# Connected subgraphs in edge-coloured graphs

Henry Liu[1]

Based on a joint survey with Shinya Fujita[2] and Colton Magnant[3]

[1]SYSU, China
[2]Yokohama City University, Japan
[3]Georgia Southern University, USA

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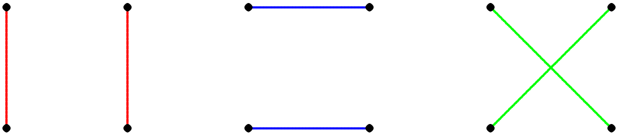

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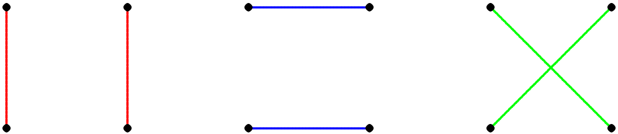

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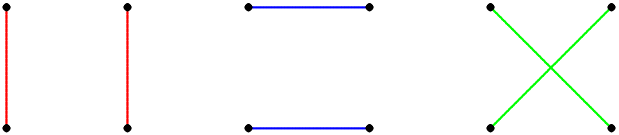

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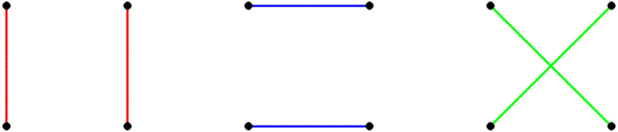

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- Implies that, if $r - 1$ is a prime power, then there is an $r$-colouring of $K_{(r-1)^2}$ such that the largest monochromatic connected subgraph has $r - 1$ vertices.

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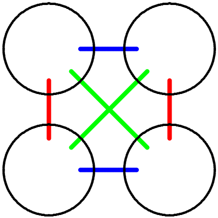

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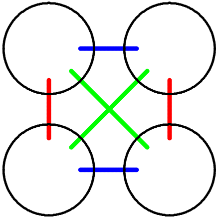

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Largest monochromatic subgraph has at most

$$(r-1)\left\lceil \frac{n}{(r-1)^2} \right\rceil < \frac{n}{r-1} + r$$

vertices, i.e. $m(n,r) < \frac{n}{r-1} + r$.

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$$\begin{aligned} \mathbb{E}Z &= \frac{1}{e(H)} \sum_{xy \in E(H)} (d(x) + d(y)) = \frac{1}{e(H)} \sum_{v \in V(H)} d(v)^2 \\ &\overset{\text{C-S}}{\geq} \frac{1}{e(H)} \left( \frac{1}{m} + \frac{1}{n} \right) e(H)^2 \geq \frac{m+n}{r}. \end{aligned}$$

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(c) *broom (i.e. a path with a star at one end) (Burr 1992).*

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Let $f(n, r)$ be the maximum integer $\ell$ such that, every $r$-colouring of $K_n$ contains a monochromatic cycle of length at least $\ell$. The affine plane construction gives $f(n, r) < \frac{n}{r-1} + r$ if $r - 1$ is a prime power. Inspired by this, they also conjectured:

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Fujita, Lesniak, Tóth (2015) showed that Conjecture 7 holds when $n$ is linear in $r$, with $r$ sufficiently large.

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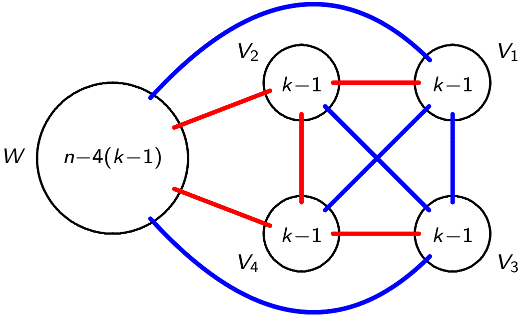

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- ~~Conjecture holds (Łuczak 2016).~~ Mistake found by Furuya.

