The Balanced Decomposition Number of $TK_4$ and Series-Parallel Graphs

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Abstract

A balanced colouring of a graph $G$ is a colouring of some of the vertices of $G$ with two colours, say red and blue, such that there is the same number of vertices in each colour. The balanced decomposition number $f(G)$ of $G$ is the minimum integer $s$ with the following property: For any balanced colouring of $G$, there is a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that, for every $i$, $V_i$ induces a connected subgraph of order at most $s$, and contains the same number of red and blue vertices. The function $f(G)$ was introduced by Fujita and Nakamigawa in 2008. They conjectured that $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$ if $G$ is a 2-connected graph on $n$ vertices. In this paper, we shall prove two partial results, in the cases when $G$ is a subdivided $K_4$, and a 2-connected series-parallel graph.

1 Introduction

In this paper, all graphs will be simple and finite. For such a graph $G$, let $V(G)$ be its vertex set and $E(G)$ be its edge set. For $X \subset V(G)$, let $X^c = V(G) \setminus X$; let $G[X]$ be the subgraph of $G$ induced by $X$; and let $N(X) = \{v \in X^c : vx \in E(G) \text{ for some } x \in X\}$ be the open neighbourhood of $X$ in $G$. For a subgraph $H \subset G$, the graph $H - X$ is the subgraph of $H$ induced by $V(H) \setminus X$. We write $H - u$ for $H - \{u\}$. For $k \in \mathbb{N}$, $G$ is a $k$-connected graph if $|V(G)| \geq k + 1$, and $G - X$ is connected for every $X \subset V(G)$ with $|X| \leq k - 1$. For

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u, v ∈ V(G), the graph distance from u to v in G is denoted by \(d_G(u, v)\). If \(P\) is a path with end-vertices \(u\) and \(v\), then \(\text{int } P\) is the path \(P - \{u, v\}\) (this is vacuous if \(|V(P)| \leq 2\).

We refer the reader to [1] for any undefined graph theoretic terms.

In 2008, Fujita and Nakamigawa [6] introduced the balanced decomposition number of a graph. For a graph \(G\), a balanced colouring of \(G\) is a pair \((R, B)\), where \(R, B \subset V(G)\), \(R \cap B = \emptyset\), and \(|R| = |B|\). We refer the vertices of \(R\) (resp. \(B\)) as the red (resp. blue) vertices, and those of \(V(G) \setminus (R \cup B)\) the uncoloured vertices. A set \(X \subset V(G)\) is a balanced set if \(|X \cap R| = |X \cap B|\), and \(G[X]\) is connected. A balanced decomposition of \(G\) is a partition \(V(G) = V_1 \cup \cdots \cup V_r\) (for some \(r \geq 1\)), such that each \(V_i\) is a balanced set. We may also write the balanced decomposition as \(P = \{V_1, \ldots, V_r\}\). The size of \(P\) is the maximum of \(|V_1|, \ldots, |V_r|\).

If \(G\) is a disconnected graph, then any balanced colouring of \(G\) with one red vertex and one blue vertex, in different components, has no possible balanced decomposition. Hence, we will only consider balanced decompositions for connected graphs.

If \(G\) is a connected graph of order \(n\), and \(k \in \mathbb{Z}\), \(0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\), define
\[
f(k, G) = \min\{s \in \mathbb{N} : \text{every balanced colouring } (R, B) \text{ of } G \text{ with } |R| = |B| = k \text{ has a balanced decomposition of size } \leq s\}.
\]

Note that \(f(k, G) \leq n\), so that \(f(k, G)\) is well-defined. The balanced decomposition number of \(G\) is then defined as
\[
f(G) = \max \left\{ f(k, G) : 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.
\]

Fujita and Nakamigawa [6] made the following conjecture.

**Conjecture 1 ([6])** If \(G\) is a 2-connected graph of order \(n\), then \(f(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1\).

The partial result when \(G = C_n\), the cycle of order \(n\), was solved [6].

**Theorem 2 ([6])** If \(n \geq 3\), then \(f(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1\).

Also, the partial result when \(G\) is a generalised \(\Theta\)-graph was solved [4]. A generalised \(\Theta\)-graph (with \(t\) paths) is a graph \(G\) which is the union of \(t \geq 2\) paths, \(Q_1, \ldots, Q_t\) say, with each having the same two end-vertices, \(x\) and \(y\) say, such that \(V(Q_i) \cap V(Q_j) = \{x, y\}\) for any \(i \neq j\). Note that the \(Q_i\) are pairwise internally vertex-disjoint paths. In addition, all but at most one of the \(Q_i\) have order at least 3. The vertices \(x\) and \(y\) are the source and sink of \(G\). We also write \(G = \Theta(Q_1, \ldots, Q_t)\).

In the proof of Theorem 3 [4], the following assertion, which contains a structural statement about balanced decompositions, was in fact proved.

**Theorem 3 ([4])** Let \(G = \Theta(Q_1, \ldots, Q_t)\) be a generalised \(\Theta\)-graph of order \(n\), where \(t \geq 2\), with source \(x\) and sink \(y\). Then \(\left\lfloor \frac{n-t+1}{2} \right\rfloor \leq f(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1\). Hence, if \(t\) is fixed, then \(f(G) = \left\lfloor \frac{n}{2} \right\rfloor + O(1)\).

Furthermore, there exists a balanced decomposition \(P\) for \(G\) of size at most \(\left\lfloor \frac{n}{2} \right\rfloor + 1\) with one of the following forms.

