0-Hecke algebra actions on quotients of polynomial rings

Jia Huang

University of Nebraska at Kearney
E-mail address: huangj2@unk.edu

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The Symmetric Group S_n

- The *symmetric group* $S_n := \{\text{bijections on } \{1, \ldots, n\}\}$ is generated by the *adjacent transpositions* $s_i = (i, i+1)$, $1 \le i \le n-1$, with quadratic relations $s_i^2 = 1$, $1 \le i \le n-1$, and braid relations

$$\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & 1 \le i \le n-2, \\ s_i s_j = s_j s_i, & |i-j| > 1. \end{cases}$$

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- The *length* of any $w \in S_n$ is $\ell(w) := \min\{k : w = s_{i_1} \cdots s_{i_k}\}$, which coincides with $\mathrm{inv}(w) := \{(i,j) : 1 \le i < j \le n, w(i) > w(j)\}$.

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- For example, $w = 3241 \in S_4$ has $\ell(w) = \mathrm{inv}(w) = 4$ and reduced repressions $w = s_2 s_1 s_2 s_3 = s_1 s_2 s_1 s_3 = s_1 s_2 s_3 s_1$.

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$$\begin{cases} (T_i+1)(T_i-q)=0, & 1 \le i \le n-1, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & 1 \le i \le n-2, \\ T_iT_j = T_jT_i, & |i-j|>1. \end{cases}$$

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- It has significance in algebraic combinatorics, knot theory, quantum groups, representation theory of p-adic groups, etc.

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- Set $q = 0$: $H_n(q) \to H_n(0)$, $T_i \to \overline{\pi}_i$, $T_w \to \overline{\pi}_w$,

$$\begin{cases} \overline{\pi}_i^2 = -\overline{\pi}_i, & 1 \le i \le n-1, \\ \overline{\pi}_i\overline{\pi}_{i+1}\overline{\pi}_i = \overline{\pi}_{i+1}\overline{\pi}_i\overline{\pi}_{i+1}, & 1 \le i \le n-2, \\ \overline{\pi}_i\overline{\pi}_j = \overline{\pi}_j\overline{\pi}_i, & |i-j| > 1. \end{cases}$$

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- Sending π_i to $-\overline{\pi}_i$ gives an algebra automorphism.

Significance of the 0-Hecke algebra

- Using the automorphism $\pi_i \mapsto -\overline{\pi}_i$ of $H_n(0)$, Stembridge (2007) gave a short derivation for the Möbius function of the *Bruhat order* of the symmetric group S_n (or more generally, any Coxeter group).

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- Norton's result provides motivations to work of Denton, Hivert, Schilling, and Thiéry (2011) on the representation theory of finite *$\mathcal{J}$-trivial monoids*.
- Krob and Thibon (1997) discovered connections between $H_n(0)$-representations and certain generalizations of symmetric functions, which is similar to the classical Frobenius correspondence between S_n-representations and symmetric functions.

Analogies between S_n and $H_n(0)$

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- Analogies between other representations of S_n and $H_n(0)$?

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- $\pi_3(x_1^3 x_2 x_3 x_4^4) = x_1^3 x_2(-x_3^2 x_4^3 - x_3^3 x_4^2)$.

The coinvariant algebra of S_n

- The *invariant ring* $\mathbb{F}[X]^{S_n} := \{f \in \mathbb{F}[X] : wf = f, \forall w \in S_n\}$ consists of all symmetric functions in $x_1, \ldots, x_n$. It is a polynomial ring $\mathbb{F}[X]^{S_n} = \mathbb{F}[e_1, \ldots, e_n]$ in the *elementary symmetric functions*

$$e_k := \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}, \quad k = 1, \ldots, n.$$

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Theorem (Chevalley–Shephard–Tod 1955, indirect proof)

The coinvariant algebra $\mathbb{F}[X]/(e_1, \ldots, e_n)$ is isomorphic to the regular representation $\mathbb{F}S_n$ of S_n, if $\mathbb{F}$ is a field of characteristic 0.

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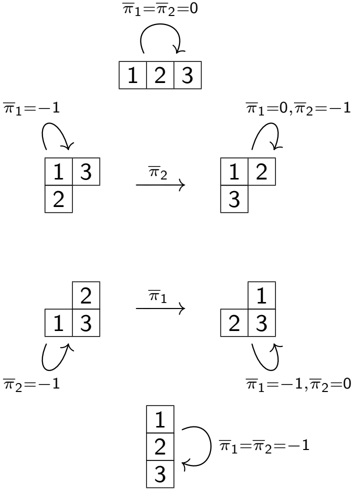
Remark

Our proof is constructive, using the *descent basis* of the coinvariant algebra given by Garsia and Stanton (1984).

