Monochromatic Structures in Edge-coloured Graphs and Hypergraphs - A survey

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Abstract

Given a graph whose edges are coloured, on how many vertices can we find a monochromatic subgraph of a certain type, such as a connected subgraph, or a cycle, or some type of tree? Also, how many such monochromatic subgraphs do we need so that their vertex sets form either a partition or a covering of the vertices of the original graph? What happens for the analogous situations for hypergraphs? In this survey, we shall review known results and conjectures regarding these questions. In most cases, the edge-coloured (hyper)graph is either complete, or non-complete but with a density constraint such as having fixed independence number. For some problems, a restriction may be imposed on the edge-colouring, such as when it is a *Gallai colouring* (i.e. the edge-colouring does not contain a triangle with three distinct colours). Many examples of edge-colourings will also be presented, each one either showing the sharpness of a result, or supporting a possible conjecture.

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1 Introduction

In this survey article, we refer to the books by Bollobás [13] and Berge [10] for any undefined terms in graph theory and hypergraph theory. For a graph G (resp. hypergraph $\mathcal{H}$), the vertex set and edge set are denoted by $V(G)$ and $E(G)$ (resp. $V(\mathcal{H})$ and $E(\mathcal{H})$). Unless otherwise stated, all graphs and hypergraphs are finite, undirected, and without multiple edges or loops. For a hypergraph $\mathcal{H}$, an edge consists of a subset of (unordered) vertices of $\mathcal{H}$, and when we wish to emphasize that an edge has t vertices, we may call such an edge a *t-edge*. A non-empty t-uniform hypergraph is *trivial* if it consists of fewer than t isolated vertices, otherwise it is *non-trivial*. In particular, a trivial graph has a single vertex. The complete graph (or clique) on n vertices is denoted by K_n, the cycle on n vertices (or of length n) is denoted by C_n, the path of length ℓ is denoted by P_ℓ, and the complete bipartite graph with class-sizes m and n is denoted by $K_{m,n}$. The t-uniform complete hypergraph on n vertices is denoted by $\mathcal{K}_n^t$. For a vertex v and disjoint vertex subsets X, Y in a graph G, the degree of v and the minimum degree of G are denoted by $\deg(v)$ and $\delta(G)$, and the bipartite subgraph induced by X and Y is denoted by (X, Y). An *independent set* in a graph G (resp. hypergraph $\mathcal{H}$) is a subset of vertices that does not contain an edge of G (resp. $\mathcal{H}$). The *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set of G, and $\alpha(\mathcal{H})$ is similarly defined for $\mathcal{H}$. For an integer $r \geq 1$, an *r-edge-colouring* of a graph G, or simply an *r-colouring*, is a function $\phi : E(G) \to \{1, 2, \ldots, r\}$. A similar definition holds for hypergraphs with $\mathcal{H}$ in place of G. Informally, an r-colouring of G (resp. $\mathcal{H}$) is an assignment where every edge of G (resp. $\mathcal{H}$) is given one of r possible colours. When we do not wish to emphasize the number of colours involved, we may simply call such an assignment an *edge-colouring*. In the case of a 2-colouring of a graph G, we often assume that the two colours are red and blue. We say that the *red subgraph* is the subgraph R of G with $V(R) = V(G)$ and consisting of the red edges, with a similar definition for the *blue subgraph* B with the blue edges. Finally, we will often deal with partitions of vertex sets of graphs and hypergraphs, with all parts as equal as possible. For a set S with n elements, we call a partition of S into $S_1, \ldots, S_p$ a *near-equal partition* if we have $||S_i| - |S_j|| \leq 1$ for every $1 \leq i, j \leq p$. Note that such a partition is unique for fixed n and p, and for every $1 \leq i \leq p$, we have $|S_i| = \lfloor \frac{n}{p} \rfloor$ or $|S_i| = \lceil \frac{n}{p} \rceil$.

The subject of edge-colourings of graphs has been a well-studied topic of graph theory over the last several decades. A landmark result is arguably *Ramsey's theorem*, published in 1930, for which the simplest version states that: *Given an integer $k \geq 2$, whenever the edges of a sufficiently large complete graph are coloured with red and blue, there is a monochromatic copy of the complete graph K_k.* Since then, a near-endless amount of related research has been carried out, leading to the foundation of several branches in the subject. For example, in *(generalised) graph Ramsey theory*, one is interested in finding monochromatic structures in edge-coloured graphs, and a central problem is to determine the *Ramsey number* $R_r(H)$, the minimum integer n such that for any r-colouring of the complete graph K_n, there exists a monochromatic copy of the graph H. On the other hand, the area of *anti-Ramsey theory*, initiated by Erdős, Simonovits and Sós [28] (1973), deals with a similar concept, where one is interested in *rainbow coloured* subgraphs (i.e. all edges of the subgraph have distinct colours) in edge-coloured graphs. A function of particular interest is the *anti-Ramsey number* $AR(n, H)$, the maximum integer r for which there exists an r-colouring of the complete graph K_n, such that there is no rainbow copy of the graph H. Another well-studied area is *Ramsey-Turán theory*, which is an area that features the connection between Ramsey and Turán type problems. Here, a function of interest is the *Ramsey-Turán number* $RT_r(n, H, k)$, which is the maximum number of edges that a graph G on n vertices can have such that, G has no independent set of k vertices, and there exists an r-colouring of G with no monochromatic copy of the graph H. This area probably emerged in the 1960s when Andrásfai [4, 5] answered some questions of Erdős that concern the function $RT_r(n, H, k)$.

Our aim in this survey is to consider a class of problems which belong to the graph Ramsey theory area. We are mainly interested in two questions. Suppose that a graph is given an edge-colouring – we sometimes call such a graph a *host graph*, and usually think of it as a rather large graph. The first question is: *On how many vertices can we find a monochromatic subgraph of a certain type, such as a connected subgraph, or a cycle, or some type of tree?* The second question is: *How many such monochromatic subgraphs do we need so that their vertex sets form either a partition or a covering of the vertices of the host graph?* We shall review known results and conjectures related to these two questions in Sections 2 and 3. In Section 4, we consider the two questions in hypergraph settings. Some of the work date as far back as the 1960s, and the research had been particularly intense since around the 1990s. We will, in particular, present many examples of edge-colourings of graphs and hypergraphs, where each edge-colouring shows either the sharpness of a result, or the support of a possible conjecture. Our results and conjectures here will have some overlaps with recent surveys of Kano and Li [67] (2008), Fujita, Magnant and Ozeki [39] (2010), and Gyárfás [47] (2011). We will attempt to minimise these overlaps, and also to emphasize more recent developments.

2 Monochromatic Structures in Graphs

In this section, we consider the problem of finding monochromatic subgraphs in edge-coloured graphs. A first result in this direction is the following observation, made a long time ago by Erdős and Rado: *A graph is either connected, or its complement is connected.* In other words, for every 2-colouring of the edges of a complete graph, there exists a monochromatic spanning connected subgraph (or equivalently, a monochromatic spanning tree). A substantial generalisation of this observation is to ask for the existence of a large monochromatic subgraph of a certain type in an edge-coloured graph. Here, we present many known results and problems related to this question. In many cases, the edge-coloured host graph is a complete graph.

2.1 Connected and k-connected subgraphs

To extend the observation of Erdős and Rado, one way is to ask what happens when $r \geq 2$ colours are used to colour the complete graph K_n. In this direction, Gerencsér and Gyárfás [43] (1967) asked for the order of a monochromatic connected subgraph that one can always find. Gyárfás and Füredi independently proved the following result (see also Liu et al. [77] for a short proof).

Theorem 2.1.1 (Gyárfás - 1977 [44]; Füredi - 1981 [40]). *Let $r \geq 2$. For every r-colouring of K_n, there exists a monochromatic connected subgraph on at least $\frac{n}{r-1}$ vertices. This bound is sharp if $r-1$ is a prime power.*

To see the sharpness, we consider the following well-known construction of an r-colouring on K_n, using affine planes. This construction will also be important for many more problems that we will encounter throughout the entire survey.

Construction 2.1.2. *Let $r-1$ be a prime power, and consider the finite affine plane $AG(2, r-1)$ over the field $\mathbb{F}_{r-1}$ (see the appendix in Section 5). Let $p_1, \ldots, p_{(r-1)^2}$ be the points and $\mathcal{P}_1, \ldots, \mathcal{P}_r$ be the parallel classes of lines of $AG(2, r-1)$. Now, take a near-equal partition of the vertex set of K_n into $(r-1)^2$ classes $V_1, \ldots, V_{(r-1)^2}$. We define the r-colouring ψ on K_n as follows. If $u \in V_i$ and $v \in V_j$ are two vertices of K_n and $1 \leq i \neq j \leq (r-1)^2$, then let $\psi(uv) = \ell$ if and only if p_i and p_j lie on the same line in the class $\mathcal{P}_\ell$ (where $1 \leq \ell \leq r$). Colour the edges inside the classes $V_1, \ldots, V_{(r-1)^2}$ arbitrarily.*

Then in the r-colouring ψ of K_n, every monochromatic connected subgraph has at most $(r-1)\lceil \frac{n}{(r-1)^2} \rceil < \frac{n}{r-1} + r$ vertices. Also, a very noteworthy feature about the r-colouring ψ is that, it can be obtained by substituting edge-coloured cliques (sometimes called *blocks*) for the vertices of an edge-coloured graph – in this case an r-colouring of $K_{(r-1)^2}$, with all edges between a pair of cliques retaining the colour of the corresponding two substituted vertices. As we shall see, this "colouring by substitution" technique will be important in many more constructions of edge-colourings.

To extend Theorem 2.1.1, we may ask for the existence of a large monochromatic subgraph with high vertex connectivity in an edge-coloured complete graph. Recall that a graph G is k-*connected* if $|V(G)| > k$, and for any set $C \subset V(G)$ with $|C| < k$, the graph $G - C$ is connected. The following question was asked by Bollobás.

Question 2.1.3 (Bollobás - 2003 [14]). *Let $1 \leq s < r$ and $k \geq 1$. Whenever we have an r-colouring of the edges of K_n, on how many vertices can we find a k-connected subgraph, using at most s colours?*

We shall focus on Bollobás' question only for the case when the desired subgraph is monochromatic, i.e. $s = 1$. For results concerning $s \geq 2$, we refer the reader to Liu et al. [76]. In order to state the results, we make the following definition. Let $m(n, r, k)$ denote the maximum integer m such that, every r-colouring of K_n contains a monochromatic k-connected subgraph on at least m vertices. Then, our task is to determine the function $m(n, r, k)$.

If $n \leq 2r(k-1)$, then we have the following construction of Matula [82] (1983).

Construction 2.1.4. *Let $n \leq 2r(k-1)$. We consider the standard decomposition of K_{2r} into r edge-disjoint "zig-zag rotational" Hamilton paths (see for example [13], Ch. I, Theorem 11). That is, let $v_1, \ldots, v_{2r}$ be the vertices of K_{2r}, and consider the Hamilton path $Q_1 = v_1 v_{2r} v_2 v_{2r-1} \cdots v_r v_{r+1}$. For $1 < \ell \leq r$, let Q_ℓ be the Hamilton path obtained from Q_1 by adding $\ell - 1$ to the indices of the vertices of Q_1, modulo $2r$. Then, $Q_1, \ldots, Q_r$ is a decomposition of K_{2r} into edge-disjoint Hamilton paths, and moreover, every vertex of K_{2r} is the end-vertex of exactly one Hamilton path. We colour the Hamilton path Q_ℓ with colour ℓ for all $1 \leq \ell \leq r$. Now, we obtain an r-colouring of K_n as follows. Partition the vertices of K_n into*

$V_1, \dots, V_{2r}$ such that $|V_i| \le k-1$ for all $1 \le i \le 2r$. For $1 \le i \ne j \le 2r$, we give all edges of (V_i, V_j) the colour of v_iv_j, and all edges inside V_i the colour of the Hamilton path in K_{2r} with v_i as an end-vertex (i.e. for $1 \le \ell \le r$, all edges inside V_ℓ and $V_{r+\ell}$ are given colour ℓ).

We see that in this r-colouring of K_n, there is no monochromatic k-connected subgraph at all, and hence $m(n, r, k) = 0$ for $n \le 2r(k-1)$.

In the case $r = 2$, Bollobás and Gyárfás [15] (2003) gave the following construction of a 2-colouring of K_n for $n > 4(k-1)$.

Construction 2.1.5. *Let $n > 4(k-1)$. We define a 2-colouring on K_n as follows. Let V_1, V_2, V_3, V_4 be four disjoint sets of vertices of K_n, each with size $k-1$, and let V_5 be the remaining vertices (note that $V_5 \ne \emptyset$). We colour all edges of (V_i, V_j) red if $\{i, j\} \in \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{1, 5\}, \{2, 5\}\}$, and blue otherwise. The edges inside the classes V_i are arbitrarily coloured.*

We see that in Constructions 2.1.4 and 2.1.5, the edge-colourings of K_n are obtained by the "colouring by substitution" technique. In the case of the former, we have substituted monochromatic cliques for the vertices of an r-coloured K_{2r}. In the latter, the 2-colouring of K_n is obtained by substituting arbitrarily 2-coloured cliques for the vertices of K_5, where the K_5 is given a 2-colouring with both colour classes forming a *bull* (i.e. the graph with two pendant edges attached to two vertices of a triangle).

In the 2-colouring of K_n in Construction 2.1.5, it is easy to check that the monochromatic k-connected subgraph with maximum order has $n-2k+2$ vertices. That is, we have $m(n, 2, k) \le n - 2k + 2$ if $n > 4(k-1)$. Inspired by Construction 2.1.4 for $r = 2$ and Construction 2.1.5, Bollobás and Gyárfás made the following conjecture.

Conjecture 2.1.6 (Bollobás, Gyárfás - 2003 [15]). *For $n > 4(k-1)$, we have $m(n, 2, k) = n - 2k + 2$.*

The two constructions imply that if Conjecture 2.1.6 is true, then the bound $n > 4(k-1)$ is the best possible. Many partial results to the conjecture are known. The conjecture has been verified by Bollobás and Gyárfás for $k = 2$ [15]; and by Liu et al. for $k = 3$ [75], and for $n \ge 13k - 15$ [77]. The best known partial result is by Fujita and Magnant.

Theorem 2.1.7 (Fujita, Magnant - 2011 [36]). *For $n > 6.5(k-1)$, we have $m(n, 2, k) = n - 2k + 2$.*

When $r \ge 3$ colours are used, Liu et al. also studied the function $m(n, r, k)$.

Theorem 2.1.8 (Liu, Morris, Prince - 2004 [77]). *Let $r \ge 3$.*

(a) $m(n, r, k) \ge \frac{n}{r-1} - 11(k^2 - k)r$.

(b) If $r-1$ is a prime power, then $m(n, r, k) < \frac{n-k+1}{r-1} + r$.

In particular, if r and k are fixed and $r-1$ is a prime power, then $m(n, r, k) = \frac{n}{r-1} + O(1)$.

To obtain part (b), we can easily modify Construction 2.1.2, as follows.

Construction 2.1.9. *For $r \geq 3$ and $n > 2r(k-1)$, we define the r-colouring ψ' on K_n as follows. Take disjoint sets of vertices $U_1, \ldots, U_r$ of K_n, each with $k-1$ vertices. Let W be the remaining vertices (note that $W \neq \emptyset$), and give the complete subgraph on W the r-colouring ψ as described in Construction 2.1.2. Now for $1 \leq i \neq j \leq r$, $u \in U_i$, $v \in U_j$ and $w \in W$, let $\psi'(uv) = \min\{i, j\}$ and $\psi'(uw) = i$. The edges inside the classes U_i are arbitrarily coloured.*

It is easy to see that in the r-colouring ψ' of K_n, the monochromatic k-connected subgraph with maximum order has at most $(r-1)\lceil \frac{n-r(k-1)}{(r-1)^2} \rceil + k - 1 < \frac{n-k+1}{r-1} + r$ vertices. Hence, $m(n,r,k) < \frac{n-k+1}{r-1} + r$ for $n > 2r(k-1)$. Moreover, Construction 2.1.4 implies that $m(n,r,k) = 0$ for $n \leq 2r(k-1)$. In view of these two constructions, Liu et al. also made the following conjecture.

Conjecture 2.1.10 (Liu, Morris, Prince - 2004 [77])**.** *Let $r \geq 3$ and $n > 2r(k-1)$. Then*

$$m(n,r,k) \geq \frac{n-k+1}{r-1}.$$

The conjecture has been verified by Liu et al. for $r = 3$ and large n.

Theorem 2.1.11 (Liu, Morris, Prince - 2004 [77])**.** *For $n \geq 480k$, we have*

$$\frac{n-k+1}{2} \leq m(n,3,k) \leq \frac{n-k+1}{2} + 1.$$

Moreover, equality holds in the lower bound if and only if $n + k \equiv 1 \pmod 4$.