(i) \(P = \{V_1, V_2, V_3\}\), where \(x \in V_1\), \(y \in V_2\), and \(V_3 \subset V(\text{int } Q_i)\) (possibly empty, whence \(P = \{V_1, V_2\}\)) for some \(i\).
(ii) \( \mathcal{P} = \{V_1, V_2\} \), where \( x, y \in V_1; V_2 \subset V(\text{int } Q_i) \) for some \( i \) with \( |V(Q_i)| \geq \left\lfloor \frac{n}{2} \right\rfloor + 2 \); and \( |V_2| = \left\lfloor \frac{n}{2} \right\rfloor \) or \( |V_2| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

(iii) \( \mathcal{P} = \{V(Q_1), V(\text{int } Q_2), \ldots, V(\text{int } Q_t)\} \).

Finally, we have partial results when the number of coloured vertices of \( G \) is small \([5, 6]\).

**Theorem 4 ([5, 6])** If \( G \) is a 2-connected graph of order \( n \geq \max(2k, 3) \), then \( f(k, G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \) for \( k = 1, 2, 3 \).

Conjecture 1 remains open. In Section 3, we shall prove the partial result in the case when \( G \) is a subdivided \( K_4 \), which we denote by \( TK_4 \).

**Theorem 5** If \( G \) is a \( TK_4 \) of order \( n \), then \( f(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

A graph is a *series-parallel (SP)* graph if it can be obtained as follows. Start with a path of length at least 1. Perform a sequence of operations of the following type successively.

(∗) Replace an edge with a generalised \( \Theta \)-graph, by identifying the vertices of the edge with the source and the sink of the generalised \( \Theta \)-graph.

The end-vertices of the initial path are the *source* and the *sink* of the SP graph.

There are many other formulations of SP graphs (see, for example \([2, 3]\)), and they are easily seen to be equivalent to the above. By a result of Duffin \([2]\), a SP graph is 2-connected if and only if it can be obtained as described above, with at least one operation, and the initial path has length 1. In Section 4, we shall prove the following case of Conjecture 1.

**Theorem 6** If \( G \) is a 2-connected series-parallel graph of order \( n \), then \( f(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

Note that Theorems 5 and 6 are exclusive from each other.

Theorem 6 has an instant corollary. A result of Elmallah and Colbourn \([3]\) says that if \( G \) is a 3-connected planar graph, then \( G \) has a spanning 2-connected SP graph.

**Corollary 7** If \( G \) is a 3-connected planar graph of order \( n \), then \( f(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

We remark that if the “2-connected” assumption on \( G \) is neglected in Theorem 6, then \( f(G) \) can vary greatly. For example, if \( G \) is a path, then \( f(G) = n \). On the other hand, if \( G \) is a generalised \( \Theta \)-graph with paths of length 2, and with a “pendant” edge attached to the source, then it can be shown that \( f(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2 \).

## 2 Tools

In this section, we develop some tools which we will need in the proofs of Theorems 5 and 6. Firstly, Lemma 8 below will be needed for both proofs.

**Lemma 8** Let \( G \) be a connected graph of order \( n \). Suppose that there is a numbering of \( V(G) \) with \( 1, \ldots, n \) such that the subgraph of \( G \) induced by any set of at least \( \left\lfloor \frac{n}{2} \right\rfloor - 1 \) consecutive vertices (modulo \( n \)) is connected. Then \( f(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \).
Proof. Let \((R, B)\) be a balanced colouring of \(G\). For \(1 \leq i \leq n\), let \(A_i\) be the vertices numbered \(i, i + 1, \ldots, i + \lfloor \frac{n}{2} \rfloor - 1\) (modulo \(n\)), and \(g(i) = \left| A_i \cap R \right| - \left| A_i \cap B \right|\). We have \(|g(i) - g(i + 1)| \leq 2\) (where \(g(n + 1) = g(1)\) by convention) for every \(i\), and \(\sum_{i=1}^{n} g(i) = 0\). Hence for some \(i\), either \(g(i) = 0\), or without loss of generality, \(g(i) = -1\) and \(g(i + 1) = 1\). If the former, then \(\{A_i, A_i^\dagger\}\) is a suitable balanced decomposition, with \(\left| A_i \right|, \left| A_i^\dagger \right| \geq \left\lceil \frac{n}{2} \right\rceil - 1\). If the latter, let \(w\) be the vertex numbered \(i + \lfloor \frac{n}{2} \rfloor\) (modulo \(n\)). Then, \(w \in R\), and \(\left( (A_i \cup \{w\}) \cap R \right) = \left| (A_i \cup \{w\}) \right| \cap B\). Hence, \(\{A_i \cup \{w\}, (A_i \cup \{w\})^c\}\) is a suitable balanced decomposition, since \(\left| A_i \cup \{w\} \right|, \left| (A_i \cup \{w\})^c \right| \geq \left\lceil \frac{n}{2} \right\rceil - 1\). \(\square\)

Next, we shall develop some ideas about SP graphs. This part of Section 2 can be interesting in its own right.

We first recall the well-known series and parallel compositions of SP graphs. Let \(G_1\) and \(G_2\) be two SP graphs, with sources \(a_1, a_2\) and sinks \(b_1, b_2\). Then, their series composition is the graph \(G_1 +_S G_2\), formed by identifying \(b_1\) and \(a_2\). Their parallel composition is the graph \(G_1 +_P G_2\), formed by identifying \(a_1, a_2\), and \(b_1, b_2\). Both of these compositions can be extended to three or more SP graphs in the obvious way. Observe that \(G_1 +_S G_2\) is connected, but not 2-connected, while \(G_1 +_P G_2\) is 2-connected.