$H_3(0) \cong \mathbb{F}[x_1, x_2, x_3]/(e_1, e_2, e_3)$



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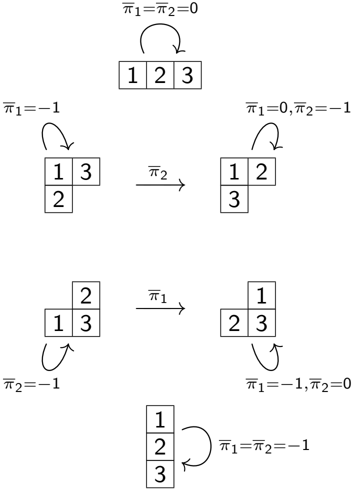

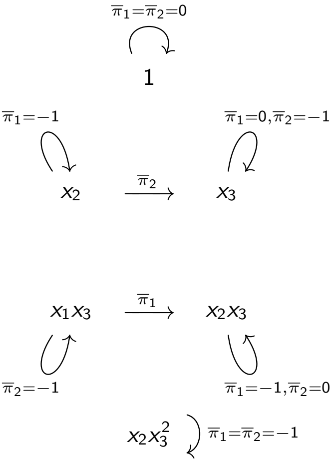

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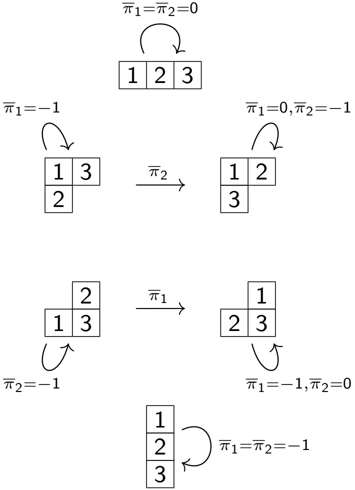

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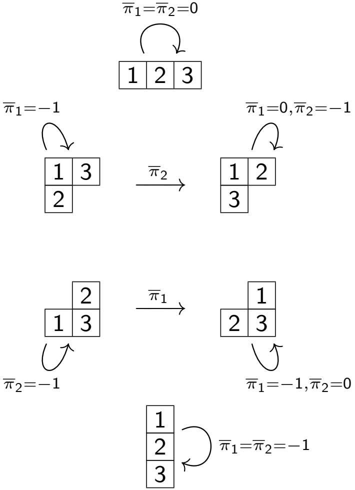

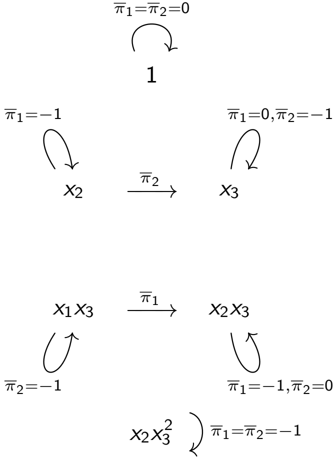

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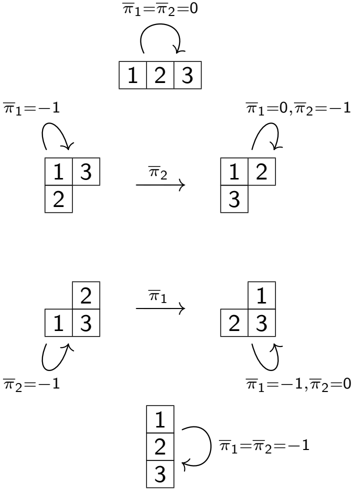
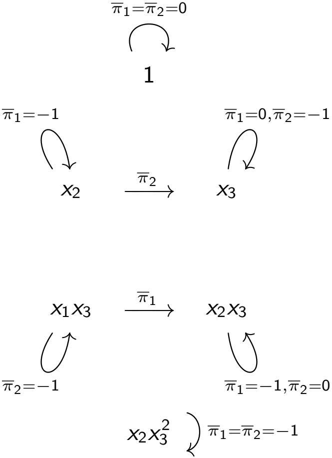
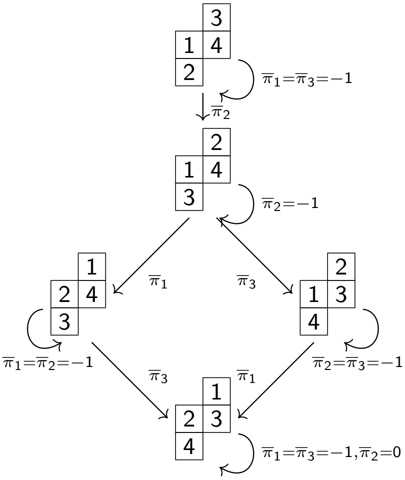




