Forbidding tight cycles in hypergraphs

Hao Huang

Emory University

Aug 13, 2018

Joint work with Jie Ma (USTC).

Forbidding cycles in graphs

A classic result

An n-vertex cycle-free graph has at most $n-1$ edges.

Forbidding cycles in graphs

A classic result

An n-vertex cycle-free graph has at most $n-1$ edges.

Proof.

- Easiest proof: see any graph theory textbook.
- An overcomplicated "proof":

$$ex(n, C_{2k}) = O(n^{1+1/k}).$$

Question

Can we generalize these results to k-uniform hypergraphs?

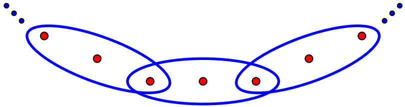
How to generalize cycles to hypergraphs?

- Berge cycle BC_ℓ^k

$$v_1, e_1, v_2, e_2, \cdots, v_m, e_m, v_1 \qquad \forall i,\ v_i, v_{i+1} \in e_i.$$

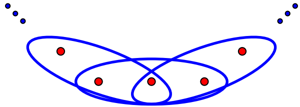
- Loose cycle LC_ℓ^k

$$(v_1, \cdots, v_k), (v_k, \cdots, v_{2k-1}), \cdots, (v_{1+(k-1)(\ell-1)}, \cdots, v_1)$$



- Tight cycle TC_ℓ^k

$$(v_1, \cdots, v_k), (v_2, \cdots, v_{k+1}), \cdots, (v_\ell, \cdots, v_{k-1})$$



Berge cycles and Loose cycles

Berge cycle:

Theorem (Györi, Lemons 2012)

For $k \geq 3, \ell \geq 4$,

$$ex(n, BC_\ell^k) = O(n^{1+\frac{1}{\lceil(\ell-1)/2\rceil}}).$$

Loose cycle:

Theorem (Kostochka, Mubayi, Verstraëte 2013)

For $k, \ell \geq 3$,

$$ex(n, LC_\ell^k) \sim \left\lfloor \frac{\ell-1}{2} \right\rfloor \binom{n}{k-1}.$$

Previous study of tight cycles

Two most important results:

Theorem (Rödl, Ruciński, Szemerédi, 2011)

For all $\gamma > 0$, every sufficiently large k-uniform hypergraph in which every $(k-1)$-set of vertices lies in at least $(1/2+\gamma)n$ edges contains a tight Hamilton cycle.

Theorem (Allen, Böttcher, Cooley, Mycroft 2017)

For every $\delta > 0$, there exists n_0 such that for any $0 < \alpha \leq 1$, every k-uniform hypergraph on $n \geq n_0$ vertices with at least $(\alpha+\delta)\binom{n}{k}$ edges contains a tight cycle of length at least αn.

Most of these works focus on the case when the extremal hypergraph is dense.

Tight-cycle-free hypergraphs

A k-uniform hypergraph is tight-cycle-free if it does not contain TC_ℓ^k for any $\ell \geq k+1$.

Denote by $f_k(n)$ the maximum number of edges in a tight-cycle-free k-uniform n-vertex hypergraph.

Observation

$$\binom{n-1}{k-1} \leq f_k(n) = O(n^k).$$

Tight-cycle-free hypergraphs

A k-uniform hypergraph is tight-cycle-free if it does not contain TC_ℓ^k for any $\ell \geq k+1$.

Denote by $f_k(n)$ the maximum number of edges in a tight-cycle-free k-uniform n-vertex hypergraph.

Observation

$$\binom{n-1}{k-1} \leq f_k(n) = O(n^k).$$

Proof. Consider a full-k-star centered at 1.

$$H = \{S : 1 \in S \subset [n] : |S| = k\}.$$

Forbidding tight cycles of fixed length

Question (Conlon)

Does there exist a constant $c > 0$ such that $ex(n, TC_\ell^3) = O(n^{2+c/\ell})$ for all ℓ which are divisible by 3?

If true, this would imply $f_3(n) = O(n^{2+o(1)})$.

Proposition

$$\Omega(n^{2.5}) = ex(n, TC_6^3) = O(n^{2.75}).$$

- Lower bound: let $|A| = |B| = |C| = n$, take C_4-free bipartite graph G on $B \cup C$. Let $E(H) = \{i \cup e : i \in A, e \in E(H)\}$.
- Upper bound: $TC_6^3 \subset K_{2,2,2}^3$.