For the rest of the paper, we assume that all SP graphs are obtained as follows. Start with a path \(G_0\) with end-vertices \(x_0, y_0\), and replace edges successively with generalised \(\Theta\)-graphs by the operation \((*)\) \(m\) times, for some \(m \geq 1\). Let \(T_1, \ldots, T_m\) be the generalised \(\Theta\)-graphs. For each \(i\), let \(x_i\) and \(y_i\) be the source and sink of \(T_i\). We make the following assumptions.

(**) No \(T_i\) replaces an edge \(e\) of some \(T_j\) \((j < i)\) which joins \(x_j\) and \(y_j\).

Otherwise, the same final SP graph can be obtained by appending \(T_j'\) instead of \(T_j\) when \(T_j\) was appended, where \(T_j'\) is the graph obtained from \(T_j\) by replacing \(e\) with \(T_i\) (by the operation \((*)\)).

(†) For any \(i\), \(T_i\) is appended as follows. \(T_i\) replaces the edge \(ab\) which appeared in some first \(T_j\) \((j < i)\), or in \(G_0\). If the former, assume that \(d_{T_j}(a, x_j) < d_{T_j}(b, x_j)\). If the latter, assume that \(d_{G_0}(a, x_0) < d_{G_0}(b, x_0)\). In both cases, identify \(x_i\) with \(a\), and \(y_i\) with \(b\).

Now, for an SP graph \(G\), we shall define a linear ordering \(\prec\) on \(V(G)\). First, for a generalised \(\Theta\)-graph \(T = \Theta(Q_1, \ldots, Q_t)\) (for some \(t \geq 2\)) with source \(a\) and sink \(b\), define a linear ordering \(\prec_T\) on \(V(T)\) as follows. We have \(u \prec_T v\) if either \(u = a\), or \(v = b\), or \(u \in V(\text{int } Q_i)\) and \(v \in V(\text{int } Q_j)\) for some \(i < j\), or \(u, v \in V(\text{int } Q_i)\) with \(d_{Q_i}(u, a) < d_{Q_i}(v, a)\) for some \(i\). Next, for \(1 \leq i \leq m\), let \(G_i\) be the SP graph obtained after \(T_1, \ldots, T_i\) have been appended (so that \(G = G_m\)). We define a linear ordering \(\prec_i\) on \(V(G_i)\) for each \(i\). Proceed inductively. Initially, define the linear ordering \(\prec_0\) on \(V(G_0)\) by \(u \prec_0 v\) if \(d_{G_0}(u, x_0) < d_{G_0}(v, x_0)\). Now for \(i \geq 1\), suppose that we have defined the linear ordering \(\prec_{i-1}\) on \(V(G_{i-1})\). The graph \(T_i\) has a linear ordering \(\prec_{T_i}\). The vertices \(x_i, y_i\) are identified with an edge \(ab \in E(G_{i-1})\), and \(ab\) first appeared either as an edge of \(G_0\), or when some \(T_j\) \((j < i)\) was appended. Define the linear ordering \(\prec_i\) on \(V(G_i)\) as follows.

- If \(u, v \notin V(T_i - \{a, b\})\) and \(u \prec_{T_i} v\) in \(G_{i-1}\), then \(u \prec_i v\).
- If \(u, v \in V(T_i - \{a, b\})\) and \(u \prec_{T_i} v\) in \(T_i\), then \(u \prec_i v\).
• Suppose that \( u \not\in V(T_i - \{a, b\}) \) and \( v \in V(T_i - \{a, b\}) \).
  - If \( ab \in E(G_0) \), or \( ab \in E(T_j) \) with \( a \neq x_j, b \neq y_j \), then \( u \prec v \) if \( u \prec_{i-1} a \) or \( u = a \) in \( G_{i-1} \), and \( v \prec u \) otherwise.
  - If \( ab \in E(T_j) \) with \( a = x_j, b \neq y_j \), then \( u \prec v \) if \( u \prec_{i-1} b \) in \( G_{i-1} \), and \( v \prec u \) otherwise.
  - If \( ab \in E(T_j) \) with \( a \neq x_j, b = y_j \), then \( u \prec v \) if \( u \prec_{i-1} a \) or \( u = a \) in \( G_{i-1} \), and \( v \prec u \) otherwise.

Finally, set \( \prec = \prec_m \). Note that \( \prec \) is well-defined, in view of (†) and (‡). In practice, the linear ordering \( \prec \) is quite simple. Figure 1 shows an example.

![Figure 1. The linear ordering \( \prec \).](image)

With the linear ordering \( \prec \) now defined, we have the following lemma.

**Lemma 9** Let \( G \) be an SP graph, with the linear ordering \( \prec \) on \( V(G) \) as defined. Then, every subgraph of \( G \) induced by an initial segment or a final segment of \( \prec \) is connected.

**Proof.** We use all the terms that we have already defined. We show inductively that the lemma holds for each \( \prec_i \) on \( G_i \). The lemma clearly holds for \( G_0 \). Now for \( 1 \leq i \leq m \), suppose that it holds for \( G_{i-1} \). \( G_i \) is obtained from \( G_{i-1} \) by replacing an edge \( ab \in E(G_{i-1}) \) with the graph \( T_i \), by identifying \( x_i, y_i \) with \( a, b \) in such a way that (†) and (‡) are satisfied. Note that \( a \prec_{i-1} b \) and \( a \prec_i b \). We also have the linear ordering \( \prec_{T_i} \) on \( V(T_i) \).

