Graph spectral conditions and structural properties

Hong-Jian Lai

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- $\lambda(G) = \lambda_1(G)$: spectral radius of G.

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- The Problem: Can spectral conditions of G be used to predict the structural properties of G?

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- where equality holds iff G is complete or an odd cycle.
- This has been extended to group colorings in X. K. Zhang's dissertation (WVU 1998).

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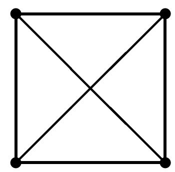
- Problem (Abiad, Brimkov, MartÍnez-Rivera, O, and Zhang, Electronic Journal of Linear Algebra, 2018) Find best possible condition on $\lambda_2(G)$ to warrant $\kappa(G) \geq k$.

Edge-Disjoint Spanning Trees

- Example



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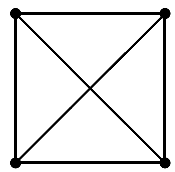
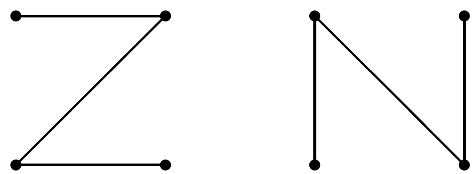


- Two edge-disjoint spanning trees ($\tau(K_4) = 2$)



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- Theorem (Nash-Williams, Tutte [J. London Math. Soc. (1961)]) For a connected graph G, $\tau(G) \geq k$ if and only if for any partition $(V_1, V_2, ..., V_t)$ of $V(G)$,

$$\frac{1}{2}\sum_{i=1}^{t} d(V_i) = \sum_{1 \leq i < j \leq t} |[V_i, V_j]_G| \geq k(t-1).$$

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- There is an equivalent version of the theorem.

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- (iii) $\forall X \subseteq E(G)$, $|X| \geq k(\omega(G - X) - 1)$.

The κ'-τ Lemma

- The κ'-τ Lemma (Gusfield, IPL 1983, and Catlin, Shao, HJL DM 2009) $\kappa'(G) \geq 2k$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.

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- Sufficiency: Any edge cut must have size at least $2k$.
- Necessity: Take a partition $(V_1, V_2, ..., V_t)$ of $V(G - X)$,

$$
\begin{aligned}
2 \sum_{1 \leq i < j \leq t} |[V_i, V_j]_{G-X}| &= \sum_{i=1}^{t} |[V_i, V - V_i]_G| - 2|X| \\
&\geq 2kt - 2k = 2k(t-1).
\end{aligned}
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Then apply Nash-Williams and Tutte Theorem.

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- Apply The κ'-τ Lemma.
- Corollary: (Cioaba, LAA 2010) If $\lambda_2(G) < d - \frac{4k-2}{d+1}$, then $\tau(G) \geq k$.

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- Conjecture (Cioaba and Wong, LAA 2012) Assume that $2 \leq 2k \leq d$. If $\lambda_2(G) < d - \frac{2k-1}{d+1}$, then $\tau(G) \geq k$.

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- Conjecture (k, δ) Let G be graph with $\delta(G) = \delta$ and $2k \leq \delta$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.

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- Theorem (G. Li and L. Shi, LAA 2013; Y. Hong, Q. Liu, and HJL, LAA 2014) For any integer $k \geq 2$ and $\delta \geq 2k$, there exists an integer $N = N(k, \delta)$ such that if $n \geq N$, then Conjecture(k, δ) holds,

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- Conjecture (Gu et al.) Let G be a graph with minimum degree $\delta \geq 2k \geq 4$. If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.

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- Theorem. (Liu, Hong, Gu, HJL, LAA 2014) Let k be an integer and G be a graph of order n and minimum degree $\delta \geq 2k$. If $\lambda_2(G,a) < (a+1)\delta - \frac{2k-1}{\delta+1}$ then $\tau(G) \geq k$.

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- Choose different values of $a \in \{0, 1, -1\}$.

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- $\lambda_i(G)$: = the ith largest eigenvalue of A.
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- Theorem. (Liu, Hong, Gu, HJL, LAA 2014)
  (1) If $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
  (2) If $q_2(G) < 2\delta - \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.
  (3) If $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$, then $\tau(G) \geq k$.

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- $N_G(u) \subseteq U$.
- $|U| \geq |\{u\} \cup N_G(u)| \geq 1 + \delta$.

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$$
\begin{aligned}
|[X,Y]|^2 \quad \geq \quad & ((a+1)\delta - \frac{d(X)}{|X|} - \lambda_2(G,a)) \cdot \\
& ((a+1)\delta - \frac{d(Y)}{|Y|} - \lambda_2(G,a))|X| \cdot |Y|.
\end{aligned}
$$

Proof of Cioaba-Wong Conjecture (i)

- Theorem Let k be an integer and G be a graph of order n and minimum degree $\delta \geq 2k$. If $\lambda_2(G, a) < (a+1)\delta - \frac{2k-1}{\delta+1}$ then $\tau(G) \geq k$.