Theorem (Verstraëte)

$$ex(n, TC_{12}^3) = O(n^{2.5}).$$

The main result

Question (Verstraëte, Sös)

Is it true that for all $k \geq 2$,

$$f_k(n) = \binom{n-1}{k-1}?$$

Theorem (H., Ma 2018+)

For every $k \geq 3$, there exists a constant $c = c(k) > 0$, such that for sufficiently large n,

$$f_k(n) \geq (1+c)\binom{n-1}{k-1}.$$

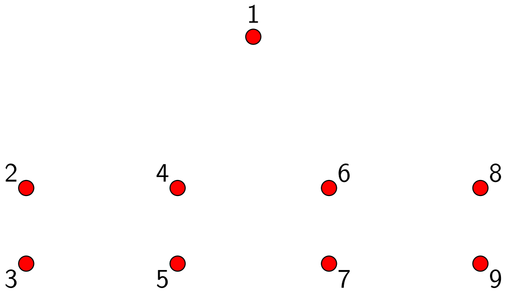
- $k = 3$: $c > \frac{1}{5}$.
- $k \geq 4$: $c > \frac{1}{\binom{O(k^3)}{k}}$.

An easy counterexample for $k = 3$

Construction for $k = 3$

Let n be an odd integer ≥ 7.

- Take a star $\mathcal{F} = \{S : 1 \in S \subset [n], |S| = 3\}$.
- Remove $(n-1)/2$ triples of the form $(1, 2i, 2i+1)$.
- Add $n-1$ triples $(2i, 2i+1, 2i+2)$, and $(2i, 2i+1, 2i+3)$.

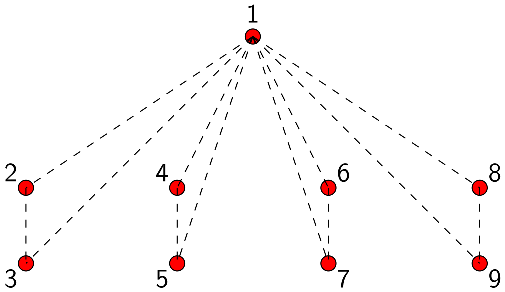


An easy counterexample for $k = 3$

Construction for $k = 3$

Let n be an odd integer ≥ 7.

- Take a star $\mathcal{F} = \{S : 1 \in S \subset [n], |S| = 3\}$.
- Remove $(n-1)/2$ triples of the form $(1, 2i, 2i+1)$.
- Add $n-1$ triples $(2i, 2i+1, 2i+2)$, and $(2i, 2i+1, 2i+3)$.

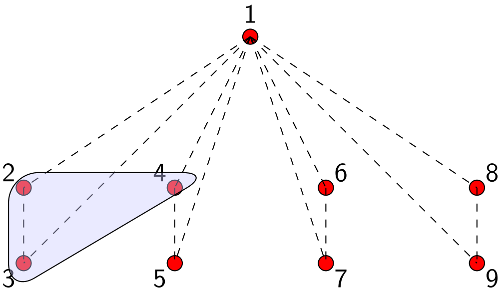


An easy counterexample for $k = 3$

Construction for $k = 3$

Let n be an odd integer ≥ 7.

- Take a star $\mathcal{F} = \{S : 1 \in S \subset [n], |S| = 3\}$.
- Remove $(n-1)/2$ triples of the form $(1, 2i, 2i+1)$.
- Add $n-1$ triples $(2i, 2i+1, 2i+2)$, and $(2i, 2i+1, 2i+3)$.

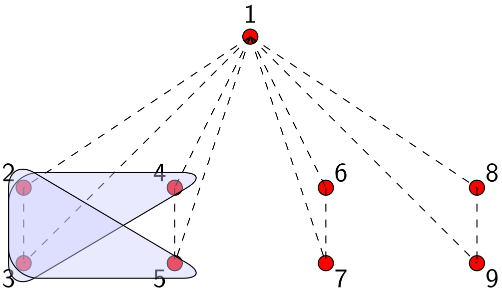


An easy counterexample for $k = 3$

Construction for $k = 3$

Let n be an odd integer ≥ 7.

- Take a star $\mathcal{F} = \{S : 1 \in S \subset [n], |S| = 3\}$.
- Remove $(n-1)/2$ triples of the form $(1, 2i, 2i+1)$.
- Add $n-1$ triples $(2i, 2i+1, 2i+2)$, and $(2i, 2i+1, 2i+3)$.

