Wang Tile and Aperiodic Sets of Tiles

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Hilbert's Entscheidungsproblem

The Entscheidungsproblem (German for "decision problem") was posed by David Hilbert in 1928. He asked for an algorithm to solve the following problem.

- Input: A statement of a first-order logic (possibly with a finite number of axioms),
- Output: "Yes" if the statement is universally valid (or equivalently the statement is provable from the axioms using the rules of logic), and "No" otherwise.

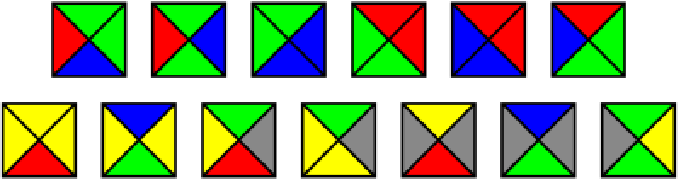
Church-Turing Theorem

Theorem (1936, Church and Turing, independently)

The Entscheidungsproblem is undecidable.

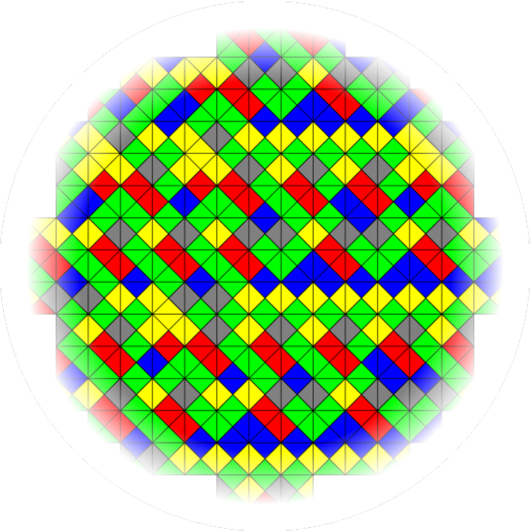
Wang Tile

In order to study the decidability of a fragment of first-order logic, the statements with the form $(\forall x)(\exists y)(\forall z)P(x, y, z)$, Hao Wang introduced the Wang Tile in 1961.



Wang's Domino Problem

Given a set of Wang tiles, is it possible to tile the infinite plane with them?



Hao Wang

Hao Wang (王浩, 1921-1995), Chinese American philosopher, logician, mathematician.

Undecidability and Aperiodic Tiling

Conjecture (1961, Wang)

If a finite set of Wang tiles can tile the plane, then it can tile the plane periodically.

If Wang's conjecture is true, there exists an algorithm to decide whether a given finite set of Wang tiles can tile the plane. (By Konig's infinity lemma)

Domino problem is undecidable

Wang's student, Robert Berger, gave a negative answer to the Domino Problem, by reduction from the Halting Problem. It was also Berger who coined the term "Wang Tiles".

Theorem (1966, Robert Berger, Memoris of AMS)

Domino problem is undecidable.

Two Key Steps in the Proof

- Turing machine can be emulated by Wang Tiles.
- There exists an aperiodic set of Wang Tiles.

Simplified Proof in 1971

Berger constructed an aperiodic set of Wang tiles which contains over 20,000 tiles.

Theorem (1971, R. M. Robinson, Invent. Math.)

Domino problem is undecidable, by constructing an aperiodic set containing 52 Wang tiles.

Robinson’s Tiles

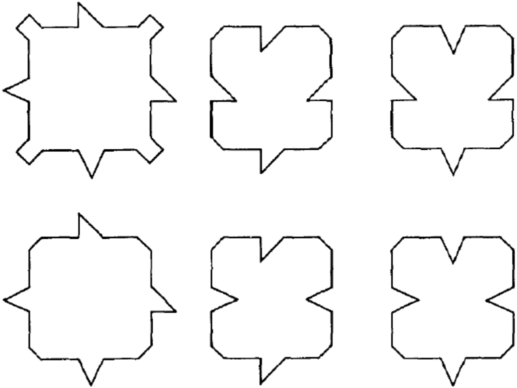


Fig. 1. Six tiles which force nonperiodicity

Reduced to 14

Theorem (1996, Jarkko Kari, Discrete Math.)

There exists an aperiodic set containing 14 Wang tiles.

J. Kari / Discrete Mathematics 160 (1996) 259–264

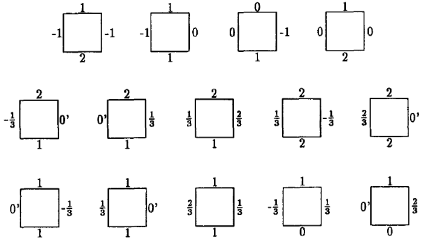


Fig. 1. Aperiodic set of 14 Wang tiles.

Kari's Technique

Most other aperiodic tilings are self-similar, Kari gave the first example of a non-self-similar aperiodic tiling.

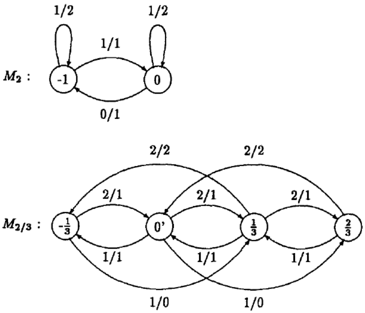


Fig. 2. Mealy machine corresponding to the aperiodic tile set.

- Emulating a Mealy machine (a kind of finite automata).
- Using the Beatty sequences.

Theorem (1996, Karel Culik II, Discrete Math.)