Observe that any initial segment and final segment of \( \prec_{T_i} \) induces a connected subgraph of \( T_i \). Also, by the definition of \( \prec_{T_i} \), \( V(T_i - \{x_i, y_i\}) \) is a single segment in \( V(G_i) \). Let \( I \subset V(G_i) \) be an initial segment of \( \prec_i \) and \( I^c = V(G_i) \setminus I \) be the corresponding final segment. It suffices to show that both \( G_i[I] \) and \( G_i[I^c] \) are connected.

• If \( a, b \not\in I \), then \( J \) is also an initial segment of \( \prec_{i-1} \) in \( V(G_{i-1}) \). Let \( I' = V(G_{i-1}) \setminus I \) be the corresponding final segment. We have \( G_i[I] = G_{i-1}[I] \), and \( G_i[I^c] \) is obtained from \( G_{i-1}[I'] \) by replacing \( ab \) with \( T_i \), so is clearly connected. A similar argument applies if \( a, b \in I \).

• If \( a \in I \) and \( b \not\in I \), then \( G_i[I] \) is formed by attaching \( T_i[I'] \) to \( G_{i-1}[J] \) at \( a \), where \( J \) is some initial segment of \( \prec_{i-1} \) in \( V(G_{i-1}) \) containing \( a \) but not \( b \), and \( J' \) is some initial segment of \( \prec_{T_i} \) in \( V(T_i) \). Clearly, \( G_i[I] \) is connected. \( G_i[I^c] \) has a similar structure, so it is also connected.

\( \square \)
3 Subdivision of $K_4$

Proof of Theorem 5. Let $G$ be a $TK_4$ of order $n$. Let $x_1, \ldots, x_4$ be the branch vertices of $G$, and $Q_1, \ldots, Q_6$ be the sub-paths joining them, with $V(Q_1) \cap V(Q_5) \cap V(Q_6) = \{x_1\}$, $V(Q_1) \cap V(Q_2) \cap V(Q_3) = \{x_2\}$, $V(Q_2) \cap V(Q_4) \cap V(Q_6) = \{x_3\}$, $V(Q_3) \cap V(Q_4) \cap V(Q_5) = \{x_4\}$. Assume that $|V(Q_i)| \geq 3$ for each $i$ (so that $n \geq 10$), otherwise the result follows from Theorem 3. Let $(R, B)$ be a balanced colouring of $G$.

Case 1. $|V(Q_i)| \leq \lfloor \frac{n}{2} \rfloor$ for all $1 \leq i \leq 6$.

We proceed by proving several claims.

Claim 10 There exist partitions $V(\text{int }Q_4) = S_1 \cup S_2$, $V(\text{int }Q_5) = T_1 \cup T_2$ and $V(\text{int }Q_6) = U_1 \cup U_2$ such that the graphs $H_1 = G[V(Q_1) \cup T_1 \cup U_2]$, $H_2 = G[V(Q_2) \cup U_1 \cup S_2]$ and $H_3 = G[V(Q_3) \cup S_1 \cup T_2]$ are connected, with $|V(H_i)| \leq \lfloor \frac{n}{2} \rfloor$ for each $i$.

Proof. Recall that $3 \leq |V(Q_i)| \leq \lfloor \frac{n}{2} \rfloor$ for each $i$. If there is no partition $V(\text{int }Q_4) = S_1 \cup S_2$ such that $G[V(Q_2) \cup S_2]$ and $G[V(Q_3) \cup S_1]$ are connected and have order at most $\lfloor \frac{n}{2} \rfloor$, then we would have $|V(G)| \geq 2 \lfloor \frac{n}{2} \rfloor + 4 > n$, a contradiction. Hence, take a suitable partition $V(\text{int }Q_4) = S_1 \cup S_2$. If partitions $V(\text{int }Q_5) = T_1 \cup T_2$ and $V(\text{int }Q_6) = U_1 \cup U_2$ do not simultaneously exist such that $G[V(Q_1) \cup T_1 \cup U_2]$, $G[V(Q_2) \cup U_1 \cup S_2]$ and $G[V(Q_3) \cup S_1 \cup T_2]$ are connected and have order at most $\lfloor \frac{n}{2} \rfloor$, then a similar counting argument gives $|V(G)| \geq 2 \lfloor \frac{n}{2} \rfloor + 2 > n$, another contradiction. □

From here, $H_1$, $H_2$ and $H_3$ are defined as in Claim 10.

Claim 11 For some $i$, there exists a balanced set $A \subset V(H_i)$, with $x_2 \in A$.

Proof. The claim holds if $x_2 \in (R \cup B)^c$. Without loss of generality, let $x_2 \in R$. If such a set $A$ does not exist, then $|V(H_i) \cap R| > |V(H_i) \cap B|$ for every $i$, so $|R| = 1 + \sum_{i=1}^{3} |V(H_i - x_2) \cap R| \geq 1 + \sum_{i=1}^{3} |V(H_i) \cap B| > |B|$, a contradiction. □

Claim 12 Let $A \subset (Q_4 \cup Q_5 \cup Q_6)^c$ be a balanced set with $x_2 \in A$, $|A| \leq \lfloor \frac{n}{2} \rfloor - 1$, and $N(A) \subset R$ or $N(A) \subset B$. Then for some $i$, there exists a balanced set $C \subset V(H_i - A)$ with $N(C) \cap V(H_i) \cap A \neq \emptyset$.