## Conjecture 8 (Bollobás, Gyárfás 2003)

*For $n > 4(k-1)$, we have $m(n,2,k) = n - 2k + 2$.*

True for:

- $k = 1$ (Erdős and Rado observation);
- $k = 2$ (Bollobás, Gyárfás 2003);
- $k = 3$ (L., Morris, Prince 2004);
- $n \geq 13k - 15$ (L., Morris, Prince 2004);
- $n > 6.5(k-1)$ (Fujita, Magnant 2011); ← Best partial result
- ~~Conjecture holds (Łuczak 2016).~~ Mistake found by Furuya.

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**Theorem 10 (L., Morris, Prince 2004)**

(a) *For $r \geq 3$, we have $m(n,r,k) \geq \frac{n}{r-1} - 11k(k-1)r$. Hence, if $k, r$ are fixed and $r-1$ is a prime power, then $m(n,r,k) = \frac{n}{r-1} + O(1)$.*

(b) *For $n \geq 480k$, we have $m(n,3,k) \geq \frac{n-k+1}{2}$.*

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*Any Gallai colouring of a complete graph can be obtained by substituting complete graphs with Gallai colourings for the vertices of a 2-coloured complete graph on at least two vertices.*

Theorem 11 is a "decomposition theorem". It is widely used to prove results about Gallai colourings.

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**Theorem 13 (Gyárfás, Simonyi 2004)**

*For every Gallai colouring of $K_n$, there is a monochromatic star with at least $\frac{2n}{5}$ vertices. This bound is sharp.*

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**Theorem 14 (Fujita, Magnant 2013)**

*Let $r \geq 3$ and $k \geq 2$. If $n \geq (r + 11)(k - 1) + 7k \log k$. Then in any Gallai colouring of $K_n$ with $r$ colours, there is a monochromatic $k$-connected subgraph on at least $n - r(k - 1)$ vertices.*

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### Problem 15

*Improve the bound $n \geq (r+11)(k-1) + 7k \log k$ in Theorem 14.*

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They remarked that this can be extended to $r$-colourings, with $\alpha(r-1)$ in the role of $\alpha$.

**Theorem 17 (Gyárfás, Sárközy 2010)**

*For every Gallai colouring of a graph $G$ with $n$ vertices and $\alpha(G) = \alpha$, there exists a monochromatic connected subgraph on at least $\frac{n}{\alpha^2+\alpha-1}$ vertices. This is close to being tight.*

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## Theorem 18 (Fujita, L., Sarkar 2016)

*Let $G$ be a graph with $n$ vertices and $\alpha(G) = \alpha$. If $n > \alpha^2 k$, then $G$ contains a $k$-connected subgraph on at least $\lceil \frac{n}{\alpha} \rceil$ vertices.*

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We remark that in Problem 19, the best that we can hope for is to improve the bound to approximately $n \geq \frac{9}{4}\alpha(k-1)$, for $\alpha \geq 3$.

Theorem 20 (Fujita, L., Sarkar 2016)

*Let $G$ be a graph with $n$ vertices and $\alpha(G) = 2$. If $n \geq 4(k-1)$, then $G$ contains a $k$-connected subgraph on at least $\left\lceil \frac{n}{2} \right\rceil$ vertices.*

Theorem 20 (Fujita, L., Sarkar 2016)

*Let $G$ be a graph with $n$ vertices and $\alpha(G) = 2$. If $n \geq 4(k-1)$, then $G$ contains a $k$-connected subgraph on at least $\lceil \frac{n}{2} \rceil$ vertices.*

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*Let $G$ be a graph with $n$ vertices and $\alpha(G) = 3$. If $n \geq \frac{27}{4}(k-1)$, then $G$ contains a $k$-connected subgraph on at least $\lceil \frac{n}{3} \rceil$ vertices.*

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Problem 22

*What happens for the edge-coloured versions of these results?*

# References

1. S. Fujita, H. Liu, C. Magnant, Monochromatic structures in edge-coloured graphs and hypergraphs - a survey, *Int. J. Graph Theory Appl.* 1 (2015) 3–56.
2. S. Fujita, H. Liu, A. Sarkar, Highly connected subgraphs of graphs with given independence number, submitted.
3. H. Liu, R. Morris, N. Prince, Highly connected monochromatic subgraphs of multicolored graphs, *J. Graph Theory* 61 (2009) 22–44.
4. `http://www.cantab.net/users/henry.liu/`

Thank you!