Proof of Cioaba-Wong Conjecture (i)

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- Approach of the proof: For any partition $(V_1, V_2, \ldots, V_t)$, want to prove $\sum_{1 \leq i < j \leq t} |[V_i, V_j]_G| \geq k(t-1)$.

Proof of Cioaba-Wong Conjecture (ii)

- Assume that $d(V_1) \leq d(V_2) \leq \ldots \leq d(V_t)$.

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- Assume that $d(V_1) \leq d(V_2) \leq \ldots \leq d(V_t)$.
- If $d(V_1) \geq 2k$, then $\sum_{1\leq i<j\leq t} |[V_i, V_j]_G| \geq kt$. Assume $d(V_1) \leq 2k-1$.
- Let $1 \leq s \leq t$ be such that $d(V_s) \leq 2k-1$ and $d(V_{s+1}) \geq 2k$ (if $s < t$).

Proof of Cioaba-Wong Conjecture (ii)

- Assume that $d(V_1) \leq d(V_2) \leq \ldots \leq d(V_t)$.
- If $d(V_1) \geq 2k$, then $\sum_{1 \leq i < j \leq t} |[V_i, V_j]_G| \geq kt$. Assume $d(V_1) \leq 2k - 1$.
- Let $1 \leq s \leq t$ be such that $d(V_s) \leq 2k - 1$ and $d(V_{s+1}) \geq 2k$ (if $s < t$).
- By U-lemma, for $1 \leq i \leq s$, $|V_i| \geq \delta + 1$.

Proof of Cioaba-Wong Conjectur (iii)

- Assumption of Theorem, for $1 \le i \le s$.

$$\lambda_2(G,a) < (a+1)\delta - \frac{2k-1}{\delta+1} \le (a+1)\delta - \frac{d(V_i)}{|V_i|}.$$

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- By Quadratic Inequality, for $2 \leq i \leq s$,

$$\begin{aligned} |[V_1, V_i]|^2 &\geq \left((a+1)\delta - \frac{d(V_1)}{|V_1|} - \lambda_2(G,a)\right) \cdot \\ &\quad \left((a+1)\delta - \frac{d(V_i)}{|V_i|} - \lambda_2(G,a)\right)|V_1| \cdot |V_i| \\ &> (2k-1-d(V_1))(2k-1-d(V_i)) \\ &\geq (2k-1-d(V_i))^2. \end{aligned}$$

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- $d(V_1) \geq \sum_{i=2}^{s} |[V_1, V_i]| \geq \sum_{i=2}^{s} \big(2k - d(V_i)\big)$.
- $$\begin{aligned} \sum_{i=1}^{t} d(V_i) &= d(V_1) + \sum_{i=2}^{s} d(V_i) + \sum_{i=s+1}^{t} d(V_i) \\ &\geq 2k(s-1) + 2k(t-s) = 2k(t-1). \end{aligned}$$

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Connectivity and eigenvalue

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  (ii) If $\mu_{n-1}(G) > \frac{(k-1)\Delta n}{2\phi(\delta,\Delta,k)}$, then $\kappa(G) \geq k$.
  (iii) If $q_2(G) < 2\delta - \frac{(k-1)\Delta n}{2\phi(\delta,\Delta,k)}$, then $\kappa(G) \geq k$.

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- New Useful Lemma. Let G be a simple connected graph with $\delta = \delta(G) \ge k \ge 2$ and girth $g = g(G) \ge 3$. Let C be a minimum vertex cut of G with $|C| = c$ and U be a connected component of $G - C$. If $c \le k - 1 < \delta$, then $|V(U)| \ge \nu(\delta, g, c)$.

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- By contradiction, we assume $\kappa(G) = c \leq k-1$.

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- Let $\bar{d_1} = \dfrac{1}{n_1} \sum_{v \in A} d_G(v)$ and $\bar{d_2} = \dfrac{1}{n_2 + c} \sum_{v \in \overline{A}} d_G(v)$.

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- to conclude

$$\lambda_2(R(aD + A)) \geq (a+1)\delta - \frac{(k-1)\Delta n}{2\nu(n-\nu)}.$$

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- Given two sequences $\theta_1 \geq \theta_2 \geq \cdots \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence interlaces the first if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$, for $1 \leq i \leq m$.

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- By interlacing (we have a contradiction)

$$\lambda_2(aD + A) \geq \lambda_2(R(aD + A)) \geq (a+1)\delta - \frac{(k-1)\Delta n}{2\nu(n-\nu)}.$$

Thank You