We also see that Theorem 2.1.8(a) gives a good bound only when $k = o(\sqrt{n})$. Liu and Person used Szemerédi's regularity lemma to obtain the following improvement.

Theorem 2.1.12 (Liu, Person - 2008 [78])**.** *Let $r \geq 3$ be fixed, and $k = o(n)$. Then, we have $m(n,r,k) \geq \frac{n}{r-1} - o(n)$. Equality holds if $r-1$ is a prime power.*

Finally, we mention a slightly different version of Question 2.1.3, for $s = 1$. Let $m^*(r,k)$ be the smallest integer n such that, any r-colouring of K_n has a monochromatic k-connected subgraph. The problem of the determination of the function $m^*(r,k)$ was proposed by Matula. Note that in this problem, one does not worry about the order of a monochromatic k-connected subgraph in r-coloured complete graphs, but only that such a subgraph exists. Matula proved the following result.

Theorem 2.1.13 (Matula - 1983 [82])**.** *Let $k \geq 2$.*

(a) $4(k-1) + 1 \leq m^*(2,k) < \left(3 + \sqrt{\frac{11}{3}}\right)(k-1) + 1.$

(b) For $r \geq 2$, we have $2r(k-1) + 1 \leq m^*(r,k) < \frac{10}{3} r(k-1) + 1$.

The lower bounds follow from Construction 2.1.4. Matula also made the following conjecture. It is a slightly weaker assertion than the combination of Conjectures 2.1.6 and 2.1.10.

Conjecture 2.1.14 (Matula - 1983 [82])**.** *For $r, k \geq 2$, we have $m^*(r,k) = 2r(k-1) + 1$.*

Matula remarked that by using some tedious arguments, Conjecture 2.1.14 holds for $k = 2, 3, 4, 5$, and that the upper bound of Theorem 2.1.13(a) can be improved to somewhat below $4.7(k-1)+1$.

Next, we consider the analogous problem when the host graph is a complete bipartite graph. We may ask for the order of a monochromatic connected subgraph that we can always find in any r-colouring of a complete bipartite graph $K_{m,n}$. For this, Gyárfás proved the following result.

Lemma 2.1.15 (Gyárfás - 1977 [44]). *Let $r \geq 1$. For every r-colouring of the complete bipartite graph $K_{m,n}$, there exists a monochromatic connected subgraph on at least $\frac{m+n}{r}$ vertices.*

Lemma 2.1.15 easily implies the first part of Theorem 2.1.1. Indeed, let K_n be given an r-colouring ($r \geq 2$). If no colour spans a connected subgraph on n vertices, then there is a colour that spans a connected component on vertex set X with $|X| < n$. By letting Y be the remaining vertices, we may apply Lemma 2.1.15 on the $(r-1)$-coloured complete bipartite subgraph (X, Y), to obtain a monochromatic connected subgraph on at least $\frac{n}{r-1}$ vertices.

To see the sharpness of Lemma 2.1.15, we may consider the following example.

Construction 2.1.16. *Take near-equal partitions of each class of $K_{m,n}$ into r sets, say $U_1, \ldots, U_r$ and $V_1, \ldots, V_r$. For $1 \leq i, j \leq r$, colour all edges of (U_i, V_j) with colour $i - j \pmod{r}$.*

Then in this r-colouring of $K_{m,n}$, the monochromatic connected subgraph with maximum order has at most $\frac{m+n}{r}+2$ vertices, since we have $|U_i| \leq \frac{m}{r}+1$ and $|V_i| \leq \frac{n}{r}+1$ for all $1 \leq i \leq r$.

Inspired by Lemma 2.1.15, Liu et al. [77] made the conjecture that the same result holds when we want to find a monochromatic k-connected subgraph, provided that the classes of $K_{m,n}$ are large: *For $r \geq 3$ and $m, n \geq rk$, any r-colouring of $K_{m,n}$ contains a monochromatic k-connected subgraph on at least $\frac{m+n}{r}$ vertices.* Note that the bound of $\frac{m+n}{r}$ does not depend on k. Lemma 2.1.15 implies that the conjecture holds for $k = 1$. The sharpness of the conjecture can again be seen by Construction 2.1.16 – note that the condition $m, n \geq rk$ implies that, the monochromatic k-connected subgraph with maximum order in the r-colouring has at most $\frac{m+n}{r}+2$ vertices.

However, we see that the case $r = 2$ is somehow omitted. Indeed, we present the following example of an r-colouring of $K_{m,n}$, which is a counter-example to Liu et al.'s conjecture for small n.

Construction 2.1.17. *Let $r, k \geq 2$, $n \leq 2r(k-1)$ and $m \geq 6(r+1)$. Partition the m-class of $K_{m,n}$ into $2r-1$ near-equal sets, say $U_1, \ldots, U_{2r-1}$, and the n-class into $2r$ near-equal sets, say $V_1, \ldots, V_{2r}$. A result of Laskar and Auerbach [72] (1976) implies that the complete bipartite graph $K_{2r-1,2r}$ can be decomposed into r edge-disjoint Hamilton paths, where every vertex of the $2r$-class is an end-vertex of exactly one Hamilton path. Colour the Hamilton paths with r distinct colours, and let $\{u_1, \ldots, u_{2r-1}\}$ and $\{v_1, \ldots, v_{2r}\}$ be the classes of the $K_{2r-1,2r}$. Now, for $1 \leq i \leq 2r-1$ and $1 \leq j \leq 2r$, we give all edges of (U_i, V_j) the colour of the edge u_iv_j.*

Then, $n \leq 2r(k-1)$ implies $|V_j| \leq k-1$ for all $1 \leq j \leq 2r$. Hence in this r-colouring of $K_{m,n}$, the monochromatic k-connected subgraph of maximum order has at most $\lceil \frac{m}{2r-1} \rceil + 2\lceil \frac{n}{2r} \rceil < \frac{m+n}{r}$ vertices (the last inequality follows from $m \geq 6(r+1)$).

In light of this construction, we revise the conjecture of Liu et al., as follows.

Conjecture 2.1.18 (Refinement of Liu, Morris, Prince - 2004 [77]). *Let $r \geq 2$ and $m \geq n > 2r(k-1)$. Then, for any r-colouring of the complete bipartite graph $K_{m,n}$, there exists a monochromatic k-connected subgraph on at least $\frac{m+n}{r}$ vertices.*

By using a bipartite version of Szemerédi's regularity lemma, Liu and Person, in response to Liu et al.'s original conjecture, obtained the following partial result.

Theorem 2.1.19 (Liu, Person - 2008 [78]). *Let $r \geq 2$ be fixed, $m \geq n \geq k$, and $k = o(n)$. Then, for every r-colouring of the compete bipartite graph $K_{m,n}$, there exists a monochromatic k-connected subgraph on at least $\frac{m+n}{r} - o(n)$ vertices.*

2.2 Cycles and regular subgraphs

Given an edge-coloured graph, we may ask for the existence of a long monochromatic cycle. Recall that the *circumference* of a graph G, denoted by $c(G)$, is the length of a longest cycle in G. Faudree et al. proved the following result.

Theorem 2.2.1 (Faudree, Lesniak, Schiermeyer - 2009 [29]). *Let G be a graph of order $n \geq 6$, and let $\overline{G}$ be the complement of G. Then $\max\{c(G), c(\overline{G})\} \geq \lceil \frac{2n}{3} \rceil$, and this bound is sharp.*

The sharpness can be easily seen by taking G to be the graph consisting of $\lfloor \frac{n}{3} \rfloor$ isolated vertices and a clique on the remaining $\lceil \frac{2n}{3} \rceil$ vertices. Note that Theorem 2.2.1 is equivalent to saying that, in any 2-colouring of K_n ($n \geq 6$), there exists a monochromatic cycle with length at least $\lceil \frac{2n}{3} \rceil$.

When $r \geq 3$ colours are used to colour K_n, we may again consider the r-colouring ψ on K_n, as described in Construction 2.1.2. Then, if $r-1$ is a prime power, the longest monochromatic cycle has length at most $\frac{n}{r-1} + r$. Inspired by the construction, Faudree et al. also made the following conjecture.

Conjecture 2.2.2 (Faudree, Lesniak, Schiermeyer - 2009 [29]). *For $r \geq 3$, let K_n be given an r-colouring. For $1 \leq i \leq r$, let G_i be the graph on n vertices induced by colour i. Then*

$$\max\{c(G_1), c(G_2), \ldots, c(G_r)\} \geq \frac{n}{r-1}.$$

Let $f(n,r)$ denote the maximum integer ℓ such that every r-colouring of K_n contains a monochromatic cycle of length at least ℓ. Then $f(n,r) < \frac{n}{r-1} + r$ if $r-1$ is a prime power, while Conjecture 2.2.2 claims that $f(n,r) \geq \frac{n}{r-1}$ whenever $r \geq 3$. Fujita [31] (2011) observed that the conjecture does not hold for small n, by considering the decomposition of K_{2r} into r edge-disjoint "zig-zag rotational" Hamilton paths as described in Construction 2.1.4, and giving each Hamilton path a distinct colour. Clearly, this r-colouring of K_{2r} implies $f(n,r) = 0$ for $n \leq 2r$, since by deleting $2r - n$ vertices from K_{2r}, there is no monochromatic cycle at all in the resulting r-colouring of K_n.

Conjecture 2.2.2 remains open for sufficiently large n – certainly, we need $n > 2r$ to hold. In the case when n is linear in r, Fujita et al. proved the following result.

Theorem 2.2.3 (Fujita, Lesniak, Tóth - 2015 [35]).

(a) For $r \geq 3$, we have $f(2r+2, r) = 3$. For $r = 1, 2$, we have $f(2r+2, r) = 4$.

(b) *For any pair of integers $s, c \geq 2$, there exists $r_0 = r_0(s, c)$ such that $f(sr + c, r) = s + 1$ for all $r \geq r_0$.*

In particular, part (b) implies that Conjecture 2.2.2 holds for $n = sr + c$ with r sufficiently large. By using a result of Erdős and Gallai [26] (1959), which is a Turán type result for the cycle, Fujita also obtained the following slightly weaker version of the conjecture.

Theorem 2.2.4 (Fujita - 2011 [31]). *For $n \geq r \geq 1$, we have $f(n, r) \geq \frac{n}{r}$.*

As a generalisation, we may ask for the largest order of a monochromatic connected d-regular subgraph, where $d \geq 2$. Thus, the case $d = 2$ corresponds to cycles. Sárközy et al. proved the following result.

Theorem 2.2.5 (Sárközy, Selkow, Song - 2013 [91]). *For every $\varepsilon > 0$ and integers $r, d \geq 2$, there exists $n_0 = n_0(\varepsilon, r, d)$ such that the following holds. For all $n \geq n_0$ and any r-colouring of K_n, there exists a monochromatic connected d-regular subgraph on at least $\frac{(1-\varepsilon)n}{r}$ vertices.*

We see that Theorem 2.2.5 can be seen as an extension of Theorem 2.2.4. The slightly surprising fact about this extension is that the value of d plays a fairly minor role, in the sense that the order of a monochromatic connected d-regular subgraph that we can find is almost as large as that of a monochromatic cycle, i.e. approximately $\frac{n}{r}$, and independent of d.

When the edge-coloured host graph is not complete, Li et al. proposed the following problem, where there is a condition on the minimum degree of the host graph.

Problem 2.2.6 (Li, Nikiforov, Schelp - 2010 [74]). *Let $0 < c < 1$ and let n be a sufficiently large integer. If G is a graph of order n with $\delta(G) > cn$, and G is given a 2-colouring with red and blue subgraphs R and B, what is the minimum possible value of $\max\{c(R), c(B)\}$?*

As opposed to just finding a single monochromatic cycle of a specified length, some authors have considered the problem of finding all cycle lengths in a specified interval. In this direction, Li et al. conjectured the following pancyclic type result for monochromatic cycles in 2-coloured graphs.

Conjecture 2.2.7 (Li, Nikiforov, Schelp - 2010 [74]). *Let $n \geq 4$ and G be a graph of order n with $\delta(G) > \frac{3n}{4}$. If G is given a 2-colouring with red and blue subgraphs R and B, then for each integer $\ell \in \left[4, \left\lceil \frac{n}{2} \right\rceil\right]$, either $C_\ell \subset R$ or $C_\ell \subset B$.*

They observed the following example, which shows that if this conjecture is true, then it is tight: Let $n = 4p$. Colour the edges of the complete bipartite graph $K_{2p,2p}$ red, and add a blue $K_{p,p}$ in each of its classes. Then we have a 2-coloured graph G with $\delta(G) = \frac{3n}{4}$, but clearly the colouring produces no monochromatic odd cycle.

In the same paper, Li et al. proved the following partial result.

Theorem 2.2.8 (Li, Nikiforov, Schelp - 2010 [74]). *Let $\varepsilon > 0$ and G be a graph of sufficiently large order n with $\delta(G) > \frac{3n}{4}$. If G is given a 2-colouring with red and blue subgraphs R and B, then for each integer $\ell \in \left[4, \left\lfloor (\frac{1}{8} - \varepsilon)n \right\rfloor\right]$, either $C_\ell \subset R$ or $C_\ell \subset B$.*

Benevides et al. offered the following slightly stronger solution but for large n. Here we need a definition. Let $n = 4p$ and G be a graph isomorphic to $K_{p,p,p,p}$. Then G is said to be 2-*bipartite* 2-*edge-coloured* if the edges of G are 2-coloured so that the graph induced by each of the colours is bipartite. Such a 2-bipartite 2-edge-coloured graph has minimum degree $\frac{3n}{4}$ but contains no monochromatic odd cycles.

Theorem 2.2.9 (Benevides, Łuczak, Scott, Skokan, White - 2012 [9])**.** *There exists a positive integer n_0 with the following property. Let G be a graph of order $n \geq n_0$ with $\delta(G) \geq \frac{3n}{4}$. Suppose that G is given a 2-colouring with red and blue subgraphs R and B. Then either $C_\ell \subset R$ or $C_\ell \subset B$ for all $\ell \in \left[4, \left\lceil \frac{n}{2} \right\rceil\right]$, or $n = 4p$, $G = K_{p,p,p,p}$ and the colouring is a 2-bipartite 2-edge-colouring.*

2.3 Subgraphs with bounded diameter

We consider the problem of finding monochromatic subgraphs with bounded diameter in edge-coloured complete graphs. For $r, D \geq 2$, define $g(n, r, D)$ to be the maximum integer m such that, for every r-colouring of K_n, there is a monochromatic subgraph with diameter at most D on at least m vertices. We remark that we neglect the case $D = 1$, since this is equivalent to the problem of the determination of the classical Ramsey number $R_r(K_m)$. The problem of the determination of the function $g(n, r, D)$ was proposed by Erdős (for $D = 2$), and Mubayi (for $D \geq 3$).

Problem 2.3.1 (Erdős - 1996 [24]; Mubayi - 2002 [83])**.** *For $r, D \geq 2$, determine the function $g(n, r, D)$.*

By simply considering the largest monochromatic star at any vertex of an r-coloured K_n, Erdős noticed that $g(n, r, 2) \geq \frac{n-1}{r} + 1$, and asked if this bound is optimal. Fowler then proved that the answer is yes, if and only if $r \geq 3$. He determined $g(n, 2, 2)$ exactly, and $g(n, r, 2)$ sharply for $r \geq 3$.

Theorem 2.3.2 (Fowler - 1999 [30])**.**

(a) $g(n, 2, 2) = \left\lceil \frac{3n}{4} \right\rceil$.

(b) For fixed $r \geq 3$, we have $g(n, r, 2) = \frac{n}{r} + O(1)$.

To see that $g(n, 2, 2) \leq \left\lceil \frac{3n}{4} \right\rceil$, we may consider the following 2-colouring of K_n. Take a near-equal partition of the vertices of K_n into sets V_1, V_2, V_3, V_4. Colour all edges of (V_i, V_{i+1}) in red for $i = 1, 2, 3$, all other edges between the classes in blue, and the edges inside the classes arbitrarily. Then, the subgraph with diameter at most 2 and of maximum order has $\left\lceil \frac{3n}{4} \right\rceil$ vertices. The proofs of $g(n, 2, 2) \geq \left\lceil \frac{3n}{4} \right\rceil$ and $g(n, r, 2) \leq \frac{n}{r} + O(1)$ for $r \geq 3$ by Fowler are more complicated.

Now, we consider $D \geq 3$. For the case $r = 2$, the following result is folkloristic, and an early citation can be found in Bialostocki et al. [12].