Proof. It suffices to prove the claim with $N(A) \subset R$. If such a set $C$ does not exist, then $|V(H_i - A) \cap R| > |V(H_i - A) \cap B|$ for every $i$. But, $|A^c \cap R| = \sum_{i=1}^{3} |V(H_i - A) \cap R| > \sum_{i=1}^{3} |V(H_i - A) \cap B| = |A^c \cap B|$, a contradiction, since $A^c$ is a balanced set. □

Claim 13 Let $A$ be a balanced set with $V(Q_1) \subset A \subset V(Q_4)^c$, $|A| \leq \lfloor \frac{n}{2} \rfloor - 1$, $N(A) \setminus \{x_3, x_4\} \subset R$ or $N(A) \setminus \{x_3, x_4\} \subset B$, and $A \not\supset V(\text{int }Q_2 \cup \text{int }Q_6)$, $A \not\supset V(\text{int }Q_3 \cup \text{int }Q_5)$.

Then, there exists a balanced set $C \subset X$ with $N(C) \cap A \neq \emptyset$, $|C| \leq \lfloor \frac{n}{2} \rfloor$ and $(A \cup C)^c$ is connected, where $X$ is either $V((Q_3 \cup Q_5 \cup \text{int }Q_4) - A)$ or $V((Q_2 \cup Q_6 \cup \text{int }Q_4) - A)$.

Proof. It suffices to prove the claim with $N(A) \setminus \{x_3, x_4\} \subset R$. Let $V_i = V(\text{int }Q_i) - A$ for $i \in \{2, 3, 5, 6\}$, and $I = \{i \in \{2, 3, 5, 6\} : V_i \neq \emptyset\}$.

If we cannot find a suitable set $C \subset V_i \cup \{x_4\}$ for some $i \in \{3, 5\}$, then $|(V_3 \cup V_5 \cup \{x_4\}) \cap R| > |(V_3 \cup V_5 \cup \{x_3\}) \cap R| > |(V_2 \cup V_6 \cup \{x_3\}) \cap B|$. Since $A^c$ is a balanced set, it is clear that there exists a partition $V(Q_4) = W_1 \cup W_2$ such that $V_3 \cup V_5 \cup W_1$ and $V_2 \cup V_6 \cup W_2$ are balanced sets. One of these has at most $\lfloor \frac{n}{2} \rfloor$ vertices and hence is a suitable set for $C$. □
Claim 14 Let $A$ be a balanced set with $V(Q_1 \cup Q_3 \cup Q_5) \setminus \{x_4\} \subseteq A \subseteq V(G - x_3)$, $|A| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, and $N(A) \setminus \{x_3\} \subseteq R$ or $N(A) \setminus \{x_3\} \subseteq B$. Then, there exists a balanced set $C \subseteq X$ with $N(C) \cap X \cap A \neq \emptyset$ and $|C| \leq \left\lfloor \frac{n}{2} \right\rfloor$, where $X$ is either $V(\text{int } Q_2 - A)$, $V(Q_4 - (A \cup \{x_3\}))$ or $V(\text{int } Q_6 - A)$. 

Proof. It suffices to prove the claim with $N(A) \setminus \{x_3\} \subseteq R$. For $i \in \{2,4,6\}$, let $X_i = V(Q_i - (A \cup \{x_3\}))$. Let $J = \{i \in \{2,4,6\} : X_i \neq \emptyset\}$. Note that $|J| \geq 2$, otherwise we have $|A| \geq n - \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil$, a contradiction. If we cannot find a suitable $C \subseteq X_i$ for some $i \in J$, then $|X_i \cap R| > |X_i \cap B|$ for each $i \in J$. But, $|A^c \cap R| \geq \sum_{i \in J} |X_i \cap R| \geq \sum_{i \in J} |X_i \cap B| + 2 > |A^c \cap B|$, a contradiction, since $A^c$ is a balanced set. 

We now describe an algorithm. Take a balanced set $A_1$ as given by Claim 11. Without loss of generality, $A_1 \subseteq V(H_1)$. We have $|A_1| \leq \left\lfloor \frac{n}{2} \right\rfloor$. 

Step 1. If $|A_1| = \left\lfloor \frac{n}{2} \right\rfloor$, stop. \{\{A_1, A_1^c\}\} is a suitable balanced decomposition for $G$. Otherwise, if $|A_1| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. If $V(Q_1) \subseteq A_1$, go to Step 3. If $A_1 = V(Q_4 \cup Q_5 \cup Q_6)^c$, stop. $V(Q_4 \cup Q_5 \cup Q_6)$ is a balanced set, so by Theorem 2, $Q_4 \cup Q_5 \cup Q_6$ has a balanced decomposition $\mathcal{P}$ with size at most $\left\lceil \frac{n}{2} \right\rceil + 1$. Hence, $\{A_1\} \cup \mathcal{P}$ is a suitable balanced decomposition for $G$. Otherwise, $A_1 \subseteq V(Q_4 \cup Q_5 \cup Q_6)^c$. If there is an uncoloured vertex $u \in N(A_1)$, or red and blue vertices $v, w \in N(A_1)$, let $A_2 = A_1 \cup \{u\}$ or $A_2 = A_1 \cup \{v, w\}$ accordingly; if not, go to Step 2. We can choose $u$, or $v$ and $w$, such that at most one of $x_1, x_3, x_4$ is appended to $A_1$. $A_2$ is another balanced set. If $|A_2| = \left\lfloor \frac{n}{2} \right\rfloor$ or $|A_2| = \left\lfloor \frac{n}{2} \right\rfloor + 1$, stop; \{\{A_2, A_2^c\}\} is a suitable balanced decomposition for $G$. Otherwise, $|A_2| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. If we have appended exactly one of $x_1, x_3, x_4$, go to Step 3, using $A_2$ for $A_1$. Otherwise, repeat Step 1, using $A_2$ for $A_1$. 

