \).

We first note that \( x_2 \) is the only neighbour of \( x_1 \) in \( N(v) \), so that

\[
\deg(x_1, \overline{N}(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 2 \geq \left\lceil \frac{n}{2} \right\rceil - 2 = |\overline{N}(v)|.
\]
We must have equality throughout, so that $n$ is odd, $\deg(x_1) = \lfloor \frac{n}{2} \rfloor + 1$, and $x_1$ is adjacent to all vertices of $\overline{N}(v)$. The exact same properties hold for $x_4$. Next, suppose that $x_2$ has $p$ neighbours in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$, where $0 \leq p \leq \lfloor \frac{n}{2} \rfloor - 3$. Let $S_2$ be the set of these $p$ neighbours. We have

$$\deg(x_2, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 3 - p = \left\lceil \frac{n}{2} \right\rceil - 3 - p. \quad (3.5)$$

Now, $x_3$ does not have a neighbour in $S_2$, otherwise there would exist a path of order 5 in $G[N(v)]$. Thus, $x_3$ has at most $|N(v)| - 4 - p = \left\lceil \frac{n}{2} \right\rceil - 3 - p$ neighbours in $N(v) \setminus \{x_1, x_2, x_3, x_4\}$. Let $S_3$ be these neighbours of $x_3$, so that $S_2 \cap S_3 = \emptyset$. We have

$$\deg(x_3, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 3 - \left( \left\lceil \frac{n}{2} \right\rceil - 3 - p \right) = p + 1. \quad (3.6)$$

Suppose that $x_2, x_3$ have a common neighbour $y_1 \in \overline{N}(v)$. Clearly from (3.5) and (3.6), at least one of $x_2, x_3$ has at least two neighbours in $\overline{N}(v)$. If $x_2$ has this property, then $x_1, x_2$ have a common neighbour $y_2 \in \overline{N}(v) \setminus \{y_1\}$. Thus, $y_1 x_3 y_1 x_2 + x_2$ is a copy of gem$_5$, and by Claim 3.4 with $z = v$, we have $\phi(G, \text{gem}_5) < e_n$. A similar argument holds if $x_3$ has at least two neighbours in $\overline{N}(v)$, with $x_4$ in place of $x_1$.

Thus, if $T_2, T_3 \subset \overline{N}(v)$ are the sets of neighbours of $x_2, x_3$ in $\overline{N}(v)$ respectively, then we may assume that $T_2 \cap T_3 = \emptyset$. Note that from (3.5) and (3.6), we have

$$\deg(x_2, \overline{N}(v)) + \deg(x_3, \overline{N}(v)) \geq \left\lfloor \frac{n}{2} \right\rfloor - 2 = |\overline{N}(v)|.$$

Thus, we must have equality above, as well as in (3.5) and (3.6). This means that $\deg(x_2) = \deg(x_3) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, and we have the partitions $N(v) \setminus \{x_1, x_2, x_3, x_4\} = S_2 \cup S_3$ and $\overline{N}(v) = T_2 \cup T_3$. Clearly there are no edges in $G[S_2 \cup S_3]$, otherwise there would exist a path of order 5 in $G[N(v)]$. Next, suppose that there is a path of order 3 in $G[\overline{N}(v)]$, say $y_1 y_2 y_3$. Suppose that $y_2 \in T_2$. Then $x_2 x_1 y_2 x_4 y_3 + y_2$ is a copy of gem$_5$, so that by Claim 3.4 with $z = x_1$, we have $\phi(G, \text{gem}_5) < e_n$. A similar argument holds if $y_2 \in T_3$. Otherwise, we have $|N(v)| - 1$ edges in $G[\overline{N}(v)]; |\overline{N}(v)|$ edges between $\{x_2, x_3\}$ and $\overline{N}(v);$ and at most $\lfloor \frac{n}{2} \rfloor |\overline{N}(v)|$ edges in $G[\overline{N}(v)]$. By (2.5) and since $n$ is
odd,
\[
e(G) \leq 2|N(v)| - 1 + |N(v)| + (|N(v)| - 2)|N(v)| + \left\lfloor \frac{1}{2}|N(v)| \right\rfloor
= 2\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor - 1 + \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \left( \left\lfloor \frac{n}{2} \right\rfloor - 2 \right) + \frac{1}{2} \left( \left\lfloor \frac{n}{2} \right\rfloor - 2 \right)
= \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lceil \frac{n+1}{4} \right\rceil
= e_n,
\]
which contradicts the assumption \(e(G) > e_n\).