Theorem 2.3.3. *We have $g(n, 2, D) = n$ for all $D \geq 3$. That is, in every 2-colouring of K_n, there exists a monochromatic spanning subgraph with diameter at most D.*

For $r, D \geq 3$, we again have the upper bound of $g(n, r, D) < \frac{n}{r-1} + r$ if $r - 1$ is a prime power, from Construction 2.1.2. Mubayi managed to prove a lower bound for $g(n, r, 3)$.

Theorem 2.3.4 (Mubayi - 2002 [83]). *For $r \geq 2$, we have*

$$g(n, r, 3) > \frac{n}{r - 1 + \frac{1}{r}}.$$

For $r = 3$ and $D \geq 4$, Mubayi also managed to compute $g(n, 3, D)$ exactly.

Theorem 2.3.5 (Mubayi - 2002 [83]). *For $D \geq 4$, we have*

$$g(n, 3, D) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4, \\ \lceil \frac{n}{2} \rceil & \text{otherwise.} \end{cases}$$

Thus in Problem 2.3.1, the determination of the function $g(n, r, D)$, in a sharpness sense for large n, remains open for $r = D = 3$, and for $r \geq 4$, $D \geq 3$.

2.4 Subgraphs with large minimum degree

In an edge-coloured graph, how large a monochromatic subgraph can we find, if the host graph and the monochromatic subgraphs have constraints on their minimum degrees? Let $h(n, c, d, r)$ be the maximum integer m such that, for every r-colouring of any graph of order n and with minimum degree at least c, there exists a monochromatic subgraph with minimum degree at least d and order at least m. Caro and Yuster proposed the problem of the determination of $h(n, c, d, r)$. They proved the following results.

Theorem 2.4.1 (Caro, Yuster - 2003 [19]).

(a) For all $d \geq 1$ and $c > 4(d - 1)$,

$$h(n, c, d, 2) \geq \frac{c - 4d + 4}{2(c - 3d + 3)} n + \frac{3d(d - 1)}{4(c - 3d + 3)}.$$

(b) For all $d \geq 1$ and $c \leq 4(d - 1)$, if n is sufficiently large, then $h(n, c, d, 2) \leq d^2 - d + 1$. In particular, $h(n, c, d, 2)$ is independent of n.

Theorem 2.4.2 (Caro, Yuster - 2003 [19]). *For all $d \geq 1$, $r \geq 2$ and $c > 2r(d - 1)$, there exists a constant C such that*

$$h(n, c, d, r) \leq \frac{c - 2r(d - 1)}{r(c - (r + 1)(d - 1))} n + C.$$

In particular, $h(n, c, d, 2) \leq \frac{c - 4d + 4}{2(c - 3d + 3)} n + C$.

We see that Theorems 2.4.1 and 2.4.2 imply that if c is fixed, then $h(n, c, d, 2)$ is determined up to a constant additive term. The theorems also show that $h(n, c, d, 2)$ transitions from a constant to a value linear in n when c increases from $4d - 4$ to $4d - 3$.

To see Theorem 2.4.1(b), it suffices to construct a 2-coloured, $4(d - 1)$-regular graph on n vertices (for n sufficiently large), with no monochromatic subgraph having minimum degree at least d and on more than $d^2 - d + 1$ vertices. Caro and Yuster gave the following construction.

Construction 2.4.3. *Let $c = 4(d-1)$ and n be sufficiently large. We first create a specific graph H on n vertices. Let the vertices be $v_1, \dots, v_n$ and connect two vertices v_i and v_j if and only if $|i-j| \leq d-1$. Hence, all the vertices $v_d, \dots, v_{n-d+1}$ have degree $2(d-1)$, and the remaining $2(d-1)$ vertices have smaller degree. We add the following $\binom{d}{2}$ edges. For all $1 \leq i \leq j \leq d-1$, we add the edge $v_i v_{jd+1}$. For example, if $d=3$ we add v_1v_4, v_1v_7 and v_2v_7. Note that these added edges are indeed new edges. The resulting graph H has n vertices and $(d-1)n$ edges. Furthermore, all the vertices have degree $2(d-1)$, except for v_{jd+1} whose degree is $2(d-1)+j$, and $v_{n-d+1+j}$ whose degree is $2(d-1)-j$, for $1 \leq j \leq d-1$. Now, note that the vertices of excess degree, namely $v_{d+1}, v_{2d+1}, \dots, v_{d^2-d+1}$, form an independent set. Hence, for n sufficiently large, there exists a $4(d-1)$-regular graph with n vertices, and a 2-colouring of it, such that each monochromatic subgraph is isomorphic to H. In the second copy, the vertex playing the role of v_{jd+1} plays the role of the vertex $v_{n-d+1+j}$ in the first copy, for $1 \leq j \leq d-1$, and vice versa.*

Observe that in the first copy of H, any subgraph with minimum degree at least d may only contain the vertices $v_1, \dots, v_{d^2-d+1}$, and thus has order at most d^2-d+1. This clearly implies Theorem 2.4.1(b).

Theorem 2.4.2 trivially holds for $d=1$. For $d \geq 2$, Caro and Yuster provided the following construction.

Construction 2.4.4. *Let $d, r \geq 2$ and $c > 2r(d-1)$. We construct an r-coloured graph with $n = r(m+d)$ vertices and minimum degree at least c, where m is an arbitrary element of some fixed infinite arithmetic sequence whose difference and first element are only functions of c, d and r. This r-coloured graph will have no monochromatic subgraph with minimum degree at least d and more vertices than the value stated in Theorem 2.4.2, and this clearly suffices for the theorem. Let m be a sufficiently large positive integer such that*

$$y = \frac{(r-1)(d-1)}{c-(r+1)(d-1)} m$$

is an integer. Let $A_1, \dots, A_r, B_1, \dots, B_r$ be pairwise disjoint sets of vertices, with $|A_i| = y$ and $|B_i| = m+d-y$ for $1 \leq i \leq r$. In each B_i, we place a graph with colour i, and with minimum degree at least $c-(r-1)(d-1)$. In each A_i, we place a $(d-1)$-degenerate graph with colour i, having precisely d vertices of degree $d-1$ and the rest are of degree $2(d-1)$. It is easy to show that such a graph exists. Denote by A'_i the $y-d$ vertices of A_i with degree $2(d-1)$ and $A''_i = A_i \setminus A'_i$. Now for each $j \neq i$, we place a bipartite graph with colour i, whose classes are A_i and $A_j \cup B_j$. In this bipartite graph, the degree of all the vertices of $A_j \cup B_j$ is $d-1$, the degrees of all the vertices of A'_i are at least $\frac{c-(r+1)(d-1)}{r-1}$, and the degrees of all vertices of A''_i are at least $\frac{c-r(d-1)}{r-1}$. This can be done for m sufficiently large since

$$(y-d)\left\lceil \frac{c-(r+1)(d-1)}{r-1} \right\rceil + d\left\lceil \frac{c-r(d-1)}{r-1} \right\rceil \leq (d-1)(m+d).$$

For m sufficiently large, we can place all of these $r(r-1)$ bipartite graphs such that their edge sets are pairwise disjoint (an immediate consequence of Hall's Theorem).

In this construction, the minimum degree of the graph is at least c. Furthermore, any monochromatic subgraph with minimum degree at least d must be completely placed within some B_i. It follows that

$$h(n,c,d,r) \leq m + d - \frac{(r-1)(d-1)}{c-(r+1)(d-1)}m = \frac{c-2r(d-1)}{r(c-(r+1)(d-1))}n + C.$$

Caro and Yuster also managed to determine $h(n,c,d,2)$ whenever c is very close to n, for n sufficiently large.

Theorem 2.4.5 (Caro, Yuster - 2003 [19])**.** *Let c and d be positive integers. For n sufficiently large, $h(n, n-c, d, 2) = n - 2d - c + 3$.*

To see the upper bound of Theorem 2.4.5 for large n, they gave the following construction.

Construction 2.4.6. *Let A, A', B be disjoint sets of vertices with $|A| = |B| = 2d + c - 3$ and $|A'| = n - 2(2d + c - 3)$. Colour all edges within $A \cup A'$ red, and all edges within B and between A' and B blue. Let $A = \{v_1, \ldots, v_{2d+c-3}\}$ and $B = \{u_1, \ldots, u_{2d+c-3}\}$. For $1 \leq i \leq 2d + c - 3$, colour the $d-1$ edges from u_i to $v_i, \ldots, v_{i+d-2}$ blue, and the $d-1$ edges to $v_{i+d-1}, \ldots, v_{i+2d-3}$ red, with the indices of the v_j taken modulo $2d + c - 3$. There are no other edges connecting A and B.*

It is easy to verify that for n sufficiently large, this graph is $(n-c)$-regular, and contains no monochromatic subgraph with minimum degree at least d and more than $n - 2d - c + 3$ vertices.

2.5 Specific trees

We recall the observation of Erdős and Rado: *Every 2-coloured complete graph contains a monochromatic spanning tree.* We may extend this observation, by insisting that the monochromatic subgraph is a specific type of tree. There are many results in this direction. Bialostocki et al. proved the following.

Theorem 2.5.1 (Bialostocki, Dierker, Voxman - 1992 [12])**.** *For every 2-colouring of K_n, there exists a monchromatic spanning tree with height at most 2.*

Theorem 2.5.2 (Bialostocki, Dierker, Voxman - 1992 [12])**.** *For every 2-colouring of K_n, there exists a monchromatic spanning subdivided star, whose centre has degree at most $\lceil \frac{n-1}{2} \rceil$.*

The same authors had also conjectured that every 2-coloured K_n contains a monochromatic spanning broom (A *broom* is a path with a star at one end). This conjecture was proved by Burr, but his proof was unfortunately unpublished. A proof of Burr's result can be found in the survey of Gyárfás [47].

Theorem 2.5.3 (Burr - 1992 [16])**.** *For every 2-colouring of K_n, there exists a monchromatic spanning broom.*

A *double star* is a graph obtained by connecting the centres of two vertex-disjoint stars with an edge. Mubayi, and Liu et al. independently proved the following result, which is an extension of Lemma 2.1.15.

Lemma 2.5.4 (Mubayi - 2002 [83]; Liu, Morris, Prince - 2004 [77]). *Let $r \geq 1$. For every r-colouring of the complete bipartite graph $K_{m,n}$, there exists a monochromatic double star with at least $\frac{m+n}{r}$ vertices.*

The sharpness of Lemma 2.5.4 can again be seen by the r-colouring of $K_{m,n}$ in Construction 2.1.16. Inspired by the lemma, Gyárfás and Sárközy studied the analogous problem of finding a monochromatic double star in an r-coloured complete graph. They noticed the lemma implies that in any r-colouring of K_n, either all colour classes induce just one component, or there is a monochromatic double star with at least $\frac{n}{r-1}$ vertices. They also asked the following question, which is the analogue of Theorem 2.1.1 for double stars.

Question 2.5.5 (Gyárfás and Sárközy - 2008 [53]). *Let $r \geq 3$. For every r-colouring of K_n, is it true that there exists a monochromatic double star on at least $\frac{n}{r-1}$ vertices?*

They managed to prove the following weaker result.

Theorem 2.5.6 (Gyárfás and Sárközy - 2008 [53]). *For $r \geq 2$ and every r-colouring of K_n, there exists a monochromatic double star on at least $\frac{n(r+1)+r-1}{r^2}$ vertices.*

For the case $r = 2$, Theorem 2.5.6 gives the existence of a monochromatic double star on at least $\frac{3n+1}{4}$ vertices in any 2-coloured K_n. By considering random graphs or Paley graphs, one can obtain a 2-colouring of K_n where the monochromatic double star of maximum order has $\frac{3n}{4} + O(1)$ vertices. The random graphs construction was in fact shown implicitly by Erdős, Faudree, Gyárfás and Schelp [25] (1989). Hence in Question 2.5.5, the constraint $r \geq 3$ is necessary.

2.6 Gallai colourings and extensions

In this subsection, we shall consider the task of finding monochromatic subgraphs in edge-coloured complete graphs by putting a restriction on the edge-colouring. In [61], Gyárfás and Simonyi defined a *Gallai colouring* to be an edge-colouring of a graph where no triangle is coloured with three distinct colours. This model of colouring dates back to Gallai's paper [42] (1967), where he studied transitively orientable graphs, and the paper was subsequently translated into English by Maffray and Preissmann [81]. Gallai colourings have since been studied (directly or indirectly) by many authors. Notably, Cameron, Edmonds and Lovász [17, 18, 79] encountered these colourings, when they extended the perfect graph theorem. Also, Körner, Simonyi and Tuza [70, 71] called such an edge-colouring a *Gallai partition*, and they found the colourings to be relevant in an information theoretic function called the *graph entropy*. Finally, Gyárfás et al. [58] introduced the *Gallai-Ramsey number*, which is the analogue of the classical Ramsey number, but restricted to Gallai colourings.

Gallai colourings generalise 2-colourings, and these two types of edge-colourings are very closely related. Indeed, a flagship result is the following decomposition theorem, which Gallai [42], and Cameron and Edmonds [17] used implicitly, and was properly restated by Gyárfás and Simonyi [61].

Theorem 2.6.1 (Gallai - 1967 [42]). *Any Gallai colouring on a complete graph can be obtained by substituting complete graphs with Gallai colourings for the vertices of a 2-coloured complete graph on at least two vertices.*

That is, given a complete graph K with a Gallai colouring, the following is true. There exists a partition of the vertices of K into sets $V_1, \dots, V_p$ (for some $p \geq 2$) such that, the edges within every V_i form a Gallai colouring, and for every $1 \leq i \neq j \leq p$, all edges of (V_i, V_j) have the same colour and can be one of only two possible colours. Such a decomposition is called a *Gallai decomposition* of K, and the Gallai coloured complete graphs on $V_1, \dots, V_p$ are the *blocks* of the Gallai decomposition. Also, the 2-coloured complete graph on p vertices, say $v_1, \dots, v_p$, with $v_i v_j$ given the colour of (V_i, V_j) for all $1 \leq i \neq j \leq p$, is the *base graph* of the decomposition.

The following result also appeared in [42]. Gyárfás and Simonyi stated the result explicitly in [61], and noted that Theorem 2.6.1 follows from it.

Theorem 2.6.2 (Gallai - 1967 [42])**.** *Every Gallai colouring with at least three colours on a complete graph K_n has a colour which induces a disconnected subgraph on n vertices.*

We see that Theorem 2.6.1 illustrates the close connection between Gallai colourings and 2-colourings. The theorem is an important tool which can be used to show that, certain results which hold for 2-colourings also hold for Gallai colourings. For instance, recall the observation of Erdős and Rado: *Every 2-coloured complete graph has a monochromatic spanning tree.* Now, given a Gallai colouring on a complete graph K, we may apply Theorem 2.6.1 to obtain a Gallai decomposition for K, and then the observation on the base graph, to obtain a monochromatic spanning tree for K. Thus, Erdős and Rado's observation extends to: *Every Gallai coloured complete graph has a monochromatic spanning tree.*

It turns out that some of the other results that we have already seen can also be extended, including Theorems 2.5.1, 2.5.3, 2.3.2(a), 2.3.3, and Theorem 2.5.6 for $r = 2$.

Theorem 2.6.3 (Gyárfás, Simonyi - 2004 [61])**.** *For every Gallai colouring of K_n, there exists a monchromatic spanning tree with height at most* 2*.*

Theorem 2.6.4 (Gyárfás, Simonyi - 2004 [61])**.** *For every Gallai colouring of K_n, there exists a monchromatic spanning broom.*

Theorem 2.6.5 (Gyárfás, Sárközy, Sebő, Selkow - 2009 [58])**.** *For every Gallai colouring of K_n, there exists a monochromatic subgraph with diameter at most* 2 *on at least $\left\lceil \frac{3n}{4} \right\rceil$ vertices. This is best possible for every n.*

Theorem 2.6.6 (Gyárfás, Sárközy, Sebő, Selkow - 2009 [58])**.** *For every Gallai colouring of K_n, there exists a monochromatic spanning subgraph with diameter at most* 3*.*

Theorem 2.6.7 (Gyárfás, Sárközy, Sebő, Selkow - 2009 [58])**.** *For every Gallai colouring of K_n, there exists a monochromatic double star with at least $\frac{3n+1}{4}$ vertices. This is asymptotically best possible.*

On the other hand, an example where such an extension does not apply is when we want to find a large monochromatic star in an edge-coloured complete graph. Given a 2-colouring of K_n, there is a monochromatic star on at least $\left\lceil \frac{n-1}{2} \right\rceil + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices, by considering the larger monochromatic star at any vertex. This bound is essentially best possible, since in the 2-colouring of K_n consisting of two red cliques of orders $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lceil \frac{n}{2} \right\rceil$, with the remaining edges blue, the monochromatic star with maximum order has $\left\lceil \frac{n}{2} \right\rceil + 1$ vertices. With a little more effort, it is not hard to show that we can always find a monochromatic star on $\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices if $n \not\equiv 3 \pmod 4$, and $\left\lceil \frac{n}{2} \right\rceil + 1$ vertices if $n \equiv 3 \pmod 4$, with each value the best possible. However, we have the following result for Gallai colourings.