$$\mathbf{ch}_t = \mathbf{s}_3 + t\mathbf{s}_{12} + t^2\mathbf{s}_{21} + t^3\mathbf{s}_{111}$$

$\alpha = (1, 2, 1)$


3
1 4
2
$\overline{\pi}_1=\overline{\pi}_3=-1$
$\overline{\pi}_2$
2
1 4
3
$\overline{\pi}_2=-1$
$\overline{\pi}_1$
$\overline{\pi}_3$
1
2 4
3
$\overline{\pi}_1=\overline{\pi}_2=-1$
2
1 3
4
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1
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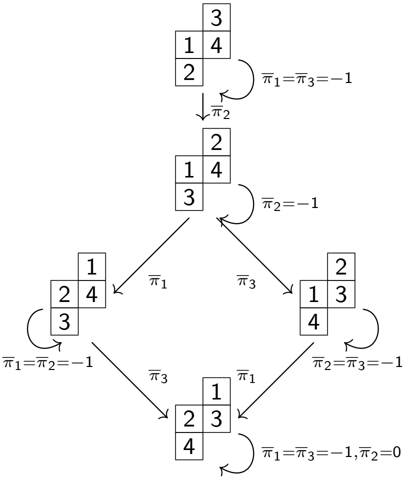

$\overline{\pi}_1{=}\overline{\pi}_3{=}{-1}$
$\overline{\pi}_2$
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$\overline{\pi}_1$
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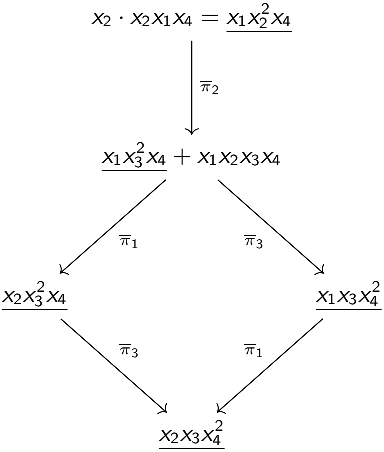
$\overline{\pi}_1{=}\overline{\pi}_3{=}{-1}, \overline{\pi}_2{=}0$



$x_2 \cdot x_2 x_1 x_4 = \underline{x_1 x_2^2 x_4}$
$\overline{\pi}_2$
$\underline{x_1 x_3^2 x_4} + x_1 x_2 x_3 x_4$
$\overline{\pi}_1$
$\overline{\pi}_3$
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A generalization of the coinvariant algebra

- Let $n \geq k \geq 1$ be two integers. Define a homogeneous ideal

$$I_{n,k} := \langle x_1^k, x_2^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle.$$

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The S_n-module structure of $R_{n,k}$

- Let $\mathcal{OP}_{n,k}$ be the set of all k-block partitions of the set $[n]$. For example, $(35|126|4) \in \mathcal{OP}_{6,3}$.

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Theorem (Haglund–Rhoades–Shimozono 2018)

As an ungraded S_n-module, $R_{n,k}$ is isomorphic to $\mathbb{C}[\mathcal{OP}_{n,k}]$. Moreover, the graded Frobenius characteristic of $R_{n,k}$ is

$$\sum_{\tau \in \mathrm{SYT}(n)} q^{\mathrm{maj}(\tau)} \binom{d - \mathrm{des}(\tau) - 1}{n-k}_q s_{\mathrm{shape}(\tau)}.$$