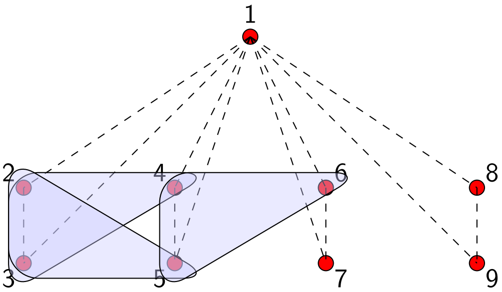


An easy counterexample for $k = 3$

Construction for $k = 3$

Let n be an odd integer ≥ 7.

- Take a star $\mathcal{F} = \{S : 1 \in S \subset [n], |S| = 3\}$.
- Remove $(n-1)/2$ triples of the form $(1, 2i, 2i+1)$.
- Add $n-1$ triples $(2i, 2i+1, 2i+2)$, and $(2i, 2i+1, 2i+3)$.

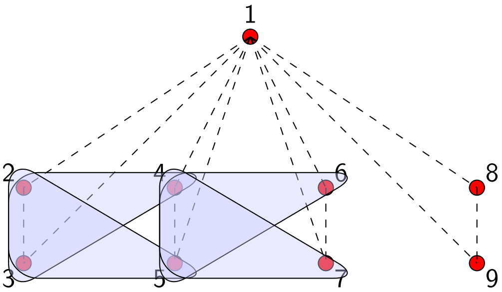


An easy counterexample for $k = 3$

Construction for $k = 3$

Let n be an odd integer ≥ 7.

- Take a star $\mathcal{F} = \{S : 1 \in S \subset [n], |S| = 3\}$.
- Remove $(n-1)/2$ triples of the form $(1, 2i, 2i+1)$.
- Add $n-1$ triples $(2i, 2i+1, 2i+2)$, and $(2i, 2i+1, 2i+3)$.

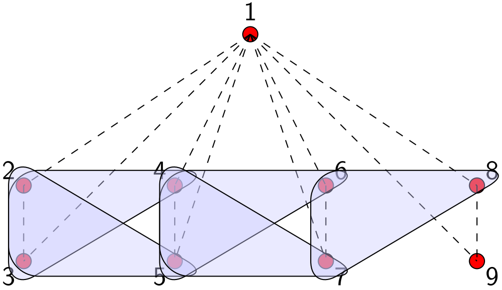


An easy counterexample for $k = 3$

Construction for $k = 3$

Let n be an odd integer ≥ 7.

- Take a star $\mathcal{F} = \{S : 1 \in S \subset [n], |S| = 3\}$.
- Remove $(n-1)/2$ triples of the form $(1, 2i, 2i+1)$.
- Add $n-1$ triples $(2i, 2i+1, 2i+2)$, and $(2i, 2i+1, 2i+3)$.

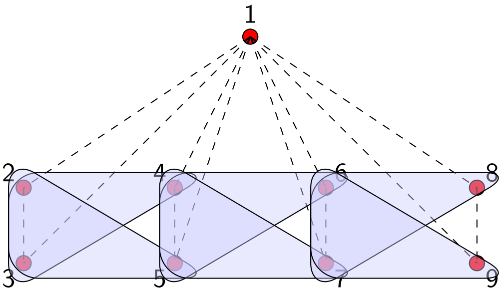


An easy counterexample for $k = 3$

Construction for $k = 3$

Let n be an odd integer ≥ 7.

- Take a star $\mathcal{F} = \{S : 1 \in S \subset [n], |S| = 3\}$.
- Remove $(n-1)/2$ triples of the form $(1, 2i, 2i+1)$.
- Add $n-1$ triples $(2i, 2i+1, 2i+2)$, and $(2i, 2i+1, 2i+3)$.



Kalai's Conjecture

However, this gives only $f_3(n) \geq \binom{n-1}{2} + \frac{n-1}{2} = (1 + o(1))\binom{n-1}{2}$.
How can we show $f_3(n) \geq (1 + c)\binom{n-1}{2}$ for positive c?