Theorem 2.6.8 (Gyárfás, Simonyi - 2004 [61]). *For every Gallai colouring of K_n, there exists a monochromatic star with at least $\frac{2n}{5}$ vertices. This bound is sharp.*

The sharpness in Theorem 2.6.8 can be seen from the following Gallai colouring of K_n. Partition the vertices of K_n into five near-equal sets $V_1, \ldots, V_5$. Colour all edges of (V_i, V_{i+1}) with colour 1, those of (V_i, V_{i+2}) with colour 2, and those inside the classes V_i with colour 3 (for all $1 \leq i \leq 5$, with indices taken modulo 5). Then, the monochromatic star of maximum order has $\lceil \frac{2n}{5} \rceil + 1$ vertices.

To generalise the concept of Gallai colourings, we may replace the role of the forbidden 3-coloured triangle. An edge-colouring of a graph F is *rainbow* if the colours of the edges of F are distinct. Then, we may consider edge-colourings of complete graphs where a rainbow coloured copy of a fixed graph F is forbidden. We shall call such an edge-colouring *rainbow F-free*.

In this direction, Fujita and Magnant considered graphs F that are close to a triangle. For $s, t \geq 0$, let $H_{s,t}$ be the graph obtained by taking a triangle, s single edges and t copies of P_2 (the path of length 2), and identifying one vertex of the triangle, say v, with one end-vertex of each single edge and each P_2. The vertex v is the *centre* of $H_{s,t}$. Note that the s single edges are pendant edges of $H_{s,t}$ (recall that a *pendant edge* of a graph F is an edge that has an end-vertex with degree 1 in F).

Fujita and Magnant considered analogues of Theorem 2.6.1 with rainbow $H_{s,0}$-free colourings. For $s = 1$, they proved the following result, which shows that in a rainbow $H_{1,0}$-free colouring, one cannot hope to have a decomposition as strong as a Gallai decomposition.

Theorem 2.6.9 (Fujita, Magnant - 2012 [37]). *For every rainbow $H_{1,0}$-free colouring of a complete graph K, one of the following holds.*

(i) $V(K)$ can be partitioned such that there are at most two colours on the edges between the parts.

(ii) There are three (different coloured) monochromatic spanning trees of K, and moreover, there exists a partition of $V(K)$ with exactly three colours on edges between parts and between each pair of parts, the edges have only one colour.

For $s \geq 2$, they also proved the following decomposition type result for $H_{s,0}$-free colourings. The decomposition is slightly weaker than the decomposition when $s = 1$ but it is still the best possible.

Theorem 2.6.10 (Fujita, Magnant - 2012 [37]). *For $s \geq 2$, in any rainbow $H_{s,0}$-free colouring of a complete graph K, there exists a partition of $V(K)$ such that between the parts, there are at most $s + 2$ colours. Furthermore, there exists an edge-colouring of a complete graph such that for every partition of the vertices, there are $s + 2$ colours between the parts.*

Next, Fujita et al. considered extending Theorem 2.6.2. A graph F is said to have the *disconnection property* if there exists an integer $r_0 = r_0(F)$ such that the following is true: For every complete graph K_n whose edges are coloured with at least r_0 colours and without a rainbow copy of F, there exists a colour which spans a disconnected subgraph on n vertices. Notice that $r_0(F) \geq |E(F)|$, since for every sufficiently large complete graph K and every $r \leq |E(F)| - 1$, there exists an r-colouring of K where every colour spans a connected subgraph, and such an

r-colouring does not contain a rainbow copy of F. If it is possible to take $r_0(F) = |E(F)|$, then F is said to have the *Gallai property.*

Let DP and GP denote, respectively, the family of graphs that have the disconnection property and the Gallai property. Note that we have the following.

- $GP \subset DP$.
- If F' is a subgraph of F and $F \in DP$, then $F' \in DP$.

Combining Theorems 2.6.2, 2.6.9 and 2.6.10, we have $H_{s,0} \in GP$ for all $s \geq 0$. Fujita et al. proved several results, which we summarise as follows.

Theorem 2.6.11 (Fujita, Gyárfás, Magnant, Seress - 2013 [34])**.**

(a) Let P_ℓ denote the path of length ℓ. Then $P_2, P_3, P_4, P_5 \in GP$.

(b) Let C_ℓ denote the cycle of length ℓ. Then $C_{2h} \notin GP$ for every $h \geq 1$.

(c) If $F \in DP$ is connected and bipartite, then F is either a tree, or a unicyclic graph, or two such components joined by an edge.

(d) For any $F \in DP$, there exists an edge $e \in E(F)$ such that $F - e$ is bipartite.

(e) If $F \in DP$ is connected, then F can be obtained from a tree by adding at most two edges.

(f) If F is a unicyclic graph such that its cycle is a triangle, then $F \in DP$. Hence, any forest belongs to DP.

(g) $H_{s,1} \in GP$ for all $s \geq 0$.

(h) $H^+_{s,0} \in GP$ for all $s \geq 0$, where $H^+_{s,0}$ is obtained from $H_{s,0}$ by adding a pendant edge to $H_{s,0}$ at a vertex of the triangle, different from the centre. $H^{2+}_{1,0} \in GP$, where $H^{2+}_{1,0}$ is a triangle with three pendant edges added, one at each vertex of the triangle.

To see part (b), Fujita et al. presented the following construction.

Construction 2.6.12. *Let A, B be disjoint sets with $|A| = |B| = 2(r-1)q + 1$. Let $A = \bigcup_{i=1}^{r-1} A_i \cup \{a\}$ and $B = \bigcup_{i=1}^{r-1} B_i \cup \{b\}$, where the sets are all disjoint and $|A_i| = |B_i| = 2q$. The edge ab and the edges within A and B are given colour r. For $i = 1, 2, \ldots, r-1$, the edges between a and B_i; b and A_i; and A_i and B_i, are given colour i. Split each A_i, B_i into two disjoint equal parts, $A_i = X_i \cup Y_i$, $B_i = U_i \cup W_i$ (with q vertices in each). For any $i \neq j \in \{1, 2, \ldots, r-1\}$, the edges between X_i and U_j; and Y_i and W_j, are given colour i; the edges between X_i and W_j; and Y_i and U_j are given colour j. This colours all edges of the complete graph on $A \cup B$.*

By taking $r = 2h$, Fujita et al. showed that the r-colouring in Construction 2.6.12 is rainbow C_{2h}-free, and every colour induces a connected subgraph.

In light of their findings, Fujita et al. also stated the following problems. The first is in response to Theorem 2.6.11(a), the second in response to Theorem 2.6.11(a) and Construction 2.6.12, and the fifth in response to Theorem 2.6.11(g).

Problem 2.6.13 (Fujita, Gyárfás, Magnant, Seress - 2013 [34])**.**

(a) Are all paths in GP?

(b) Is $C_4 \in DP$?

(c) Are odd cycles in GP? Or in DP?

(d) Do we have $GP = DP$?

(e) Is $H_{s,t} \in GP$ for all $s, t \geq 0$?

Finally, Fujita and Magnant proved the following result about the existence of monochromatic k-connected subgraphs in Gallai colourings.

Theorem 2.6.14 (Fujita, Magnant - 2013 [38])**.** *Let $r \geq 3$ and $k \geq 2$. If $n \geq (r+11)(k-1) + 7k \log k$, then for every Gallai colouring of K_n using r colours, there exists a monochromatic k-connected subgraph on at least $n - r(k-1)$ vertices.*

Theorem 2.6.14 can be seen as an extension of Theorem 2.1.7. It also illustrates the striking difference that the Gallai colouring condition imposes. If we consider r and k to be fixed, then Theorem 2.6.14 gives, in a Gallai coloured K_n, the existence of a monochromatic k-connected subgraph on $n - O(1)$ vertices, i.e. nearly all vertices of K_n. But Theorem 2.1.8(b) (hence Construction 2.1.9) implies that we cannot have more than $\frac{n}{r'-1} + O(1)$ vertices for such a subgraph in an r-coloured K_n, where $r' \leq r$ is the largest integer such that $r' - 1$ is a prime power.

We end this subsection with the following two results, where rainbow coloured paths are avoided in edge-coloured complete graphs.

Theorem 2.6.15 (Thomason, Wagner - 2007 [95])**.** *Let $r \geq 4$ and $k \geq 1$. If $n \geq 2k$, then every rainbow P_4-free r-colouring of K_n contains a monochromatic k-connected subgraph on at least $n - k + 1$ vertices (where P_4 denotes the path of length 4).*

Theorem 2.6.16 (Fujita, Magnant - 2013 [38])**.** *Let $r \geq \max\{\frac{k}{2} + 8, 15\}$ and $k \geq 1$. If $n \geq (r+11)(k-1) + 7k \log k + 2r + 3$, then every rainbow P_5-free r-colouring of K_n contains a monochromatic k-connected subgraph on at least $n - 7k + 2$ vertices (where P_5 denotes the path of length 5).*

2.7 Host graphs with given independence number

We give a brief review of some results of Gyárfás and Sárközy, concerning the existence of monochromatic connected subgraphs in edge-coloured graphs with given independence number.

Theorem 2.7.1 (Gyárfás, Sárközy - 2010 [54])**.** *Let G be a graph on n vertices and with independence number $\alpha(G) = \alpha$. Then, for every 2-colouring of G, there exists a monochromatic connected subgraph on at least $\lceil \frac{n}{\alpha} \rceil$ vertices. This result is sharp.*

The sharpness in Theorem 2.7.1 can be easily seen by taking G to consist of α cliques of near-equal orders, i.e. each clique has $\lfloor \frac{n}{\alpha} \rfloor$ or $\lceil \frac{n}{\alpha} \rceil$ vertices. Gyárfás and Sárközy remarked that Theorem 2.7.1 can be extended to r-colourings, with $\alpha(r-1)$ in the role of α.

They also considered using Gallai colourings, and obtained the following result.

Theorem 2.7.2 (Gyárfás, Sárközy - 2010 [54]). *Let G be a graph on n vertices and with independence number $\alpha(G) = \alpha$. Then, for every Gallai colouring of G, there exists a monochromatic connected subgraph on at least $\frac{n}{\alpha^2+\alpha-1}$ vertices.*

They noted that the bound of $\frac{n}{\alpha^2+\alpha-1}$ in Theorem 2.7.2 is not far from the truth, and provided Construction 2.7.3 below which shows that we cannot have more than $\frac{(c\log\alpha)n}{\alpha^2}$ for the maximum order of a monochromatic connected subgraph (for some constant c). Hence, the bound of $\lceil \frac{n}{\alpha} \rceil$ in Theorem 2.7.1 does not extend to Gallai colourings.

Construction 2.7.3. *Consider a triangle-free graph G' with $\alpha(G') = \alpha$ and with the maximum number of vertices. That is, G' has $p = R(3, \alpha+1) - 1$ vertices, where $R(3, \alpha+1)$ is the Ramsey number of a triangle versus a $K_{\alpha+1}$ clique. A famous result of Kim [68] (1995) implies that p is almost quadratic, its order of magnitude is $\frac{\alpha^2}{\log\alpha}$. We give G' an edge-colouring where all edges have distinct colours. Now, we define an edge-coloured graph G on n vertices, by substituting Gallai coloured cliques for the vertices of G', with the vertex sets of the cliques forming a near-equal partition of $V(G)$.*

Then, we have a Gallai colouring of G, and $\alpha(G) = \alpha$. Moreover, the monochromatic connected subgraph of G with maximum order has at most $2\lceil \frac{n}{p} \rceil = \frac{(c\log\alpha)n}{\alpha^2}$ vertices, where c is a constant coming from Kim's estimate of $R(3, \alpha+1)$.

Gyárfás and Sárközy thus posed the following problem.

Problem 2.7.4 (Gyárfás, Sárközy - 2010 [54]). *Determine the function $f(\alpha)$, the largest value such that for every Gallai coloured graph G on n vertices with independence number $\alpha(G) = \alpha$, there exists a monochromatic connected subgraph on at least $f(\alpha)n$ vertices.*

From Theorem 2.7.2 and Construction 2.7.3, we have

$$\frac{1}{\alpha^2+\alpha-1} \le f(\alpha) \le \frac{c\log\alpha}{\alpha^2}.$$

For $\alpha = 2$, Gyárfás and Sárközy gave the following construction.

Construction 2.7.5. *Consider the graph H_8 on eight vertices which is the complement of the Wagner graph, i.e. $V(H_8) = \{v_1, \dots, v_8\}$ and $E(H_8) = \{v_iv_{i\pm2}, v_iv_{i\pm3} : 1 \le i \le 8\}$, where indices are taken modulo 8. Define the edge-colouring on H_8 where for $1 \le i \le 8$, the edges v_iv_{i+2} and v_iv_{i-3} have colour i. As before, define an edge-coloured graph G on n vertices, by substituting Gallai coloured cliques for the vertices of H_8, with the vertex sets of the cliques forming a near-equal partition of $V(G)$.*

Then, we have a Gallai colouring of G, and $\alpha(G) = 2$. Moreover, the monochromatic connected subgraph of G with maximum order has at most $3\lceil \frac{n}{8} \rceil$ vertices. Hence, we have the following.

Lemma 2.7.6 (Gyárfás, Sárközy - 2010 [54]). $\frac{1}{5} \le f(2) \le \frac{3}{8}$.

Finally, the following result about the existence of a monochromatic double star in a Gallai coloured graph was also proved.

Theorem 2.7.7 (Gyárfás, Sárközy - 2010 [54]). *Let G be a graph on n vertices and with independence number $\alpha(G) = \alpha$. Then, for every Gallai colouring of G, there exists a monochromatic double star on at least $\frac{n}{\alpha^2+\alpha-2/3}$ vertices.*

3 Partitioning and Covering by Monochromatic Structures

We consider problems of the following form: *Let $\mathcal{F}$ be some family of graphs. For example, $\mathcal{F}$ may be the family of all cycles, or paths, or trees, etc. Whenever we have an r-colouring of the edges of a graph G, how many monochromatic subgraphs do we need so that they form a partition or a covering of the vertices of G, with each subgraph belonging to $\mathcal{F}$?*

Throughout this section, the empty graph (with no vertices), a single vertex and a single edge will also be considered as cycles, of lengths 0, 1 and 2 respectively – they may be considered to be "degenerate cycles". In the case when $\mathcal{F}$ is the family of all cycles, we define the *cycle partition number* (resp. *cycle covering number*) of an r-coloured graph G to be the minimum integer m such that, for any r-colouring of the edges of G, the vertices of G can be partitioned into (resp. covered by) at most m monochromatic cycles. We can make similar definitions when $\mathcal{F}$ is the family of all paths, or the family of all trees, and obtain the analogous terms *path partition number*, *path covering number*, *tree partition number*, and *tree covering number* of an r-coloured graph G.

We shall survey many results and open problems, including those that concern these partition and covering numbers, as well as when $\mathcal{F}$ is some other family of graphs. Most of the research done have been towards the case when G is a complete graph.

3.1 Partitioning and covering by cycles

In this subsection, we shall mainly consider the problems of partitioning and covering the vertices of an r-coloured complete graph by monochromatic cycles (recall that we include the degenerate cycles of lengths 0, 1 and 2). We will also consider analogous problems with some other r-coloured host graphs.

We begin by considering the case for 2-coloured complete graphs. A long-standing conjecture of Lehel (1979), which was first published by Ayel [6], is the following: *For every 2-colouring of K_n with red and blue, there exists a partition of the vertices into a red cycle and a blue cycle.* Ayel also proved the conjecture for some special types of 2-colourings of K_n.

In support of a positive solution, Gyárfás noted two results in his survey [45] that may be considered as partial results. Furthermore, algorithms for producing these subgraphs were provided.

Theorem 3.1.1 (Gerencsér, Gyárfás - 1967 [43, 45]). *Every 2-colouring of K_n with red and blue contains a Hamilton cycle that is either monochromatic, or is the union of a red path and a blue path.*

Theorem 3.1.2 (Gyárfás - 1983 [45]). *For every 2-colouring of K_n with red and blue, the vertices can be covered by one red cycle and one blue cycle such that, the two cycles have at most one vertex in common.*

Many partial results to Lehel's conjecture were then proved. Łuczak, Rödl and Szemerédi [80] (1998) proved that the conjecture holds for all sufficiently large n, by making use of Szemerédi's regularity lemma. The bound on n was later improved to $n \geq 2^{18000}$ by Allen [3] (2008), without using the regularity lemma. The conjecture was finally settled by Bessy and Thomassé, who used an ingenious argument which involved Theorems 3.1.1 and 3.1.2.