A 0-Hecke analogue

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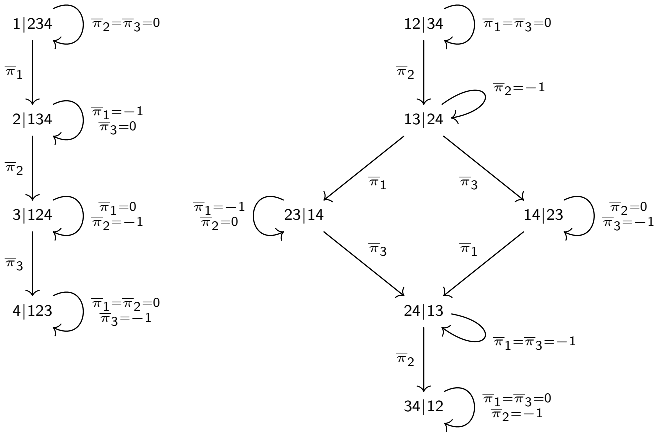
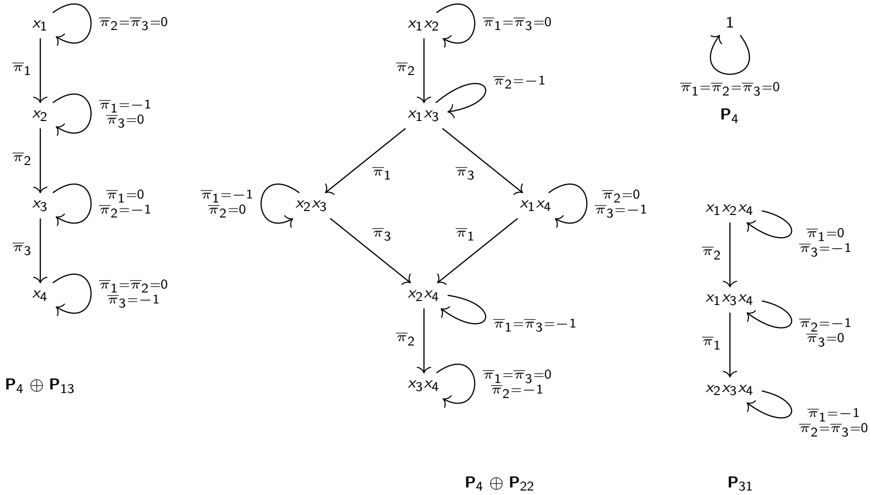
A decomposition of $\mathbb{F}[\mathcal{OP}_{4,2}]$



$\mathcal{OP}_{13} \cong \mathbf{P}_4 \oplus \mathbf{P}_{13}$ $\qquad$ $\mathcal{OP}_{22} \cong \mathbf{P}_4 \oplus \mathbf{P}_{22}$ $\qquad$ $\mathcal{OP}_{31} \cong \mathbf{P}_4 \oplus \mathbf{P}_{31}$

A decomposition of $S_{4,2}$



Graded characteristics of $S_{n,k}$

Theorem (H.–Rhoades 2018)

The graded $H_n(0)$-module $S_{n,k}$ corresponds

$$\sum_{\alpha \models n} t^{\mathrm{maj}(\alpha)} \begin{bmatrix} n-\ell(\alpha) \\ k-\ell(\alpha) \end{bmatrix}_t \mathbf{s}_\alpha \quad \textit{inside } \mathbf{NSym}$$

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Remark

This result connects to the *Delta Conjecture* of Haglund, Remmel, and Wilson (2016) in the theory of Macdonald polynomials.

More quotients of the polynomial ring

Theorem (DeConcini, Garsia, Procesi, Hotta, Springer, Tanisaki)

- *For any* $\mu \vdash n$, $\mathbb{C}[X]$ *has a homogeneous* S_n*-stable ideal* J_μ *generated by certain elementary symmetric functions in partial variable sets.*

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- *The graded Frobenius characteristic of $R_\mu = \mathbb{C}[X]/J_\mu$ is the modified Hall-Littlewood symmetric function*

$$\widetilde{H}_\mu(x;t) = \sum_\lambda t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda$$

where $n(\mu) = \mu_2 + 2\mu_3 + 3\mu_4 + \cdots$ and $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.

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Theorem (H. 2014)

- *The ideal J_μ is $H_n(0)$-stable if and only if $\mu = (1^k, n-k)$ is a hook. Assume μ is a hook below.*

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$H_n(0)$-action on $R_\mu = \mathbb{C}[X]/J_\mu$

Theorem (H. 2014)

- *The ideal J_μ is $H_n(0)$-stable if and only if $\mu = (1^k, n-k)$ is a hook. Assume μ is a hook below.*
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- We are currently investigate a two-parameter family of quotients of the Stanley-Reisner ring (with Daniël Kroes).

Thank you!