Step 2. We have either $N(A_1) \subseteq R$ or $N(A_1) \subseteq B$. The set $A_1$ satisfies the conditions of Claim 12, and we can find a balanced set $C$ as described. If $|A_1 \cup C| \geq \left\lceil \frac{n}{2} \right\rceil$, stop. We have a suitable balanced decomposition $\{A_1, C, (A_1 \cup C)^c\}$ for $G$. Otherwise, $|A_1 \cup C| \leq \left\lceil \frac{n}{2} \right\rceil - 1$. If we have exactly one of $x_1, x_3, x_4$ in $A_1 \cup C$, go to Step 3, using $A_1 \cup C$ for $A_1$. Otherwise, go back to Step 1, using $A_1 \cup C$ for $A_1$. 

Step 3. Re-label the $Q_i$, $x_j$ and $H_k$ by cycling $Q_1 \to Q_2 \to Q_3 \to Q_1$, $Q_4 \to Q_5 \to Q_6 \to Q_4$, $x_1 \to x_3 \to x_4 \to x_1$, and $H_1 \to H_2 \to H_3 \to H_1$, so that $V(Q_1) \subseteq A_1 \subseteq V(Q_4)^c$ in the re-labelling. If $A_1 \supseteq V(\text{int } Q_3 \cup \text{int } Q_5)$ or $A_1 \supseteq V(\text{int } Q_2 \cup \text{int } Q_6)$, go to Step 5. Otherwise, if there is an uncoloured vertex $u \in N(A_1) \setminus \{x_3, x_4\}$, or red and blue vertices $v, w \in N(A_1) \setminus \{x_3, x_4\}$, let $A_3 = A_1 \cup \{u\}$ or $A_3 = A_1 \cup \{v, w\}$ accordingly; if not, go to Step 4. If $|A_3| = \left\lfloor \frac{n}{2} \right\rfloor$ or $|A_3| = \left\lfloor \frac{n}{2} \right\rfloor + 1$, stop. We have a suitable balanced decomposition $\{A_3, A_3^c\}$ for $G$. Otherwise, $|A_3| \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. If $A_3 \supseteq V(\text{int } Q_3 \cup \text{int } Q_5)$ or $A_3 \supseteq V(\text{int } Q_2 \cup \text{int } Q_6)$, go to Step 5, using $A_3$ for $A_1$. Otherwise, repeat Step 3, using $A_3$ for $A_1$. 

Step 4. We have either $N(A_1) \setminus \{x_3, x_4\} \subseteq R$ or $N(A_1) \setminus \{x_3, x_4\} \subseteq B$. The set $A_1$ satisfies the conditions of Claim 13, and we can find a balanced set $C$ as described. If $|A_1 \cup C| \geq \left\lceil \frac{n}{2} \right\rceil$, stop. We have a suitable balanced decomposition $\{A_1, C, (A_1 \cup C)^c\}$ for $G$. Otherwise, $|A_1 \cup C| \leq \left\lceil \frac{n}{2} \right\rceil - 1$. If we have exactly one of $x_3, x_4$ in $A_1 \cup C$, go to Step 5, using $A_1 \cup C$ for $A_1$. Otherwise, go back to Step 3, using $A_1 \cup C$ for $A_1$. 

Step 5. Re-label the $Q_i$, $x_j$ and $H_k$ by $Q_2 \leftrightarrow Q_3$, $Q_5 \leftrightarrow Q_6$, $x_3 \leftrightarrow x_4$ and $H_2 \leftrightarrow H_3$, so that $V(Q_1 \cup Q_3 \cup Q_5) \setminus \{x_4\} \subseteq A_1 \subseteq V(G - x_3)$ in the re-labelling. If there is an
uncoloured vertex $u \in N(A_1) \setminus \{x_3\}$, or red and blue vertices $v, w \in N(A_1) \setminus \{x_3\}$, let $A_4 = A_1 \cup \{u\}$ or $A_4 = A_1 \cup \{v, w\}$ accordingly. If $|A_4| = \lfloor \frac{n}{2} \rfloor + 1$, stop. We have a suitable balanced decomposition $\{A_4, A'_4\}$ for $G$. Otherwise, $|A_4| \leq \lfloor \frac{n}{2} \rfloor - 1$.

Repeat Step 5, using $A_4$ for $A_1$. If $N(A_1) \setminus \{x_3\} \subset R$ or $N(A_1) \setminus \{x_3\} \subset B$, then $A_1$ satisfies the conditions of Claim 14, and we can find a balanced set $C$ as described. If $|A_1 \cup C| \geq \lfloor \frac{n}{2} \rfloor$, stop. We have a suitable balanced decomposition $\{A_1, C, (A_1 \cup C)^c\}$ for $G$. Otherwise, $|A_1 \cup C| \leq \lfloor \frac{n}{2} \rfloor - 1$. Repeat Step 5, using $A_1 \cup C$ for $A_1$.

The algorithm must terminate, since whenever we append new vertices, we are increasing the number of vertices in $A_1$. When the algorithm terminates, we will obtain a suitable balanced decomposition for $G$.

**Case 2. Without loss of generality, $|V(Q_1)| \geq \lfloor \frac{n}{2} \rfloor + 1$.**

Number the vertices of $G$ with $1, \ldots, n$ as follows. Start at $x_1$ and move along $Q_1$ to $x_2$. Then, move along $Q_2$ to the vertex adjacent to $x_3$. Then, move along $Q_3$ from the vertex adjacent to $x_3$ to the vertex adjacent to $x_4$. Then, move along $Q_4$ from $x_3$ to $x_4$. Then, move along $Q_5$ from the vertex adjacent to $x_4$ to the vertex adjacent to $x_5$. Finally, move along $Q_6$, from the vertex adjacent to $x_3$ to the vertex adjacent to $x_1$.