Theorem 3.1.3 (Bessy, Thomassé - 2010 [11]). *For every 2-colouring of K_n with red and blue, there exists a partition of the vertices into a red cycle and a blue cycle.*

Recently, Schelp [92] suggested the strengthening of certain Ramsey type problems from complete graphs to graphs of given minimum degree. Inspired by the problems in [92], Balogh et al. made the following conjecture, which is a version of Lehel's conjecture and Theorem 3.1.3 with a minimum degree condition.

Conjecture 3.1.4 (Balogh, Barát, Gerbner, Gyárfás, Sárközy - 2014 [7]). *Let G be a graph on n vertices with $\delta(G) > \frac{3n}{4}$. Then, for any 2-colouring of G with red and blue, there exists a partition of the vertex set $V(G)$ into a red cycle and a blue cycle.*

They provided the following example, which shows that the condition $\delta(G) \geq \frac{3n}{4}$ is sharp.

Construction 3.1.5. *Consider the 2-colouring of the cycle C_4 where the colours of the edges alternate in red and blue. Now, let G be a 2-coloured graph on n vertices, obtained by substituting arbitrarily 2-coloured cliques (with red and blue) for the vertices of the C_4, with the vertex sets of the cliques forming a near-equal partition of $V(G)$.*

Then in such a 2-coloured graph G, there is no partition of the vertex set $V(G)$ into a red cycle and a blue cycle, while the minimum degree is $\lfloor \frac{3n}{4} \rfloor - 1$.

Going one step further, Barát and Sárközy made the following stronger conjecture, which replaces the minimum degree condition by an Ore type condition (i.e. a condition with a lower bound on the degree sum of any two non-adjacent vertices in a graph).

Conjecture 3.1.6 (Barát, Sárközy - 2014 [8]). *Let G be a graph on n vertices such that for any two non-adjacent vertices x and y, we have $\deg(x) + \deg(y) > \frac{3n}{2}$. Then, for any 2-colouring of G with red and blue, there exists a partition of the vertex set $V(G)$ into a red cycle and a blue cycle.*

Again, Construction 3.1.5 shows that the Ore type condition in Conjecture 3.1.6 is sharp, since in the construction, the value of $\deg(x) + \deg(y)$ is $\lfloor \frac{3n}{2} \rfloor - 2$ or $\lceil \frac{3n}{2} \rceil - 2$, for any two non-adjacent vertices x and y. Barát and Sárközy also managed to prove an asymptotic version of the conjecture.

Theorem 3.1.7 (Barát, Sárközy - 2014 [8]). *For all $\eta > 0$, there exists $n_0 = n_0(\eta)$ such that the following holds. Let G be a graph on $n \geq n_0$ vertices such that for any two non-adjacent vertices x and y, we have $\deg(x) + \deg(y) \geq (\frac{3}{2} + \eta)n$. Then, for every 2-colouring of G with red and blue, there exist a red cycle and a blue cycle which are vertex-disjoint, and together they cover at least $(1 - \eta)n$ vertices of G.*

Theorem 3.1.7 improved an earlier result of Balogh et al. [7], which had the minimum degree condition $\delta(G) > (\frac{3}{4} + \eta)n$ in place of the Ore type condition.

Next, we consider the analogous situation when $r \geq 2$ colours are used to colour K_n. Erdős et al. proved the following result.

Theorem 3.1.8 (Erdős, Gyárfás, Pyber - 1991 [27]). *Let $r \geq 2$. For every r-colouring of K_n, there exists a partition of the vertices into at most $cr^2 \log r$ monochromatic cycles (for some constant c).*

A notable fact about Theorem 3.1.8 is that the number of monochromatic cycles depends only on r, and not on n. Thus, we may simply call the minimum number of monochromatic cycles the *cycle partition number*, and denote it by $p(r)$. The function $p(r)$ is the same for every complete graph K_n, irrespective of the value of n. We have $p(r) \leq cr^2 \log r$.

Erdős et al. [27] conjectured that $p(r) = r$. We remark that in this conjecture, the monochromatic cycles do not necessarily have all r distinct colours, and hence strictly speaking, it is not quite a generalisation of Lehel's conjecture (which asked for a partition of a 2-coloured complete graph into two cycles with *distinct* colours). Erdős et al. noted that their conjecture is best possible, as the following example shows.

Construction 3.1.9. *Partition the vertices of a large complete graph into $A_1, \ldots, A_r$, where the sequence $|A_i|$ grows fast enough. For $u \in A_i$, $v \in A_j$ where $1 \leq i \leq j \leq r$, colour the edge uv with colour i.*

Then in this example, the vertices of the complete graph cannot even be covered by fewer than r monochromatic paths.

By using Szemerédi's regularity lemma, Gyárfás et al. improved the bound in Theorem 3.1.8, for all sufficiently large complete graphs.

Theorem 3.1.10 (Gyárfás, Ruszinkó, Sárközy, Szemerédi - 2006 [51])**.** *For $r \geq 2$, there exists $n_0 = n_0(r)$ such that, for all $n \geq n_0$ and every r-colouring of K_n, the vertices can be partitioned into at most $100r \log r$ monochromatic cycles.*

Later, the same authors, again using the regularity lemma, proved the following partial results for 3-colourings.

Theorem 3.1.11 (Gyárfás, Ruszinkó, Sárközy, Szemerédi - 2011 [52])**.** *In every 3-colouring of K_n, all but $o(n)$ of the vertices can be partitioned into three monochromatic cycles.*

Theorem 3.1.12 (Gyárfás, Ruszinkó, Sárközy, Szemerédi - 2011 [52])**.** *There exists n_0 such that, for all $n \geq n_0$ and every 3-colouring of K_n, the vertices can be partitioned into at most 17 monochromatic cycles.*

Gyárfás et al. noted that Theorem 3.1.11 fails if we insist that the monochromatic cycles must have distinct colours, by considering the r-colouring ψ on K_n with $r = 3$ in Construction 2.1.2, but with all edges within the four classes having the same colour. In this 3-colouring, at most $\frac{3}{4}$ of the vertices can be covered by three vertex-disjoint cycles having different colours. They also proved the following result, which shows that this example is essentially best possible.

Theorem 3.1.13 (Gyárfás, Ruszinkó, Sárközy, Szemerédi - 2011 [52])**.** *In every 3-colouring of K_n, at least $(\frac{3}{4} - o(1))n$ vertices can be partitioned into three monochromatic cycles having distinct colours.*

A breakthrough for Erdős et al.'s conjecture was finally made by Pokrovskiy, who surprisingly managed to disprove it for all $r \geq 3$.

Theorem 3.1.14 (Pokrovskiy - 2014 [84])**.** *For $r \geq 3$, there exist infinitely many r-coloured complete graphs whose vertices cannot be partitioned into r monochromatic cycles.*

Pokrovskiy's construction for Theorem 3.1.14 is inductive in r, and is as follows.

Construction 3.1.15. *Let $m \geq 1$ be an integer. First, partition the vertex set of K_{43m} into four classes A_1, A_2, A_3, A_4, with $|A_1| = 10m$, $|A_2| = |A_4| = 13m$, and $|A_3| = 7m$. Colour the edges of (A_1, A_2) and (A_3, A_4) with colour 1; those of (A_1, A_3), (A_2, A_4), and within A_3 and A_4 with colour 2; and those of (A_1, A_4), (A_2, A_3), and within A_1 and A_2 with colour 3. Denote this 3-coloured complete graph by H_3^m.*

Now, let $r \geq 4$. We obtain the r-coloured complete graph H_r^m from the $(r-1)$-coloured complete graph H_{r-1}^{5m}, by taking a copy of H_{r-1}^{5m} and a further $2m$ vertices, and giving all the new edges colour r. Note that H_r^m is an r-coloured complete graph on $|V(H_{r-1}^{5m})| + 2m$ vertices. For every fixed $r \geq 3$, we have now defined a sequence of r-coloured complete grpahs H_r^m.

Finally, we obtain the r-coloured complete graph J_r^m from H_r^m, by taking a copy of H_r^m and a further r vertices $v_1, \ldots, v_r$. Then, all edges from v_i to H_r^m $(1 \leq i \leq r)$ are given colour i, the edge v_1v_2 is given colour 3, all edges v_1v_j $(j \geq 3)$ are given colour 2, and all remaining edges v_iv_j $(2 \leq i < j)$ are given colour 1. Note that J_r^m is an r-coloured complete graph on $|V(H_r^m)| + r$ vertices.

For fixed $r \geq 3$, Pokrovskiy proved that every r-coloured complete graph J_r^m satisfies Theorem 3.1.14. He noted that in all of his examples of r-colourings of K_n, it is possible to cover $n-1$ of the vertices of K_n with r vertex-disjoint monochromatic cycles. He also remarked that Construction 3.1.15 can be generalised to work for all $n \geq n_0$, where $n_0 = n_0(r)$, by essentially replacing "m is an integer" with "m is a real number". Thus, this strengthens Theorem 3.1.14 to that it holds for all sufficiently large n.

Pokrovskiy then offered the following weaker versions of Erdős et al.'s original conjecture. The first is a relaxation that the vertex-disjoint monochromatic cycles cover all but a constant number of vertices of K_n. The second removes the constraint that the monochromatic cycles covering K_n are vertex-disjoint.

Conjecture 3.1.16 (Pokrovskiy - 2014 [84])**.** *For each $r \geq 3$, there exists a constant $c = c(r)$ such that in every r-colouring of K_n, there are r vertex-disjoint monochromatic cycles covering $n - c$ vertices of K_n.*

Conjecture 3.1.17 (Pokrovskiy - 2014 [84])**.** *For $r \geq 3$ and every r-colouring of K_n, there are r monochromatic cycles (not necessarily disjoint) covering all vertices of K_n. That is, the cycle covering number of an r-coloured K_n is r.*

Again, if Conjecture 3.1.17 is true, then it is optimal, in view of Construction 3.1.9.

Next, we consider the analogous problem when the r-coloured host graph is a balanced complete bipartite graph $K_{n,n}$. Observing from Theorem 3.1.8 that the cycle partition number of K_n depends only on r, Erdős et al. asked if the same is true for $K_{n,n}$.

Question 3.1.18 (Erdős, Gyárfás, Pyber - 1991 [27])**.** *Does the cycle partition number of an r-coloured balanced complete bipartite graph $K_{n,n}$ depend only on r (and not on n)?*

Erdős et al. also remarked that, by using a lemma in [27], it can be shown that the vertices of any r-coloured $K_{n,n}$ can be covered by at most cr^2 (not necessarily disjoint) monochromatic cycles (for some constant c).

By considering the concept of "uniformity" in bipartite graphs, Haxell gave a positive answer to Question 3.1.18 for large r.

Theorem 3.1.19 (Haxell - 1997 [64]). *The cycle partition number of an r-coloured balanced complete bipartite graph $K_{n,n}$ is $O((r \log r)^2)$, for r sufficiently large.*

We end this subsection with a generalisation due to Sárközy, where the host graph G has a given independence number α. Similar to Theorem 3.1.8, Sárközy managed to prove an upper bound for the cycle partition number of such a graph G whose edges are r-coloured, with the upper bound depending only on r and α.

Theorem 3.1.20 (Sárközy - 2011 [87]). *Let G be a graph with independence number $\alpha(G) = \alpha$. Then, for every r-colouring of G, the vertices of G can be partitioned into at most $25(\alpha r)^2 \log(\alpha r)$ monochromatic cycles.*

Hence, we can define $p(r, \alpha)$ to be the minimum number of monochromatic cycles needed to partition the vertex set of any r-coloured graph G with $\alpha(G) = \alpha$. The function $p(r, \alpha)$ is a generalisation of the cycle partition number $p(r)$. We have $p(r, \alpha) \leq 25(\alpha r)^2 \log(\alpha r)$. Sárközy also made the conjecture in [87] that $p(r, \alpha) = \alpha r$ for every $r, \alpha \geq 1$, which generalises the conjecture of Erdős et al. that $p(r) = r$. Sárközy's conjecture, if true, is easily seen to be optimal, by considering a graph consisting of α sufficiently large cliques, with each given an r-colouring as described in Construction 3.1.9. We see that the conjecture is true for $r = 2$ and $\alpha = 1$ (Theorem 3.1.3 of Bessy and Thomassé), but false for $r \geq 3$ and $\alpha \geq 1$, since we can take a graph with α cliques, where each one is given the r-colouring of Construction 3.1.15 provided by Pokrovskiy. The conjecture is also known to be true for $r = 1$ and every $\alpha \geq 1$, which is the following result of Pósa.

Theorem 3.1.21 (Pósa - 1963 [85]). *The vertices of a graph G can be partitioned into at most $\alpha(G)$ cycles.*

Hence, the remaining open case is $r = 2$ and $\alpha \geq 2$, which we state below.

Conjecture 3.1.22 (Sárközy - 2011 [87]). *For every $\alpha \geq 2$, we have $p(2, \alpha) = 2\alpha$.*

For Sárközy's original conjecture, Balogh et al. suggested that the following weaker version may be true. It is a stronger version of Conjecture 3.1.16 by Pokrovskiy.

Conjecture 3.1.23 (Balogh, Barát, Gerbner, Gyárfás, Sárközy - 2014 [7]). *Let $r, \alpha \geq 1$. Then there exists a constant $c = c(r, \alpha)$ such that, for every r-colouring of a graph G on n vertices with independence number $\alpha(G) = \alpha$, there are αr vertex-disjoint monochromatic cycles covering at least $n - c$ vertices of G.*

Pokrovskiy's example implies that $c \geq \alpha$ must be true. Balogh et al. managed to prove the following asymptotic version for $r = 2$ and every $\alpha \geq 1$.

Theorem 3.1.24 (Balogh, Barát, Gerbner, Gyárfás, Sárközy - 2014 [7]). *For all $\eta > 0$ and integers $\alpha \geq 1$, there exists $n_0 = n_0(\eta, \alpha)$ such that the following holds. For every 2-colouring of a graph G on $n \geq n_0$ vertices with independence number $\alpha(G) = \alpha$, there are at most 2α vertex-disjoint monochromatic cycles covering at least $(1 - \eta)n$ vertices of G.*

3.2 Partitioning and covering by paths or trees

We now consider the analogous problems of partitioning and covering r-coloured graphs by monochromatic paths or trees. By Theorem 3.1.8, the path partition, path covering, tree partition, and tree covering numbers of an r-coloured K_n all depend only on r, and we may simply call these functions the *path partition number*, the *path covering number*, the *tree partition number*, and the *tree covering number*. By Construction 3.1.9, the path partition and covering numbers are both at least r.

Gyárfás made the following conjectures about the path partition and covering numbers, which are slightly weaker versions of Erdős et al.'s conjecture about the cycle partition number. The second conjecture is a weaker version of Conjecture 3.1.17 by Pokrovskiy.

Conjecture 3.2.1 (Gyárfás - 1989 [46]). *The vertices of every r-coloured complete graph K_n can be partitioned into r monochromatic paths. That is, the path partition number is r.*

Conjecture 3.2.2 (Gyárfás - 1989 [46]). *The vertices of every r-coloured complete graph K_n can be covered by r monochromatic paths. That is, the path covering number is r.*

He had also proved in [46] that the path covering number is a function of r (and not n), and conjectured that the same is true for the path partition number. This is true by Erdős et al.'s result (Theorem 3.1.8).

Rado had already proved a countably infinite version of Conjecture 3.2.1.

Theorem 3.2.3 (Rado - 1978 [86]). *The vertices of every r-coloured countably infinite complete graph can be partitioned into r monochromatic finite or one-way infinite paths.*

The case $r = 2$ of Conjecture 3.2.1 (and Conjecture 3.2.2) is true, and is a result of Gerencsér and Gyárfás (Theorem 3.1.1). More recently, Pokrovskiy settled the case $r = 3$.

Theorem 3.2.4 (Pokrovskiy - 2014 [84]). *For every 3-colouring of K_n, the vertices can be partitioned into three monochromatic paths.*

In Pokrovskiy's proof of Theorem 3.2.4, he splits into two cases: whether or not the 3-colouring of K_n is 4-partite. A 3-colouring of K_n is *4-partite* if there is a partition of the vertices of K_n into A_1, A_2, A_3, A_4 such that, all edges of (A_1, A_4) and (A_2, A_3) have colour 1; those of (A_2, A_4) and (A_1, A_3) have colour 2; and those of (A_3, A_4) and (A_1, A_2) have colour 3. The remaining edges (within the classes) are arbitrarily coloured. Note that this 3-colouring has a similar structure to the edge-colouring ψ of K_n that was described in Construction 2.1.2, in the case $r = 3$. The following result was proved, which strengthens Theorem 3.2.4.