There exists an aperiodic set containing 13 Wang tiles.

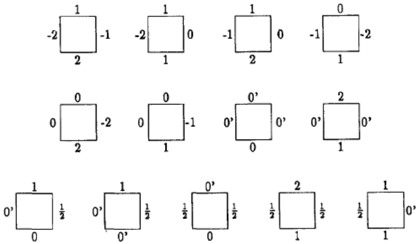


Fig. 5. An aperiodic set of 13 Wang tiles.

Theorem (2015, Emmanuel Jeandel and Michael Rao)

There exists an aperiodic set containing 11 Wang tiles. Moreover, there is no aperiodic set of Wang tiles with 10 tiles or less, and there is no aperiodic set of Wang tiles with less than 4 colors.

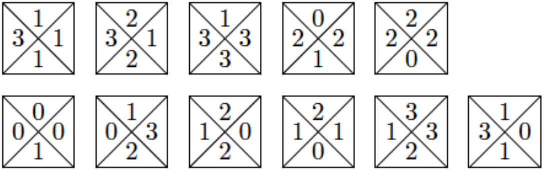
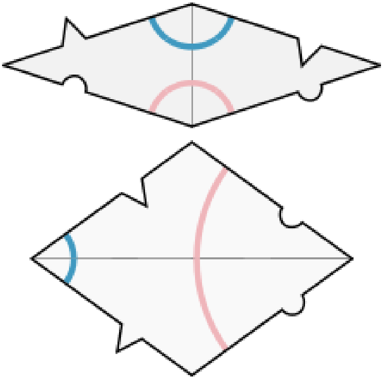


Figure 4: Wang set $\mathcal{T}'$.

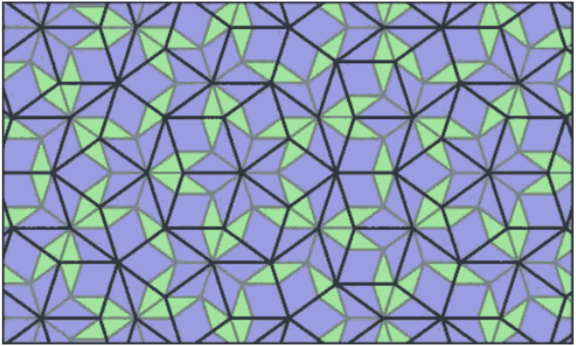
Aperiodic Tiling Forced by Shape Only

The aperiodic tiling of Wang tiles are forced by shape and colors (or pattern on the tiles).

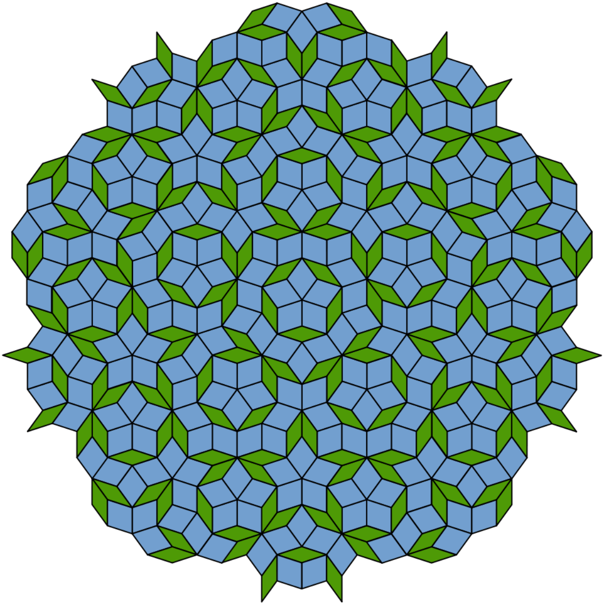
Penrose's Tiles (1974)



Self-similarity of Penrose Tiling



A Penrose Tiling with 5-fold Symmetry



The Discovery of Quasicrystal

- The study of aperiodic tilings was initiated by Wang in the early 1960s.
- Penrose tiling was found in 1970s.
- Quasicrystal with 5-fold symmetry was first observed in 1982 by Dan Shechtman.
- One day in the 1980s, C. N. Yang(杨振宁) called Wang and told Wang that his work on aperiodic tiling found applications in crystallography.
- Dan Shechtman was awarded the Nobel Prize in Chemistry in 2011.

One Tile

Theorem (2011, Socolar and Taylor, JCTA)

There exists an aperiodic hexagonal tile (allowing reflection).

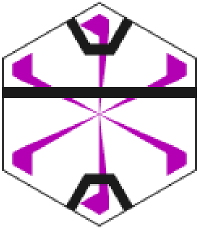


Fig. 3. Enforcing nonperiodicity by shape alone with a multiply-connected prototile. All the patches of a single color, taken together, form a single tile. (For a color image, the reader is referred to the web version of this article.)

The Einstein Problem

The Einstein (German for "one stone") problem is still open. Is there a single tile (homeomorphic to the unit disk) that can tessellate the plane only non-periodically?

References

R. M. Robinson, Undecidability and Nonperiodicity for Tiling of the Plane. *Inventiones Mathematicae* **12** (1971) 177-209.

Emmanuel Jeandel and Michael Rao, An aperiodid set of 11 Wang tiles. arXiv:1506.06492 [cs.DM]

J. E. S. Socolar and J. M. Taylor, An aperiodic hexagonal tile. *J. of Combin. Theory Series A* **118** (2011) 2207-2231.

Thank you!