This numbering satisfies the condition of Lemma 8. Indeed, let $A \subset V(G)$ be a set of consecutive vertices (modulo $n$), with first vertex $v$, and $|A| \geq \lfloor \frac{n}{2} \rfloor - 1$. Every initial segment induces a connected subgraph of $G$. If $v \in V(Q_1)$, then $G[A]$ is connected. If $v \in V(Q_1)^c$, then note that $|V(Q_1)^c| \leq n - (\lfloor \frac{n}{2} \rfloor + 1) \leq |A|$. Hence, either $A = V(Q_1)^c$, or $A$ is the union of an initial segment and a final segment. In either case, $G[A]$ is connected. Hence by Lemma 8, we have $f(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.

The proof of Theorem 5 is now complete. □

4 Series-Parallel Graphs

**Proof of Theorem 6.** Let $G$ be a 2-connected SP graph of order $n$. $G$ can be obtained as described in Section 2, where (i) and (ii) are satisfied. Let $T_1, \ldots, T_m$ be the generalised $\Theta$-graphs, for some $m \geq 1$. Let $T_i = \Theta(Q_1, \ldots, Q_t)$, for some $t \geq 2$, with source $x$ and sink $y$. For a subgraph $F \subset T_1$, let $(F) \subset G$ be the subgraph of $G$ that $F$ has been transformed to.

Let $(R, B)$ be a balanced colouring of $G$. We shall prove a stronger assertion. There exists a balanced decomposition $\mathcal{P}$ for $G$ of size at most $\lfloor \frac{n}{2} \rfloor + 1$, with one of the following forms.

(i) $\mathcal{P} = \{V_1, V_2, V_3\}$, where $x \in V_1$, $y \in V_2$, and $V_3 \subset V((Q_i) - \{x, y\})$ (possibly empty, whence $\mathcal{P} = \{V_1, V_2\}$) for some $i$.

(ii) $\mathcal{P} = \{V_1, V_2\}$, where $x, y \in V_1$, and $V_2 \subset V((Q_i) - \{x, y\})$ for some $i$ with $|V((Q_i))| \geq \lfloor \frac{n}{2} \rfloor + 2$, and $|V_2| = \lfloor \frac{n}{2} \rfloor$ or $|V_2| = \lfloor \frac{n}{2} \rfloor + 1$.

(iii) $\mathcal{P} = \{V((Q_1)), V((Q_2) - \{x, y\}), \ldots, V((Q_t) - \{x, y\})\}$.

**Case 1.** $|V((Q_i))| \leq \lfloor \frac{n}{2} \rfloor + 1$ for all $1 \leq i \leq t$.

We use induction on $m$. By Theorem 3, the stronger assertion holds for $m = 1$. Now let $m \geq 2$ and suppose that the result holds for any 2-connected SP graph that can be obtained from $m - 1$ applications of the operation $(\ast)$.  

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Let $T_m = \Theta(R_1, \ldots, R_n)$ for some $s \geq 2$, with source $a$ and sink $b$. $T_m$ has a linear ordering $\prec_{T_m}$ as described in Section 2. Obtain the graph $H$ from $G$ as follows. Replace $T_m$ with a path $P$ of order $|V(T_m)|$ by identifying the end-vertices of $P$ with $a$ and $b$, and with the vertex $u \in V(T_m)$ corresponding to the vertex $u' \in V(P)$ by $d_P(u', a) + 1$ being the position of $u$ in $\prec_{T_m}$. Also, let $u'$ inherit the colour of $u$, and let $\prec_P$ be the corresponding linear ordering on $V(P)$.

The graph $H$ can be obtained by $m - 1$ applications of the operation $(\ast)$, so by induction, $H$ has a balanced decomposition $\mathcal{P}'$ of size at most $\lfloor \frac{n}{3} \rfloor + 1$, with one of the forms (i) to (iii) as described above. If $\mathcal{P}'$ is of form (iii), then $\mathcal{P}'$ is a suitable balanced decomposition for $G$ of form (iii), in view of (i) (since $\{a, b\} \neq \{x, y\}$). If $\mathcal{P}'$ is of form (i) or (ii), then the path $P$ is partitioned into at most three sub-paths. If $P$ is divided into one or two sub-paths, then by Lemma 9, $\mathcal{P}'$ is still a balanced decomposition in $G$ and is of form (i) or (ii). If $P$ is divided into three sub-paths and $\mathcal{P}'$ is of form (ii), then $\mathcal{P}' = \{V_1, V_2\}$ as described.

The end-vertices of $P$ must be in $V_1$. This means that $|V(P)| \geq |V_2| + 2 \geq \lfloor \frac{n}{3} \rfloor + 2$, a contradiction. Now, assume that $\mathcal{P}'$ is of form (i). Then, $\mathcal{P}' = \{V_1, V_2, V_3\}$ as described. $G[V_1]$ has the following structure: Take $H[V_1]$, remove an initial segment of $P$ (w.r.t. $\prec_P$), and replace with the corresponding initial segment of $V(T_m)$ (w.r.t. $\prec_{T_m}$). By Lemma 9, $G[V_1]$ is connected. Similarly, $G[V_2]$ is also connected. Now, $V_3$ is a middle segment of $V(T_m)$ (w.r.t. $\prec_{T_m}$), so $G[V_3]$ consists of possibly several disjoint paths, each one being a sub-path of $R_j$ for some $j$.

We now describe an algorithm.