Theorem 3.2.5 (Pokrovskiy - 2014 [84]). *Suppose that we have a 3-colouring of K_n.*

(a) If the 3-colouring is not 4-partite, then the vertices of K_n can be partitioned into three monochromatic paths with different colours.

(b) If the 3-colouring is 4-partite, then the vertices of K_n can be partitioned into three monochromatic paths, at most two of which have the same colour.

Conjectures 3.2.1 and 3.2.2 remain open for $r \geq 4$, and the best known upper bound for both the path partition number and path covering number is $100r \log r$ for all sufficiently large n, the same as the one in Theorem 3.1.10 by Gyárfás et al.

Now, we consider the path partition number of an r-coloured balanced complete bipartite graph $K_{n,n}$. For the case $r = 2$, we say that a 2-colouring of $K_{n,n}$ with red and blue is *split* if the classes can be partitioned as $X_1 \cup X_2$ and $Y_1 \cup Y_2$ such that, $X_1, X_2, Y_1, Y_2 \neq \emptyset$; all edges of (X_1, Y_1) and (X_2, Y_2) are red; and those of (X_1, Y_2) and (X_2, Y_1) are blue. Gyárfás and Lehel proved the following result.

Theorem 3.2.6 (Gyárfás, Lehel - 1973 [45, 49]). *Suppose that we have a 2-colouring of $K_{n,n}$. If the 2-colouring is not split, then there exist two disjoint monochromatic paths with different colours which cover all, except possibly one, of the vertices of $K_{n,n}$.*

Pokrovskiy proved the following slight extension.

Theorem 3.2.7 (Pokrovskiy - 2014 [84]). *Suppose that we have a 2-colouring of $K_{n,n}$. There exists a partition of the vertices of $K_{n,n}$ into two monochromatic paths with different colours if and only if the 2-colouring is not split.*

Pokrovskiy remarked that there are split 2-colourings of $K_{n,n}$ which cannot be partitioned into two monochromatic paths, even when we are allowed to repeat colours. Indeed, any split 2-colouring with X_1, X_2, Y_1, Y_2 satisfying $||X_1| - |Y_1|| \geq 2$ and $||X_1| - |Y_2|| \geq 2$ will have this property. Also, it is easy to see that every 2-coloured $K_{n,n}$ which is split can be partitioned into three monochromatic paths. Hence by Theorem 3.2.6 or 3.2.7, we immediately have the following result.

Theorem 3.2.8. *The path partition number of a 2-coloured balanced complete bipartite graph $K_{n,n}$ is 3.*

For r-colourings of $K_{n,n}$, Pokrovskiy made the following conjecture.

Conjecture 3.2.9 (Pokrovskiy - 2014 [84]). *The path partition number of an r-coloured balanced complete bipartite graph $K_{n,n}$ is $2r - 1$.*

Pokrovskiy gave the following example of an r-colouring of $K_{n,n}$ which shows that the conjecture is optimal. Here, $n = \sum_{i=1}^{r}(10^i + i)$. Let X and Y be the classes of $K_{n,n}$. Partition X into $X_1, \dots, X_r$ and Y into $Y_1, \dots, Y_r$, where $|X_i| = 10^i + i$ and $|Y_i| = 10^i + r + 1 - i$. The edges of (X_i, Y_j) are coloured with colour $i + j \pmod r$. Then, it can be shown that this r-coloured $K_{n,n}$ cannot be partitioned into $2r - 2$ monochromatic paths.

The best known upper bound for the path partition number of an r-coloured $K_{n,n}$ is again $O((r \log r)^2)$ for large r, due to Haxell (Theorem 3.1.19).

Next, we consider tree partition and covering numbers of r-coloured graphs. Recall that when the r-coloured host graph is a complete graph K_n, then these two functions depend only on r. The case $r = 2$ is simply the observation of Erdős and Rado that every 2-coloured K_n contains a monochromatic spanning tree, i.e. the tree partition and covering numbers of a 2-coloured K_n are both 1.

For general $r \geq 2$, it is obvious that the tree covering number is at most r, since the monochromatic stars at any vertex of an r-coloured K_n form a good covering. Construction 2.1.2 implies that if $r-1$ is a prime power and $n \geq (r-1)^2(r-2)$, then the tree partition and covering numbers are both at least $r-1$. Indeed, any monochromatic connected subgraph in the r-colouring ψ of K_n has at most $(r-1)\lceil \frac{n}{(r-1)^2} \rceil < \frac{n}{r-2}$ vertices, and hence any partition or covering by monochromatic trees would require at least $r-1$ trees.

In view of these, Erdős et al. made the following conjecture.

Conjecture 3.2.10 (Erdős, Gyárfás, Pyber - 1991 [27]). *For $r \geq 2$, the tree partition number is $r-1$.*

In support of this conjecture, they managed to settle the case $r = 3$.

Theorem 3.2.11 (Erdős, Gyárfás, Pyber - 1991 [27]). *The tree partition number of a 3-coloured complete graph K_n is 2.*

Haxell and Kohayakawa then came close to proving Conjecture 3.2.10 for large n.

Theorem 3.2.12 (Haxell, Kohayakawa - 1996 [65]). *Let $r \geq 1$ and $n \geq 3r^4 r!(1-\frac{1}{r})^{3(1-r)} \log r$. Then, the vertices of an r-coloured complete graph K_n can be partitioned into at most r monochromatic trees with distinct colours.*

In [64], Haxell also remarked that the method in [65] shows that, the tree partition number for an r-coloured balanced complete bipartite graph $K_{n,n}$ is at most $2r$, for n sufficiently large with respect to r. Before that, Erdős et al. had remarked in [27] that cr^2 is an upper bound (for some constant c).

Finally, Fujita et al. considered the tree partition number of an r-coloured graph G, where G has given independence number. They proved the following result, which is an extension of Theorem 2.7.1.

Theorem 3.2.13 (Fujita, Furuya, Gyárfás, Tóth - 2012 [32]). *For any 2-colouring of a graph G, the vertex set $V(G)$ can be partitioned into at most $\alpha(G)$ monochromatic trees.*

In fact, Fujita et al. proved a stronger version of Theorem 3.2.13 for hypergraphs, which we will see in Section 4. They also proved the following result, which is a version using Gallai colourings.

Theorem 3.2.14 (Fujita, Furuya, Gyárfás, Tóth - 2012 [32]). *For every integer $\alpha \geq 1$ there exists an integer $g = g(\alpha)$ such that the following holds. If G is a graph with $\alpha(G) = \alpha$ and G is given a Gallai colouring, then the vertex set $V(G)$ can be partitioned into at most g monochromatic trees.*

Theorem 3.2.14 extends a result of Gyárfás, Simonyi and Tóth [62] (2012) that in any Gallai colouring of a graph G, the number of monochromatic trees covering all the vertices is bounded in terms of $\alpha(G)$.

Now, we consider the tree covering number of an r-coloured graph G. We have Conjecture 3.2.15 below, which is an equivalent formulation of a long standing conjecture of Ryser (1971). The conjecture appeared in the Ph.D. thesis of Henderson [66]. A stronger form of the conjecture was also made independently by Lovász at around the same time.

Conjecture 3.2.15 (Ryser - 1971; Lovász - 1971, appeared in Henderson [66])**.** *Let $r \geq 2$. Then, for any r-colouring of the edges of a graph G, the vertex set $V(G)$ can be covered by at most $\alpha(G)(r-1)$ monochromatic trees.*

In particular, the tree covering number is $r-1$.

The sharpness of Conjecture 3.2.15 for graphs of sufficiently large order again follows from Construction 2.1.2, if $r-1$ is a prime power. Indeed, for graphs with independence number α, we can take G to consist of α vertex-disjoint cliques, each with at least $(r-1)^2(r-2)$ vertices, and give each clique the r-colouring ψ. Then as before, any partitioning or covering of G by monochromatic trees would require at least $\alpha(r-1)$ trees.

Many partial results to the conjecture are known. The case $r=2$ is equivalent to Kőnig's theorem. After partial results by Haxell [63], and Szemerédi and Tuza [94], the case $r=3$ was solved by Aharoni [1], relying on an interesting topological method that he established with Haxell in [2].

Theorem 3.2.16 (Aharoni - 2001 [1])**.** *For any 3-colouring of a graph G, the vertex set $V(G)$ can be covered by at most $2\alpha(G)$ monochromatic trees.*

For the case $\alpha(G)=1$, i.e. for complete graphs, the cases $r=3,4,5$ were proved, respectively, by Gyárfás [44] (1977); Duchet [23] (1979) and Tuza [96] (1978); and Tuza [96] (1978). Hence, we have the following result.

Theorem 3.2.17. *For $r=2,3,4,5$, the tree covering number is $r-1$.*

All remaining cases of Conjecture 3.2.15 are still open.

We end this subsection by considering a bipartite version of the tree covering number, discovered by Gyárfás and Lehel. For this, they made the following conjecture.

Conjecture 3.2.18 (Gyárfás, Lehel - 1977 [44, 73])**.** *Let $r \geq 2$. Then, in every r-colouring of a complete bipartite graph $K_{m,n}$, the vertices can be covered by at most $2r-2$ monochromatic trees.*

It is easy to see that the vertices of any r-coloured $K_{m,n}$ can be covered by at most $2r-1$ monochromatic trees. Indeed, let u and v be two vertices in opposite classes of $K_{m,n}$, and take the monochromatic double star with centres u and v, along with the remaining monochromatic stars centred at u and v (there are at most $2r-2$ such stars).

On the other hand, Gyárfás [44] (see also Chen et al. [20]) provided the following example of an r-coloured complete bipartite graph $K_{m,n}$ which requires at least $2r-2$ monochromatic trees to cover the vertex set.

Construction 3.2.19. *Let the classes of $K_{m,n}$ be A and B, where $|A| = m = r-1$ and $|B| = n = r!$. Label the vertices of A with $\{1,\dots,r-1\}$ and those of B with the $(r-1)$-permutations of the elements of $\{1,\dots,r\}$. For $i \in A$ and $\pi = j_1 j_2 \cdots j_{r-1} \in B$, let the colour of the edge $i\pi$ be j_i.*

Since each vertex in B is incident with $r-1$ edges of distinct colours, every monochromatic tree is a star with $(r-1)!$ leaves centred in A. This means that there is a vertex cover with $2r-2$ monochromatic trees: just take the r monochromatic stars centred at vertex $r-1$, and add one edge from each vertex $i = 1, 2, \ldots, r-2$ of A. Gyárfás [44] (see also Chen et al. [20]) showed that this particular 2-coloured $K_{m,n}$ cannot be covered with less than $2r-2$ monochromatic trees.

Hence, if Conjecture 3.2.18 is true, then it is best possible. In other words, the conjecture claims that the tree covering number of an r-coloured $K_{m,n}$ is $2r-2$. Chen et al. managed to prove the conjecture for small r.

Theorem 3.2.20 (Chen, Fujita, Gyárfás, Lehel, Tóth - 2012 [20])**.** *Let $r = 2, 3, 4, 5$. Then, for every r-colouring of a complete bipartite graph $K_{m,n}$, the vertex set can be covered by at most $2r-2$ monochromatic trees.*

3.3 Partitioning by regular subgraphs

We consider the problem of partitioning an edge-coloured graph into monochromatic connected d-regular subgraphs and single vertices, where $d \geq 2$. This can be seen as a generalisation of the problem of estimating the cycle partition number of an edge-coloured graph, which is roughly the case $d = 2$, with degenerate cycles playing a minor role. We note that it is important to allow the possibility of having single vertices in such a partition (such vertices can be considered as degenerate d-regular graphs). For example, suppose that we have an edge-colouring of a graph G where there is a vertex v and a colour, say red, such that an edge is red if and only if it is incident with v. Then, we will require v to occur as a single vertex, in any partition of the vertex set $V(G)$ into monochromatic connected d-regular subgraphs and single vertices.

The generalisation from cycles to d-regular subgraphs was suggested by Servatius [93]. We define the *d-regular partition number* of an r-coloured graph G to be the minimum integer m such that, for any r-colouring of the edges of G, the vertices of G can be partitioned into at most m monochromatic connected d-regular subgraphs and single vertices. Sárközy and Selkow were the first to prove a result, in the case where the edge-coloured host graph is complete.

Theorem 3.3.1 (Sárközy, Selkow - 2000 [89])**.** *There exists a constant c such that, for every $r, d \geq 2$ and any r-colouring of a complete graph K_n, the vertices can be partitioned into at most $r^{c(r \log r + d)}$ monochromatic connected d-regular subgraphs and single vertices.*

Again, similar to the consequences of Theorems 3.1.8 and 3.1.20, the d-regular partition number of an r-coloured K_n is independent of n. Using Szemerédi's regularity lemma and with the help of Theorem 2.2.5, Sárközy et al. then improved the bound on the d-regular partition number in Theorem 3.3.1, for sufficiently large n.

Theorem 3.3.2 (Sárközy, Selkow, Song - 2013 [91])**.** *Let $r, d \geq 2$. Then, there exists $n_0 = n_0(r, d)$ such that, for all $n \geq n_0$ and every r-colouring of K_n, the vertex set of K_n can be partitioned into at most $100r \log r + 2rd$ monochromatic connected d-regular subgraphs and single vertices.*

Sárközy et al. [91] provided the following construction, which shows that the bound of $100r \log r + 2rd$ in Theorem 3.3.2 is close to being the best possible, especially if r is small compared to d.

Construction 3.3.3. *Let $n > (r-1)(d-1)$. Let $A_1, \ldots, A_{r-1}$ be disjoint vertex sets of K_n, each with size $d-1$, and let A_r be the remaining vertices. Define an r-colouring on K_n as follows.*

Assign colour 1 to all edges containing a vertex from A_1. Then, assign colour 2 to all edges containing a vertex from A_2 and not in colour 1. We continue in this fashion, until we have assigned colour $r-1$ to all edges containing a vertex from A_{r-1} and not in colour $1, \dots, r-2$. Finally, assign colour r to all the edges within A_r.

Then in this edge-colouring of K_n, it is not hard to show that in any partition of the vertex set of K_n into monochromatic connected d-regular subgraphs and single vertices, all vertices of $A_1 \cup \cdots \cup A_{r-1}$ must occur as single vertices. Hence, the d-regular partition number of an r-coloured K_n is at least $(r-1)(d-1)+1$.

Sárközy and Selkow also considered the d-regular partition number of an r-coloured balanced complete bipartite graph $K_{n,n}$, and proved the following result.

Theorem 3.3.4 (Sárközy, Selkow - 2000 [89])**.** *There exists a constant c such that, for every $r, d \geq 2$ and any r-colouring of a balanced complete bipartite graph $K_{n,n}$, the vertices can be partitioned into at most $r^{c(r \log r + d)}$ monochromatic connected d-regular subgraphs and single vertices.*

Finally, Sárközy et al. considered the situation when the host graph has fixed independence number. They proved the following result, which contains Theorem 3.3.1.

Theorem 3.3.5 (Sárközy, Selkow, Song - 2011 [90])**.** *There exists a constant c such that the following holds. Let $r, d \geq 2$, and G be a graph with independence number $\alpha(G) = \alpha$. Then, for every r-colouring of G, the vertex set $V(G)$ can be partitioned into at most $(\alpha r)^{c(\alpha r \log(\alpha r) + d)}$ monochromatic connected d-regular subgraphs and single vertices.*

A lower bound of $\alpha((r-1)(d-1)+1)$ for the corresponding d-regular partition number can be easily seen, by considering the graph which consists of α cliques, each with more than $(r-1)(d-1)$ vertices, and given the r-colouring as defined in Construction 3.3.3. It would be desirable to improve the large gap between the two bounds. Sárközy et al. suggested that the lower bound of $\alpha((r-1)(d-1)+1)$ may be closer to the truth.

4 Monochromatic Structures in Hypergraphs

In this section, we shall review problems that concern the existence of a monochromatic subhypergraph in an edge-coloured hypergraph, as well as problems associated with partitioning and covering of the host hypergraph by such subhypergraphs. These extend the analogous situations for graphs, which we have already seen in Sections 2 and 3. As in the case for graphs, we will consider situations where the monochromatic subhypergraphs are cycles, paths, and connected hypergraphs. Throughout, all of our edge-coloured host hypergraphs will be non-trivial and t-uniform, for some $t \geq 2$, and hence such a t-uniform host hypergraph has at least t vertices. For many problems, the host hypergraph is a t-uniform complete hypergraph on n vertices, which we denote by $\mathcal{K}_n^t$ (with $n \geq t$).

4.1 Connected subhypergraphs and Berge cycles

We begin by recalling the most standard definitions of hypergraph paths, cycles, and connected hypergraphs.