Kalai's Conjecture

For every integer k and ℓ, suppose H is a k-uniform n-vertex hypergraph not containing TP_ℓ^k, then

$$e(H) \leq \frac{\ell - 1}{k}\binom{n}{k-1}.$$

- Note that $TP_\ell^k \notin K_{k+\ell-2}^k$.
- Take an (almost) Steiner system S of $(k + \ell - 2)$-sets, such that every $(k-1)$-tuple appears at most once.
- Replace each set in S by a copy of $K_{k+\ell-2}^k$. This gives

$$e(H) = \frac{\binom{n}{k-1}}{\binom{k+\ell-2}{k-1}} \cdot \binom{k+\ell-2}{k} = \frac{\ell-1}{k}\binom{n}{k-1}.$$

A naive approach:

Take the previous construction H (on say n vertices). For large N, take an (almost) Steiner system S of n-subsets of $[N]$, with each pair appearing at most once. Replace each set in S by a copy of H.

The resulted hypergraph G is tight-cycle-free. But

$$e(G) \sim \frac{\binom{N}{2}}{\binom{n}{2}} e(H).$$

Unless $e(H)$ is already $> (1 + c)\binom{n}{2}$, this approach would not work.

Improvement of the constant factor (II)

Definition. The t-shadow of a hypergraph H, denoted by $\partial_t(H)$, is defined as the following:

$$\partial_t(H) = \{S : |S| = t, S \subset e \text{ for some } e \in E(H)\}.$$

Lemma

Let H be a full-k-star with the vertex set $\{0\} \cup [n]$ and the center 0. Let $G_1, \cdots, G_t$ be subhypergraphs of H, and $F_1, \cdots, F_t$ be k-uniform hypergraphs on $[n]$. Suppose

(i) The hypergraphs $H_i := (H \setminus G_i) \cup F_i$ are tight-cycle-free.

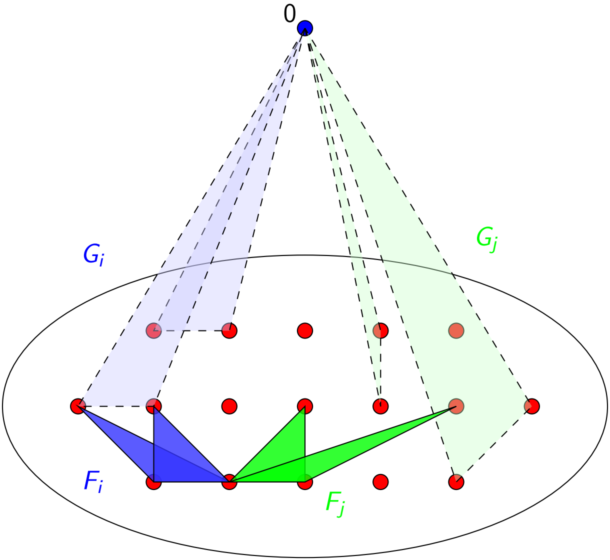
(ii) $\partial_{k-1}(F_i) \cap \partial_{k-1}(F_j) = \varnothing$ for all $1 \le i < j \le t$.

Then

$$H' := \left(H \setminus \bigcup_{i=1}^{t} G_i\right) \cup \left(\bigcup_{i=1}^{t} F_i\right)$$

is also tight-cycle-free.

Visualizing the Lemma



Improvement of the constant factor (III)

Recall that the previous example (when $n = 7$) corresponds to here $n = 6$, and

$$F \sim \{(1,2,3),(1,2,4),(3,4,5),(3,4,6),(5,6,1),(6,1,2)\}$$

$$G \sim \{(0,1,2),(0,3,4),(0,5,6)\}$$

The 2-shadow of F is

$$\partial_2(F) \sim K_6^2$$

One can pack $(1-o(1))\frac{\binom{n}{2}}{\binom{6}{2}}$ copies of F in K_n, which gives a tight-cycle-free 3-uniform hypergraph H' on $n+1$ vertices and

$$\binom{n-1}{2} + (1-o(1))\frac{\binom{n}{2}}{\binom{6}{2}} \cdot 3 = (1 + \frac{1}{5} + o(1))\binom{n-1}{2}$$

edges.