**Case 3.** The order of the longest path in \(G[N(v)]\) is 3.

Let \(x_1x_2\) be such a path in \(G[N(v)]\). We consider the following subcases.

**Subcase 3.1.** \(x_1x_2 \in E(G)\).

We have \(\deg(x,N(v)) = 2\), for otherwise \(G[N(v)]\) would contain a \(P_4\). Thus
\[
\deg(x,N(v)) \geq \left\lceil \frac{n}{2} \right\rceil + 1 - 3 \geq \left\lfloor \frac{n}{2} \right\rfloor - 3 = |N(v)| - 1.
\]
Similar inequalities hold for \(x_1, x_2\). If \(\deg(x, \overline{N(v)}) = |N(v)|\), then there exist \(y_1, y_2 \in \overline{N(v)}\) such that, \(y_i\) is a common neighbour of \(x, x_i\) for \(i = 1, 2\). Then \(y_1x_1v_2y_2 + x\) is a copy of \(gem_5\), and by Claim 3.4 with \(z = v\), we have \(\phi(G, gem_5) < e_n\). Otherwise, we have \(\deg(x, \overline{N(v)}) = |N(v)| - 1\), whence \(n\) is odd and \(\deg(x) = \left\lfloor \frac{n}{2} \right\rfloor + 1\). We may assume that \(x, x_1\) have a common neighbour \(y \in \overline{N(v)}\). Now \(vx_2x_1y\) is a path of order 4 in \(G[N(x)]\), and we are done by applying Case 1 or Case 2 with \(x\) in place of \(v\).

**Subcase 3.2.** \(x_1x_2 \notin E(G)\).

Let \(N(v) = \{x, x_1, x_2, \ldots, x_{\lfloor n/2 \rfloor}\}\). For \(i = 1, 2\), we have
\[
\deg(x_i, \overline{N(v)}) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 2 \geq \left\lfloor \frac{n}{2} \right\rfloor - 2 = |\overline{N(v)}|.
\]
We must have equality to hold throughout, whence \(n\) is odd, \(\deg(x_1) = \deg(x_2) = \left\lfloor \frac{n}{2} \right\rfloor + 1\), and both \(x_1, x_2\) are adjacent to all vertices of \(\overline{N(v)}\). If \(x\) has neighbours \(y_1, y_2 \in \overline{N(v)}\), then we are done as in Subcase 3.1. If \(x\) has exactly one neighbour \(y \in \overline{N(v)}\), then we have
\[
\deg(x, N(v) \setminus \{x, x_1, x_2\}) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 - 4 \geq 1,
\]
and we may assume that \(xx_3 \in E(G)\). Then \(x_1yx_3x_2 + x\) is a copy of \(gem_5\), and we have \(\phi(G, gem_5) < e_n\) by Claim 3.4 with \(z = v\). Otherwise, suppose that \(x\) does not
have a neighbour in \( \overline{N}(v) \). We may apply the exact same argument as in Subcase 2.2 of Theorem 3.1 to deduce that, \( x_i \) is adjacent to all vertices of \( \overline{N}(v) \) for all \( 1 \leq i \leq \lceil \frac{n}{2} \rceil \), and \( G[\overline{N}(v)] \) must contain a path of order 3, say \( y_1y_2y_3 \). Then \( x_1y_1x_2y_3x_3 + y_2 \) is a copy of \( \text{gem}_5 \), and by Claim 3.4 with \( z = x_2 \), we have \( \phi(G, \text{gem}_5) < e_n \).

**Case 4.** The longest path in \( G[N(v)] \) has order 2.

Note that this is indeed the remaining case, since \( \deg(x, N(v)) \geq 2m - 1 = 1 \) for all \( x \in N(v) \). Moreover \( N(v) \) induces a perfect matching in \( G \). By a similar argument as in (3.7), we must have \( n \) odd, and for every \( x \in N(v) \), we have \( \deg(x) = \lceil \frac{n}{2} \rceil + 1 \) and \( x \) is adjacent to all vertices of \( \overline{N}(v) \). Thus, we can find an edge \( x_1x_2 \) in \( G[N(v)] \) and a common neighbour \( y \in \overline{N}(v) \) of \( x_1, x_2 \). Now since \( vx_2y \) is a path of order 3 in \( G[N(x_1)] \), we are done by applying Case 1, Case 2 or Case 3 with \( x_1 \) in place of \( v \).

The induction step is complete, and this completes the proof of Theorem 3.3.  

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