A hypergraph $\mathcal{P}$ is a *Berge path* of length $\ell \geq 0$ if $\mathcal{P}$ consists of distinct vertices $v_1, \dots, v_{\ell+1}$ and distinct edges $e_1, \dots, e_\ell$ such that for $1 \leq i \leq \ell$, we have $v_i, v_{i+1} \in e_i$. Similarly, a hypergraph $\mathcal{C}$ is a *Berge cycle* of length $\ell \geq 2$ if $\mathcal{C}$ consists of distinct vertices $v_1, \dots, v_\ell$, called the *core* of $\mathcal{C}$, and distinct edges $e_1, \dots, e_\ell$ such that for $1 \leq i \leq \ell$, we have $v_i, v_{i+1} \in e_i$ (with $v_{\ell+1} = v_1$). Note that in both cases, any other incidences are permitted, and for fixed $\ell \geq 2$, Berge paths and cycles are generally not unique, even when we restrict to t-uniformity where $t \geq 3$. Of course, the 2-uniform cases (with $\ell \geq 3$ for Berge cycles) just reduce to simple paths and cycles.

A hypergraph $\mathcal{H}$ is *connected* if for any two vertices of $\mathcal{H}$, there exists a Berge path in $\mathcal{H}$ containing them. Equivalently, $\mathcal{H}$ is connected if the *shadow graph* $G_{\mathcal{H}}$, which is the simple graph where $V(G_{\mathcal{H}}) = V(\mathcal{H})$ and $E(G_{\mathcal{H}}) = \{xy : xy \subset e \text{ for some } e \in E(\mathcal{H})\}$, is connected.

The earliest appearances of these three definitions are possibly in the book of Berge ([10], Ch. 17). Here, we are interested in problems that concern the existence of monochromatic copies of these hypergraphs in edge-coloured hypergraphs.

Our first aim is to consider the extension of Theorem 2.1.1 to hypergraphs. That is, whenever we have an r-colouring of the edges of the t-uniform complete hypergraph $\mathcal{K}_n^t$, how large a monochromatic connected subhypergraph can we always find? Let $h(n, r, t)$ be the largest integer m such that, whenever we have an r-colouring of $\mathcal{K}_n^t$, there is a monochromatic connected subhypergraph with at least m vertices. Hence, we would like to determine the function $h(n, r, t)$. The case $t = 2$ is the case for simple graphs, which was considered in Theorem 2.1.1. Füredi and Gyárfás extended Theorem 2.1.1 as follows.

Theorem 4.1.1 (Füredi, Gyárfás - 1991 [41])**.** *In every r-colouring of the edges of $\mathcal{K}_n^t$, there is a monochromatic connected subhypergraph on at least $\frac{n}{q}$ vertices, where q is the smallest integer satisfying $r \leq 1 + q + q^2 + \cdots + q^{t-1}$. That is, we have $h(n, r, t) \geq \frac{n}{q}$.*

Moreover, for fixed r, q, t, we have the sharp result of $h(n, r, t) = \frac{n}{q} + O(1)$ if $r = 1 + q + q^2 + \cdots + q^{t-1}$ and q is a prime power, where an affine space of dimension t over the field $\mathbb{F}_q$ exists.

We see that Theorem 4.1.1 implies that if $1 \leq r \leq t$, then $q = 1$, and hence $h(n, r, t) = n$. That is, for $1 \leq r \leq t$, every r-coloured $\mathcal{K}_n^t$ contains a monochromatic spanning connected subhypergraph. This result had already been proved by Gyárfás [44] (1977).

To see the sharpness in Theorem 4.1.1, we have the following construction, which uses the existence of affine spaces and is a generalisation of Construction 2.1.2.

Construction 4.1.2. *Let $t, q \geq 2$ and $r \geq 3$ such that $r = 1 + q + q^2 + \cdots + q^{t-1}$ and q is a prime power. Consider the finite affine space $AG(t, q)$ over the field $\mathbb{F}_q$ (see the appendix in Section 5). Let $p_1, \dots, p_{q^t}$ be the points and $\mathcal{P}_1, \dots, \mathcal{P}_r$ be the parallel classes of hyperplanes of $AG(t, q)$. Now, take a near-equal partition of the vertex set of $\mathcal{K}_n^t$ into q^t classes $V_1, \dots, V_{q^t}$. We define an r-colouring ψ on $\mathcal{K}_n^t$ as follows. Let e be an edge of $\mathcal{K}_n^t$, and $e' = \{p_i : 1 \leq i \leq q^t$, and there exists a vertex of e in the class $V_i\}$. We let $\psi(e) = \ell$, where $1 \leq \ell \leq r$ is such that there is a hyperplane in $\mathcal{P}_\ell$ containing all vertices of e'. In particular, this means that the edges inside the classes $V_1, \dots, V_{q^t}$ are arbitrarily coloured.*

Now, if r, q, t are fixed, then in the r-colouring ψ of $\mathcal{K}_n^t$, every monochromatic connected subhypergraph has at most $q^{t-1}\lceil \frac{n}{q^t} \rceil < \frac{n}{q} + q^{t-1} = \frac{n}{q} + O(1)$ vertices.

Applying Theorem 4.1.1 with $t = 3$ and $q = 1, 2$, we have $h(n, 3, 3) = n$ and $h(n, 7, 3) = \frac{n}{2} + O(1)$. Filling in the gap, Gyárfás [44] determined $h(n, 4, 3)$, and Gyárfás and Haxell [48] determined $h(n, 5, 3)$ and $h(n, 6, 3)$. These results are summarised as follows.

Theorem 4.1.3 (Gyárfás - 1977 [44]; Gyárfás, Haxell - 2009 [48])**.**

(a) $h(n,4,3) = \frac{3n}{4} + O(1)$.

(b) $h(n,5,3) = \frac{5n}{7} + O(1)$.

(c) $h(n,6,3) = \frac{2n}{3} + O(1)$.

In general, for fixed t, we see that for the values of r where sharpness is attained in Theorem 4.1.1, the gaps are rather large. It would be desirable to attempt to fill in these gaps.

Next, we consider the problem of finding large monochromatic Berge cycles in edge-coloured complete hypergraphs. We have already considered the case for graphs in Subsection 2.2. We say that a Berge cycle $\mathcal{C}$ in a hypergraph $\mathcal{H}$ on n vertices is *Hamiltonian* if $\mathcal{C}$ has length n. Then, a question we may ask is: *What conditions on an edge-coloured t-uniform complete hypergraph $\mathcal{K}_n^t$ will guarantee the existence of a monochromatic Hamiltonian Berge cycle?*

We see that in the case for simple graphs ($t = 2$), there is little to consider, since the construction after Theorem 2.2.1 shows that in a 2-colouring of K_n, we cannot be guaranteed to have a monochromatic cycle with length greater than $\lceil \frac{2n}{3} \rceil$. In the hypergraphs setting, Gyárfás et al. made the following conjecture.

Conjecture 4.1.4 (Gyárfás, Lehel, Sárközy, Schelp - 2008 [50])**.** *Let $t \geq 2$ be fixed and n be sufficiently large. Then, every $(t-1)$-colouring of $\mathcal{K}_n^t$ contains a monochromatic Hamiltonian Berge cycle.*

We have the following results, showing that the conjecture is solved for $t = 3$, and very nearly solved for $t = 4$.

Theorem 4.1.5 (Gyárfás, Lehel, Sárközy, Schelp - 2008 [50])**.** *Let $n \geq 5$. Then, every 2-colouring of $\mathcal{K}_n^3$ contains a monochromatic Hamiltonian Berge cycle.*

Theorem 4.1.6 (Gyárfás, Sárközy, Szemerédi - 2010 [60])**.** *Let $n \geq 140$. Then, every 3-colouring of $\mathcal{K}_n^4$ contains a monochromatic Berge cycle of length at least $n - 10$.*

Theorem 4.1.6 improved an earlier result of Gyárfás et al. [50], which was stated in asymptotic form. For general t, we have the following partial results to Conjecture 4.1.4. In the first result, the number of colours used to colour $\mathcal{K}_n^t$ is much less than $t-1$. The second result is an asymptotic version of the conjecture, where the number of colours used is at most $t - \lfloor \log_2 t \rfloor$.

Theorem 4.1.7 (Gyárfás, Lehel, Sárközy, Schelp - 2008 [50])**.** *Let $t \geq 4$ be fixed and n be sufficiently large. Then, every $\lfloor \frac{t-1}{2} \rfloor$-colouring of $\mathcal{K}_n^t$ contains a monochromatic Hamiltonian Berge cycle.*

Theorem 4.1.8 (Gyárfás, Lehel, Sárközy, Schelp - 2008 [50])**.** *For all $\eta > 0$ and integers $t, r \geq 2$ with $t \geq r + \log_2(r+1)$, there exists $n_0 = n_0(\eta, t, r)$ such that for every $n \geq n_0$, every r-colouring of $\mathcal{K}_n^t$ contains a monochromatic Berge cycle of length at least $(1-\eta)n$.*

Gyárfás et al. [50] also noted that the number of colours, $t-1$, cannot be increased in Conjecture 4.1.4. They pointed out that Gyárfás and Sárközy [55], who considered the situation when t colours are used, gave an example of a t-colouring of $\mathcal{K}_n^t$ where the maximum length of a monochromatic Berge cycle is at most $\lceil \frac{(2t-2)n}{2t-1} \rceil$. This example will be presented in Construction 4.1.12 in a more general form. Gyárfás and Sárközy made the following conjecture, which claims that the value of $\lceil \frac{(2t-2)n}{2t-1} \rceil$ is essentially best possible.

Conjecture 4.1.9 (Gyárfás, Sárközy - 2011 [55])**.** *For all $\eta > 0$ and integer $t \geq 2$, there exists $n_0 = n_0(\eta, t)$ such that for every $n \geq n_0$, every t-colouring of $\mathcal{K}_n^t$ contains a monochromatic Berge cycle of length at least $(\frac{2t-2}{2t-1} - \eta)n$.*

We see that the case for graphs ($t = 2$) was solved by Faudree et al. (Theorem 2.2.1). Gyárfás and Sárközy managed to settle the case $t = 3$.

Theorem 4.1.10 (Gyárfás, Sárközy - 2011 [55])**.** *For all $\eta > 0$, there exists $n_0 = n_0(\eta)$ such that for every $n \geq n_0$, every 3-colouring of $\mathcal{K}_n^3$ contains a monochromatic Berge cycle of length at least $(\frac{4}{5} - \eta)n$.*

Dorbec et al. considered a generalisation, by imposing a restriction on the Berge cycles. They introduced the notion of s-tight Berge cycles, as follows. For $s \geq 2$, a Berge cycle is *s-tight* if it has core $v_1, \ldots, v_\ell$ and distinct edges $e_1, \ldots, e_\ell$ such that, for all $1 \leq i \leq \ell$, we have $v_i, v_{i+1}, \ldots, v_{i+s-1} \in e_i$ (with indices taken modulo ℓ). Hence, the case $s = 2$ reduces to ordinary Berge cycles. Dorbec et al. made the following conjecture, which is a generalisation of Conjecture 4.1.4.

Conjecture 4.1.11 (Dorbec, Gravier, Sárközy - 2008 [22])**.** *Let $2 \leq r, s \leq t$ be fixed such that $r+s \leq t+1$, and n be sufficiently large. Then, every r-colouring of $\mathcal{K}_n^t$ contains a monochromatic Hamiltonian s-tight Berge cycle.*

Dorbec et al. noted that if the conjecture is true, then it is best possible. They provided Construction 4.1.12 below, which Gyárfás and Sárközy [55] also presented for $s = 2$ and $r = t$. The construction shows that if $2 \leq r, s \leq t$ and $r + s > t + 1$, then there is an r-colouring of $\mathcal{K}_n^t$ where, the maximum length of a monochromatic s-tight Berge cycle is at most $\lceil \frac{s(r-1)n}{s(r-1)+1} \rceil$. Hence for Berge cycles ($s = 2$) and $r = t$, this value, obtained by Gyárfás and Sárközy, becomes $\lceil \frac{(2t-2)n}{2t-1} \rceil$.

Construction 4.1.12. *Let $2 \leq r, s \leq t$ and $r+s > t+1$. Let $A_1, \ldots, A_{r-1}$ be disjoint vertex sets from $\mathcal{K}_n^t$, each of size $\lfloor \frac{n}{s(r-1)+1} \rfloor$. The t-edges not containing a vertex from A_1 are given colour 1. The t-edges that are not coloured yet and do not contain a vertex from A_2 are given colour 2. Continuing in this fashion, the t-edges that are not coloured yet with colours $1, \ldots, r-2$ and do not contain a vertex from A_{r-1} are given colour $r-1$. Finally, the t-edges that contain a vertex from all $r-1$ sets $A_1, \ldots, A_{r-1}$ are given colour r.*

Then in this r-colouring of $\mathcal{K}_n^t$, an s-tight Berge cycle in colour i for $1 \leq i \leq r-1$ has length at most $\lceil \frac{s(r-1)n}{s(r-1)+1} \rceil$, since the subhypergraph induced by the edges in colour i leaves out A_i (a set of size $\lfloor \frac{n}{s(r-1)+1} \rfloor$) completely. Also, note that for any s ($> t - r + 1$) consecutive vertices in the core of an s-tight Berge cycle in colour r (if such a cycle exists), at least one vertex lies in $A_1 \cup \cdots \cup A_{r-1}$. Otherwise, there is an edge of the cycle containing s vertices outside of $A_1 \cup \cdots \cup A_{r-1}$, as well as a vertex from each of $A_1, \ldots, A_{r-1}$ (since the edge has colour r). This gives $t \geq s + r - 1$, a contradiction. Thus, the cycle has length at most $s(r-1)\lfloor \frac{n}{s(r-1)+1} \rfloor \leq \lceil \frac{s(r-1)n}{s(r-1)+1} \rceil$.

Conjecture 4.1.11 is exclusive from the preceding results when $s \geq 3$. For this, Dorbec et al. proved that the conjecture holds for $r = 2$, $s = 3$ and $t = 5$. They also proved the partial result where the condition $r + s > t + 1$ is significantly weakened, with the sum $r + s$ essentially being replaced by the product rs.

Theorem 4.1.13 (Dorbec, Gravier, Sárközy - 2008 [22]). *Let $n \geq 7$. Then, every 2-colouring of $\mathcal{K}_n^5$ contains a monochromatic Hamiltonian 3-tight Berge cycle.*

Theorem 4.1.14 (Dorbec, Gravier, Sárközy - 2008 [22]). *Let $2 \leq r, s \leq t$ be fixed such that $rs + 1 \leq t$, and $n \geq 2(s+1)tr^2$. Then, every r-colouring of $\mathcal{K}_n^t$ contains a monochromatic Hamiltonian s-tight Berge cycle.*

Gyárfás et al. then managed to prove the smallest non-trivial case of Conjecture 4.1.11: $r = 2$, $s = 3$ and $t = 4$.

Theorem 4.1.15 (Gyárfás, Sárközy, Szemerédi - 2010 [59]). *There exists n_0 such that for $n \geq n_0$, every 2-colouring of $\mathcal{K}_n^4$ contains a monochromatic Hamiltonian 3-tight Berge cycle.*

Theorem 4.1.15 improved an earlier result by the same authors [60] that, for $n \geq 15$, every 2-colouring of $\mathcal{K}_n^4$ contains a monochromatic 3-tight Berge cycle with length at least $n - 10$.

We end this subsection by considering an analogue of Gallai colourings for hypergraphs. We say that an edge-colouring of the t-uniform complete hypergraph $\mathcal{K}_n^t$ is a *t-Gallai colouring* if no complete subhypergraph $\mathcal{K}_{t+1}^t$ has distinct coloured edges, i.e. there is no rainbow coloured t-simplex. Hence, a 2-Gallai colouring is just a Gallai colouring. This generalisation was suggested by Gyárfás and Lehel in 2007. They observed the fact that: *Every Gallai coloured complete graph has a monochromatic connected subgraph* (mentioned in Subsection 2.6) does not extend to t-Gallai colourings of $\mathcal{K}_n^t$. Define $f_t(n)$ to be the maximum integer m such that, for every t-Gallai colouring of $\mathcal{K}_n^t$, there exists a monochromatic connected subhypergraph on at least m vertices. Gyárfás and Lehel proposed the following problem.

Problem 4.1.16 (Gyárfás, Lehel - 2007; appeared in Chua et al. [21]). *For $n \geq t \geq 2$, determine the function $f_t(n)$.*

Chua et al. proved the following result for $t = 3$.

Theorem 4.1.17 (Chua, Gyárfás, Hossain - 2013 [21]).

(a) $\lceil \frac{n+3}{2} \rceil \leq f_3(n) \leq \lceil \frac{4n}{5} \rceil$, *and this determines $f_3(n)$ for $3 \leq n \leq 6$.*

(b) $f_3(7) = 6$.