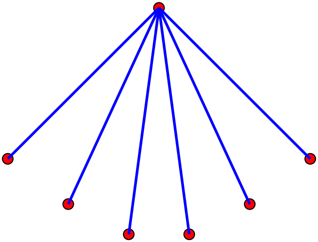
Construction for general k (I)

Lemma

Let H be the full k-star, if we

- add new edges $e_1, \cdots, e_t \subseteq V(H) \setminus \{0\}$ such that $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \emptyset$ for all $i \neq j$, and
- delete edges $\{0\} \cup f_i$ for all $1 \leq i \leq t$, where f_i is any $(k-1)$-subset of e_i,

then the resulting k-uniform hypergraph H' remains tight-cycle-free and has the same number of edges with the full-k-star.



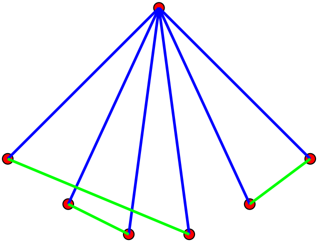
Construction for general k (I)

Lemma

Let H be the full k-star, if we

- add new edges $e_1, \cdots, e_t \subseteq V(H) \setminus \{0\}$ such that $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \varnothing$ for all $i \neq j$, and
- delete edges $\{0\} \cup f_i$ for all $1 \leq i \leq t$, where f_i is any $(k-1)$-subset of e_i,

then the resulting k-uniform hypergraph H' remains tight-cycle-free and has the same number of edges with the full-k-star.



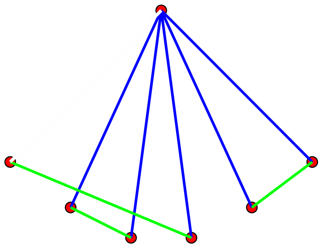
Construction for general k (I)

Lemma

Let H be the full k-star, if we

- add new edges $e_1, \cdots, e_t \subseteq V(H) \setminus \{0\}$ such that $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \emptyset$ for all $i \neq j$, and
- delete edges $\{0\} \cup f_i$ for all $1 \leq i \leq t$, where f_i is any $(k-1)$-subset of e_i,

then the resulting k-uniform hypergraph H' remains tight-cycle-free and has the same number of edges with the full-k-star.



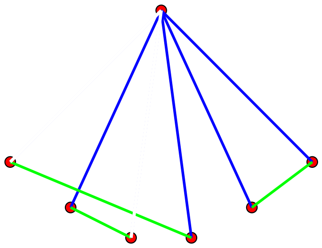
Construction for general k (I)

Lemma

Let H be the full k-star, if we

- add new edges $e_1, \cdots, e_t \subseteq V(H) \setminus \{0\}$ such that $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \emptyset$ for all $i \neq j$, and
- delete edges $\{0\} \cup f_i$ for all $1 \leq i \leq t$, where f_i is any $(k-1)$-subset of e_i,

then the resulting k-uniform hypergraph H' remains tight-cycle-free and has the same number of edges with the full-k-star.



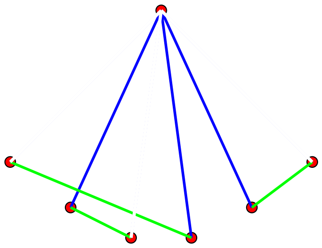
Construction for general k (I)

Lemma

Let H be the full k-star, if we

- add new edges $e_1, \cdots, e_t \subseteq V(H) \setminus \{0\}$ such that $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \varnothing$ for all $i \neq j$, and
- delete edges $\{0\} \cup f_i$ for all $1 \leq i \leq t$, where f_i is any $(k-1)$-subset of e_i,

then the resulting k-uniform hypergraph H' remains tight-cycle-free and has the same number of edges with the full-k-star.



Construction for general k (II)

Goal

Find $F \subset \binom{[n]}{k}$ and $G \subset H$, such that

- $F = \{e_1, \cdots, e_T\} \cup e_0$, with $e_0 = [k]$.
- $G = \{\{0\} \cup f_i\}_{i=1,\cdots,T}$ such that $f_i \subset e_i$ and $|f_i| = k - 1$.
- $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \varnothing$ for $i > j \geq 1$.
- $(H \setminus G) \cup F$ is still tight-cycle-free.

We know that $(H \setminus G) \cup (F \setminus e_0)$ is tight-cycle-free. So a potential tight cycle must contain the edge e_0.

Take $f_1, \cdots, f_k$ to be $f_i = [k] \setminus \{i\}$, and $e_i = f_i \cup \{k + i\}$.