Step 1. If $|V_1| = \lfloor \frac{n}{3} \rfloor$ or $|V_1| = \lfloor \frac{n}{3} \rfloor + 1$, stop. We have a suitable balanced decomposition $\{V_1, V_1^\prime\}$ for $G$. Otherwise, $|V_1| \leq \lfloor \frac{n}{3} \rfloor - 1$. If $N(V_1) \cap V(T_m - b) \subset R$ or $N(V_1) \cap V(T_m - b) \subset B$, go to Step 2. Otherwise, there is an uncoloured vertex $u \in N(V_1) \cap V(T_m - b)$, or red and blue vertices $v, w \in N(V_1) \cap V(T_m - b)$. Let $V_1' = V_1 \cup \{u\}$ or $V_1' = V_1 \cup \{v, w\}$, and $V_2' = V_3 \setminus \{u\}$ or $V_2' = V_3 \setminus \{v, w\}$ accordingly. Repeat Step 1, using $V_1'$ for $V_1$, and $V_2'$ for $V_3$.

Step 2. Note that $G[V_3]$ consists of paths $A_1, \ldots, A_r$, where for each $i$, $A_i \subset \text{int} \, R_j$ for some $j$, and $A_i$ has one end-vertex adjacent to a vertex in $V_1$, the other adjacent to a vertex in $V_2$.

Let $a_1, \ldots, a_r$ be the end-vertices adjacent to vertices in $V_1$. Since $N(V_1) \cap V(T_m - b) \subset R$ (resp. $N(V_1) \cap V(T_m - b) \subset B$), we have $a_1, \ldots, a_r \in R$ (resp. $a_1, \ldots, a_r \in B$). Since $|V_3 \cap R| = |V_3 \cap B|$, for some $i$ and $x \in V(A_i)$, the path $Q = a_i \cdots x \subset A_i$ satisfies $|V(Q) \cap R| = |V(Q) \cap B|$. If $|V_1 \cup V(Q)| \geq \lfloor \frac{n}{3} \rfloor + 2$, stop. $\{V_1, V(Q), (V_1 \cup V(Q))^{\ast}\}$ is a suitable balanced decomposition for $G$, since $|V(Q)| < |V(T_m)| \leq \lfloor \frac{n}{3} \rfloor$. Otherwise, return to Step 1, using $V_1 \cup V(Q)$ for $V_1$ and $V_3 \setminus V(Q)$ for $V_3$.

This algorithm must terminate, and when it does so, we have a balanced decomposition of size at most $\lfloor \frac{n}{3} \rfloor + 1$ for $G$, and with a structure of form (i) or (ii). This completes the proof of Case 1.

Case 2. Without loss of generality, $|V(\langle Q_1 \rangle)| \geq \lfloor \frac{n}{3} \rfloor + 2$.

For $a, b \in V(Q_1)$, let $a \cdots b \subset Q_1$ be the sub-path with end-vertices $a$ and $b$. Let $Q_1 = u_1 \cdots u_s$ for some $s \geq 3$, where $u_1 = x$, $u_s = y$, and let $Q' = \langle Q_2 \rangle +_p \cdots +_p \langle Q_1 \rangle$. Note that $|V(Q')| \leq \lfloor \frac{n}{3} \rfloor$. We have

(a) either $|V(\langle u_j u_{j+1} \rangle)| \leq \lfloor \frac{n}{3} \rfloor$ for every $1 \leq j < s$,
of consecutive vertices (modulo \(n = G_r \geq i < j \preceq \)) such that (a) holds. We can describe this structure as follows. There are vertices \(v_1, \ldots, v_s \in V(G)\), where \(s \geq 3\), and SP graphs \(F_1, \ldots, F_q\) such that for every \(1 \leq i \leq q\), \(F_i\) has source \(v_i\) and sink \(v_{i+1}\) (where \(v_{q+1} = v_1\) by convention), \(|V(F_i)| \leq \left\lceil \frac{n}{2} \right\rceil\), and \(G\) is the union of the \(F_i\) in this way. Number \(V(G)\) with \(1, \ldots, n\) as follows. Each \(F_i\) has a linear ordering \(\prec_i\) as described in Section 2. We have \(u\) precedes \(v\) in the numbering if either \(u = v_i\), or \(u, v \in V(F_i - v_i)\) and \(u \prec_i v\) for some \(i\), or \(u \in V(F_i - v_i)\) and \(v \in V(F_j - v_j)\) for some \(i < j\). This numbering satisfies the condition of Lemma 8. Indeed, let \(A \subset V(G)\) be a set of consecutive vertices (modulo \(n\)) with \(|A| \geq \left\lceil \frac{n}{2} \right\rceil - 1\). Since \(|V(F_i)| \leq \left\lceil \frac{n}{2} \right\rceil\) for every \(i\), the vertices of \(v_1, \ldots, v_p\) in \(A\) are consecutive. Without loss of generality, \(v_1, \ldots, v_p \in A\), where \(p \geq 1\). Then, \(A = V(F_1 \cup \cdots \cup F_{p-1}) \cup I \cup J\), where \(I\) is an initial segment of \(F_p - v_{p+1}\) (w.r.t. \(\prec_p\)), and \(J\) is a final segment of \(F_q - v_q\) (w.r.t. \(\prec_q\)). By Lemma 9, \(G[I]\) and \(G[J]\) are both connected, so \(G[A]\) is also connected. Hence, applying Lemma 8, we have \(f(G) \leq \left\lceil \frac{n}{2} \right\rceil + 1\), and we are done for Case 2.

This completes the proof of Theorem 6. \(\square\)

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