4.2 Partitioning and covering by monochromatic subhypergraphs

In this subsection, we consider problems about partitioning or covering of edge-coloured hypergraphs by monochromatic subhypergraphs. These monochromatic subhypergraphs will be cycles, paths, and connected hypergraphs.

Here, we will use another type of path and another type of cycle. Let $t \geq 2$. A t-uniform hypergraph $\mathcal{P}$ is a *loose path* of length $\ell \geq 0$ if $\mathcal{P}$ consists of distinct vertices $v_1, \ldots, v_{\ell-1}$ and distinct edges $e_1, \ldots, e_\ell$ such that for $1 \leq i < j \leq \ell$, we have $e_i \cap e_j = \{v_i\}$ if $j - i = 1$, and $e_i \cap e_j = \emptyset$ otherwise. Similarly, a t-uniform hypergraph $\mathcal{C}$ is a *loose cycle* of length $\ell \geq 2$ if $\mathcal{C}$ consists of distinct vertices $v_1, \ldots, v_\ell$ and distinct edges $e_1, \ldots, e_\ell$ such that, if $\ell = 2$, then $t \geq 3$ and $e_1 \cap e_2 = \{v_1, v_2\}$. Otherwise, if $\ell \geq 3$, then for $1 \leq i \leq \ell$, we have $e_i \cap e_{i+1} = \{v_i\}$ (where $e_{\ell+1} = e_1$), and $e_i \cap e_j = \emptyset$ for every $1 \leq i \neq j \leq \ell$ with $|j - i| \not\equiv 1 \pmod{\ell}$.

As in Section 3, we have degenerate cases of loose paths and loose cycles. For fixed $t \geq 2$, we will also regard a set of less than t vertices as a loose path, and a single t-edge or a set of less than t vertices as a loose cycle.

Gyárfás and Sárközy proved some results concerning the partitioning and covering of the vertex set of an edge-coloured t-uniform complete hypergraph $\mathcal{K}_n^t$, using monochromatic Berge paths, loose paths or loose cycles.

Theorem 4.2.1 (Gyárfás, Sárközy - 2013 [56])**.** *For every t-colouring of $\mathcal{K}_n^t$, there exists a partition of the vertices into monochromatic Berge paths with distinct colours.*

Theorem 4.2.2 (Gyárfás, Sárközy - 2013 [56])**.** *For every 2-colouring of $\mathcal{K}_n^t$, there exist two vertex-disjoint monochromatic loose paths of distinct colours such that they cover all but at most $2t-5$ vertices.*

Theorem 4.2.3 (Gyárfás, Sárközy - 2013 [56])**.** *Let $r \geq 1$ and $t \geq 3$. There exists a constant $c' = c'(r,t)$ such that in every r-colouring of $\mathcal{K}_n^t$, the vertices can be partitioned into at most c' monochromatic vertex-disjoint loose cycles.*

By Theorem 4.2.3, we see that the number of loose cycles is independent of n, and we can define $c(r,t)$ to be the minimum number of monochromatic loose cycles needed to partition the vertex set of any r-coloured t-uniform complete hypergraph $\mathcal{K}_n^t$. This is a generalisation of the consequence of Theorem 3.1.8 that the cycle partition number $p(r)$ is well-defined. The proof of Theorem 4.2.3 by Gyárfás and Sárközy used the method of Erdős et al. [27] and the linearity of Ramsey numbers of hypergraphs with bounded degree, and this gave quite a weak upper bound for $c(r,t)$ (exponential in r and t). Sárközy used a version of Rödl and Schacht's regularity lemma for hypergraphs to obtain the following improvement, for sufficiently large n.

Theorem 4.2.4 (Sárközy - 2014 [88])**.** *Let $r,t \geq 2$. There exists $n_0 = n_0(r,t)$ such that if $n \geq n_0$, then for every r-colouring of $\mathcal{K}_n^t$, the vertices can be partitioned into at most $50rt\log(rt)$ vertex-disjoint monochromatic loose cycles.*

Gyárfás and Sárközy also made the following conjecture in relation to Theorem 4.2.2.

Conjecture 4.2.5 (Gyárfás, Sárközy - 2013 [56])**.** *For every 2-colouring of $\mathcal{K}_n^t$, there exist two disjoint monochromatic loose paths of distinct colours such that they cover all but at most $t-2$ vertices. This estimate is sharp for sufficiently large n.*

They provided the following construction which shows that if true, Conjecture 4.2.5 is best possible (for sufficiently large n).

Construction 4.2.6. *Let X and Y be two disjoint sets of vertices with $|X| = m(t-1)+1$ and $|Y| = 2(t-1)$, where $m \geq 4(t-1)$. Consider the t-uniform complete hypergraph on $X \cup Y$. We colour all t-edges within X red, and all remaining t-edges blue.*

In the construction, any red loose path leaves at least $|Y| > 2t-3$ vertices uncovered. Also, any blue loose path has at most $2|Y|$ edges and hence does not cover at least $|X| - (2|Y|(t-1) - |Y|+1) \geq 2t-3$ vertices of X, since $m \geq 4(t-1)$. Now, any two monochromatic vertex-disjoint loose paths with distinct colours together do not cover at least $t-2$ vertices. Indeed, if at least one path is trivial, then at least $(2t-3)-(t-1) = t-2$ vertices are uncovered by the two paths. Otherwise, the two paths together cover $p(t-1)+2$ vertices for some p. However, we have $(m+2)(t-1)+1$ vertices, thus at least $t-2$ vertices remain uncovered.

Next, as in the case for graphs, we can consider a situation when the edge-coloured host hypergraph is not complete, by fixing the independence number of the hypergraph. In this direction, Gyárfás and Sárközy extended Theorem 4.2.3 as follows.

Theorem 4.2.7 (Gyárfás, Sárközy - 2014 [57]). *Let $r \geq 1$, $t \geq 2$ and $\alpha \geq t-1$. There exists a constant $c' = c'(r,t,\alpha)$ such that, for every r-colouring of a t-uniform hypergraph $\mathcal{H}$ with independence number $\alpha(\mathcal{H}) = \alpha$, the vertex set $V(\mathcal{H})$ can be partitioned into at most c' monochromatic vertex-disjoint loose cycles.*

As before, we can define $c(r,t,\alpha)$ to be the minimum number of monochromatic loose cycles needed to partition the vertex set of any r-coloured t-uniform hypergraph $\mathcal{H}$ with independence number $\alpha(\mathcal{H}) = \alpha$. The upper bound for $c(r,t,\alpha)$ obtained by Gyárfás and Sárközy is again quite weak, and it would be desirable to find a good upper bound.

We see that for the function $c(r,t,\alpha)$, the cases $t=2$, $\alpha=1$; $t=2$; and $\alpha=1$ give previous functions $p(r)$, $p(r,\alpha)$ and $c(r,t)$ respectively. Gyárfás and Sárközy made the following conjecture for general t and α. It is an extension of the theorem of Pósa (Theorem 3.1.21) to t-uniform hypergraphs.

Conjecture 4.2.8 (Gyárfás, Sárközy - 2014 [57]). *For $t \geq 2$ and $\alpha \geq 1$, we have $c(1,t,\alpha) = \alpha$. That is, for every t-uniform hypergraph $\mathcal{H}$, the vertex set $V(\mathcal{H})$ can be partition into at most $\alpha(\mathcal{H})$ loose cycles.*

Gyárfás and Sárközy proved a result that is weaker than Conjecture 4.2.8 (but still extends Pósa's theorem), replacing loose cycles by *weak cycles* where only cyclically consecutive edges intersect but their intersection size is not restricted.

Theorem 4.2.9 (Gyárfás, Sárközy - 2014 [57]). *The vertex set of every t-uniform hypergraph $\mathcal{H}$ can be partitioned into at most $\alpha(\mathcal{H})$ vertex-disjoint vertices, edges and weak cycles.*

Now, we consider problems about partitioning and covering of edge-coloured hypergraphs by monochromatic connected subhypergraphs. We remark that this can be seen as an extension of the situation for edge-coloured graphs where we partition or cover their vertex sets by monochromatic trees, which is equivalent to using monochromatic connected subgraphs.

We have the following result of Fujita et al. about partitioning, which is the extension of Theorem 3.2.13 as we have remarked earlier.

Theorem 4.2.10 (Fujita, Furuya, Gyárfás, Tóth - 2012 [32]). *For $t \geq 2$ and every 2-colouring of a t-uniform hypergraph $\mathcal{H}$, the vertex set $V(\mathcal{H})$ can be partitioned into at most $\alpha(\mathcal{H}) - t + 2$ monochromatic connected subhypergraphs.*

Next, we consider results about covering. We recall the case of the conjecture of Ryser and Lovász (Conjecture 3.2.15) for r-coloured complete graphs, i.e. the tree covering number is $r-1$. Somewhat surprisingly, Király proved that an analogue of the conjecture holds for hypergraphs.

Theorem 4.2.11 (Király - 2014 [69]). *For $t \geq 3$ and every r-colouring of the t-uniform complete hypergraph $\mathcal{K}_n^t$, the vertex set can be covered by at most $\lceil \frac{r}{t} \rceil$ monochromatic connected subhypergraphs. This result is best possible for all sufficiently large n.*

The case $1 \leq r \leq t$ in Theorem 4.2.11 was already known, and is a result of Gyárfás [44] as mentioned in the remark after Theorem 4.1.1.

In [69], Király provided a construction of an r-colouring of $\mathcal{K}_n^t$, which shows that the bound of $\lceil \frac{r}{t} \rceil$ in Theorem 4.2.11 is best possible. However, his construction is valid only for $r \equiv 1 \pmod{t}$ and a specific value of n. Here, we modify his construction slightly to present an r-colouring of $\mathcal{K}_n^t$ for all $r > t \geq 2$ and sufficiently large n, such that the vertices of $\mathcal{K}_n^t$ cannot be covered by fewer than $\lceil \frac{r}{t} \rceil$ monochromatic connected subhypergraphs.

Construction 4.2.12. *Let $r > t \geq 2$, with $r = ct + b$ for some $c \geq 1$ and $1 \leq b \leq t$. Let $n \geq \binom{r}{c}$. We colour the edges of $\mathcal{K}_n^t$ with r colours as follows. Let $X = \{1, \ldots, r\}$, and assign c-subsets of X to the vertices of $\mathcal{K}_n^t$ such that every c-subset appears. Now, for an edge e of $\mathcal{K}_n^t$, the union of the c-subsets at the vertices of e has size at most ct, and hence does not contain some $i \in X$. We colour the edge e with such a colour i. This gives an r-colouring of $\mathcal{K}_n^t$.*

Then in this r-colouring of $\mathcal{K}_n^t$, any set of $\lceil \frac{r}{t} \rceil - 1 = c$ monochromatic connected subhypergraphs do not cover all the vertices of $\mathcal{K}_n^t$. Indeed, if we take all edges whose colours are in some set $Y \subset X$ with $|Y| = c$, then any vertex of $\mathcal{K}_n^t$ labelled by Y is not covered by the edges.

Fujita et al. then considered extensions of Király's result, where the host hypergraph $\mathcal{H}$ is t-uniform and has independence number t.

Theorem 4.2.13 (Fujita, Furuya, Gyárfás, Tóth - 2014 [33])**.** *Let $t \geq 2$, and $\mathcal{H}$ be a t-uniform hypergraph on at least $t + 1$ vertices, with $\alpha(\mathcal{H}) = t$.*

(a) For every t-colouring of $\mathcal{H}$, the vertex set $V(\mathcal{H})$ can be covered by at most two monochromatic connected subhypergraphs.

(b) If $t \geq 3$, then for every $(t+1)$-colouring of $\mathcal{H}$, the vertex set $V(\mathcal{H})$ can be covered by at most three monochromatic connected subhypergraphs.

Fujita et al. remarked that in part (b), the case $t = 2$ does not hold, since the construction mentioned after Conjecture 3.2.15 (with $r = 3$ and $\alpha = 2$) is an example of a 3-coloured graph G with $\alpha(G) = 2$, which needs at least four monochromatic connected subgraphs to cover the vertex set $V(G)$. On the other hand, Aharoni's result (Theorem 3.2.16) shows that four monochromatic connected subgraphs is best possible. They also provided the following examples, showing that both (a) and (b) are sharp.

Construction 4.2.14. *For (a), a trivial example is a hypergraph containing a complete t-uniform complete hypergraph and one isolated vertex, with any t-colouring. A less trivial example is a t-uniform hypergraph with vertices partitioned into t classes $V_1, \ldots, V_t$, and having all t-edges that do not meet all classes. The colour of an edge e is any i such that e does not meet the class V_i.*

For (b), consider the following $(t+1)$-coloured t-uniform hypergraph. Take disjoint vertex sets $V_1, \ldots, V_{t+1}$ and an isolated vertex, and add all t-edges in $V_1 \cup \cdots \cup V_{t+1}$. The colour of an edge e is any i such that e does not meet the class V_i.

Then in all three constructions, the hypergraph has independence number t. In the first two examples, one monochromatic connected subhypergraph does not cover all the vertices, and in the last example, two monochromatic connected subhypergraphs are insufficient.

5 Appendix: Finite Affine Planes and Affine Spaces

It is a well-known result from algebra that there exists a finite field with q elements if and only if q is a prime power. Moreover, if q is a prime power, then a finite field with q elements is unique up to isomorphism, and we denote this field by $\mathbb{F}_q$.

Let $t, q \geq 2$ be integers, where q is a prime power. The *finite affine space of dimension* t *over the field* $\mathbb{F}_q$, denoted by $AG(t,q)$, is a finite geometry which consists of a vector space of q^t points, and a collection of *hyperplanes*, each of which is a coset of a subspace with dimension $t-1$. The collection of hyperplanes can be partitioned into $\frac{q^t-1}{q-1} = 1+q+q^2+\cdots+q^{t-1}$ classes, so that each class contains exactly q pairwise parallel hyperplanes, forming a partition of the q^t points.

More precisely, $AG(t,q)$ may be defined as follows. The field $\mathbb{F}_q$ is equipped with addition and multiplication operations with identities 0 and 1. Consider the grid $\mathbb{F}_q^t = \{(x_1,\ldots,x_t) : x_i \in \mathbb{F}_q$ for $1 \leq i \leq t\}$. Two non-zero vectors $(a_1,\ldots,a_t)$ and $(b_1,\ldots,b_t)$ of $\mathbb{F}_q^t$ are *equivalent* if there exists $c \in \mathbb{F}_q \setminus \{0\}$ such that $(b_1,\ldots,b_t) = (ca_1,\ldots,ca_t)$. It is easy to see that this defines an equivalence relation on the non-zero vectors of $\mathbb{F}_q^t$, and there are $r = \frac{q^t-1}{q-1} = 1+q+q^2+\cdots+q^{t-1}$ equivalence classes. Let $(a_{11},\ldots,a_{t1}),\ldots,(a_{1r},\ldots,a_{tr}) \in \mathbb{F}_q^t$ be representatives from these equivalence classes. Now for $1 \leq j \leq r$ and $c \in \mathbb{F}_q$, a *hyperplane* is a set $H_{j,c} = \{(x_1,\ldots,x_t) \in \mathbb{F}_q^t : a_{1j}x_1 + \cdots + a_{tj}x_t = c\}$. Note that each $H_{j,0}$ (for $1 \leq j \leq r$) is a vector subspace of $\mathbb{F}_q^t$ with dimension $t-1$, and every $H_{j,c}$ is a coset of $H_{j,0}$ (for $c \in \mathbb{F}_q$). We define $AG(t,q)$ to consist of the points of $\mathbb{F}_q^t$, and the set of all hyperplanes $H_{j,c}$. Then, many properties are satisfied by $AG(t,q)$, which include the following.

- Every hyperplane contains q^{t-1} points.
- Every point is contained in exactly $r = 1+q+q^2+\cdots+q^{t-1}$ hyperplanes.
- Every set of t points lies in a unique hyperplane.
- There are a total of $qr = q(1+q+q^2+\cdots+q^{t-1})$ hyperplanes. Moreover, the family of all hyperplanes can be partitioned into families $\mathcal{P}_1,\ldots,\mathcal{P}_r$, where $\mathcal{P}_j = \{H_{j,c} : c \in \mathbb{F}_q\}$ for $1 \leq j \leq r$. Each family $\mathcal{P}_j$ contains exactly q hyperplanes which form a partition of the q^t points of $\mathbb{F}_q^t$. The families $\mathcal{P}_1,\ldots,\mathcal{P}_r$ are called the *parallel classes of hyperplanes.*

In particular, if $t = 2$, then the finite geometry $AG(2,q)$ is the *finite affine plane over* $\mathbb{F}_q$. In this case, the hyperplanes become *lines.*

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