Construction for general k (III)

An example: $(k = 4)$

$$\begin{aligned} & & e_0 &= \{1,2,3,4\} \\ f_1 &= \{2,3,4\}, & e_1 &= \{2,3,4,5\} \\ f_2 &= \{1,3,4\}, & e_2 &= \{1,3,4,6\} \\ f_3 &= \{1,2,4\}, & e_3 &= \{1,2,4,7\} \\ f_4 &= \{1,2,3\}, & e_4 &= \{1,2,3,8\} \end{aligned}$$

Observation. A tight cycle must contain $\pi(1), \pi(2), \pi(3), \pi(4)$ as consecutive vertices (π is a permutation). If it does not contain any of e_1 to e_4. Then it can only be a TC_5^4 with 0 as the other vertex. This is not possible by the choice of f_i.

Suppose the vertex after $\pi(4)$ is t, then $t \neq 0$, and

$$\{\pi(2), \pi(3), \pi(4), t\} \in F,$$

which means $t = \pi(1) + 4$. The five vertices we have so far cannot form a tight cycle!

Construction for general k (IV)

Now we have

$$s, \pi(1), \pi(2), \pi(3), \pi(4), \pi(1)+4,$$

Similarly, $s \neq 0$, and hence $s = \pi(4)+4$. The only possible tight cycle is

$$\pi(4)+4, \pi(1), \pi(2), \pi(3), \pi(4), \pi(1)+4, 0.$$

To destroy them, we remove all the triples

$$f_{k+\alpha} = \{\pi(4), \pi(1)+4, \pi(4)+4\},$$

and add back some $e_{k+\alpha} \supset f_{k+\alpha}$. One can choose e's carefully to make sure their 3-shadows are disjoint.

Construction for general k (V)

The final families: $e_0 = \{1,2,3,4\}$

$f_1 = \{2,3,4\}$, $e_1 = \{2,3,4,5\}$
$f_2 = \{1,3,4\}$, $e_2 = \{1,3,4,6\}$
$f_3 = \{1,2,4\}$, $e_3 = \{1,2,4,7\}$
$f_4 = \{1,2,3\}$, $e_4 = \{1,2,3,8\}$
$f_5 = \{1,6,5\}$, $e_5 = \{1,6,5,9\}$
$f_6 = \{1,7,5\}$, $e_6 = \{1,7,5,10\}$
$f_7 = \{1,8,5\}$, $e_7 = \{1,8,5,11\}$
$f_8 = \{2,5,6\}$, $e_8 = \{2,5,6,12\}$
$f_9 = \{2,7,6\}$, $e_9 = \{2,7,6,13\}$
$f_{10} = \{2,8,6\}$, $e_{10} = \{2,8,6,14\}$
$f_{11} = \{3,5,7\}$, $e_{11} = \{3,5,7,15\}$
$f_{12} = \{3,6,7\}$, $e_{12} = \{3,6,7,16\}$
$f_{13} = \{3,8,7\}$, $e_{13} = \{3,8,7,17\}$
$f_{14} = \{4,5,8\}$, $e_{14} = \{4,5,8,18\}$
$f_{15} = \{4,6,8\}$, $e_{15} = \{4,6,8,19\}$
$f_{16} = \{4,7,8\}$, $e_{16} = \{4,7,8,20\}$

Future research problems (I)

Question

What can we say about $ex(n, TC_\ell^k)$?

Smallest non-trivial case: TC_6^3.

Question

- Can one show for ℓ divisible by k, $ex(n, TC_\ell^k)$ is "monotone decreasing" in ℓ?
- Or, is it true that for fixed ℓ_1 divisible by k, if ℓ_2 is sufficiently large and divisible by k, then $ex(n, TC_{\ell_1}^k) \gg ex(n, TC_{\ell_2}^k)$?

Future research problems (II)

Kalai's Conjecture

For every integer k and ℓ, suppose H is a k-uniform n-vertex hypergraph not containing TP_ℓ^k, then

$$e(H) \le \frac{\ell - 1}{k}\binom{n}{k-1}.$$

- Patkós (2012):

$$e(H) \le \sum_{j=2}^{\ell} \frac{j-1}{k-j+1}\binom{n}{k-1}.$$

- Füredi, Jiang, Kostochka, Mubayi, Verstraëte (2017):

$$e(H) \le \frac{(\ell-1)(k-1)}{k}\binom{n}{k-1}.$$

- Füredi, Jiang, Kostochka, Mubayi, Verstraëte (2018+): Kalai's conjecture holds for TP_4^3.

Thank you!